Math 5286H

Midterm 2

No collaboration is allowed. This test is open-book and open-library but no electronic sources may be consulted.

This test is due in-class on **Friday, March 12**.

1. (a) For any ring R with an ideal $I \subsetneq R$, prove that there is a bijective correspondence between maximal ideals of R/I and maximal ideals of R that contain I.

   (b) Find all maximal ideals of the ring $\mathbb{C}[x, y]/(x^5 - 8y^2, y - 2)$.

2. Suppose that R is an integral domain, $\mathbb{F}$ is a field, and $\phi : R \to \mathbb{F}$ is a ring homomorphism. Let K be the field of fractions of R. Show that there exists a homomorphism $\phi' : K \to \mathbb{F}$ such that $\phi'(r/1) = \phi(r)$ for all $r \in R$ if and only if $ker(\phi) = 0$.

3. Let $S \subset \mathbb{C}^2$ be the set of elements $\{(0, n) | n \in \mathbb{Z}\}$.

   (a) Determine the ideal $I(S) \subset \mathbb{C}[x, y]$ of functions vanishing on S.

   (b) Determine the variety $V(I(S)) \subset \mathbb{C}^2$.

4. Factor the following elements into primes in the following unique factorization domains.

   (a) $x^8 + x$ in $\mathbb{Z}/2[x]$.

   (b) $12 + 24i$ in $\mathbb{Z}[i]$.

   (c) $4x^3 + 18x + 6$ in $\mathbb{Z}[x]$.

   (d) $x^3 - y^3$ in $\mathbb{Q}[x, y]$.

5. A prime ideal P of R is called *minimal* if there are no prime ideals Q with $0 \subsetneq Q \subsetneq P$.

   (a) If R has factorization into irreducibles and $a \in R$ is a prime element, show that the ideal (a) is a minimal prime ideal.

   (b) If R is a UFD, show conversely that every minimal prime ideal is of the form (a) for some prime element $a \in R$.