Math 5286H

Midterm 1

No collaboration is allowed. This test is open-book and open-library but no electronic sources may be consulted.

This test is due in-class on **Friday, February 12**.

1. (a) Describe all possible ring homomorphisms $\mathbb{Z}[x]/(x^2+1) \to \mathbb{Z}/8$.
   (b) Describe all possible ring homomorphisms $\mathbb{Z}[x]/(x^5-x) \to \mathbb{Z}/5$.
2. (a) Show that the element $\sqrt{2}+1$ is a unit in the ring $\mathbb{Z}[\sqrt{2}]$.
   (b) Prove that the ring $\mathbb{Z}[\sqrt{2}]$ has infinitely many units.
   (c) Show that the Gaussian integers $\mathbb{Z}[i]$ have only finitely many units.
3. Let $a, b, c, d \in \mathbb{Z}$. Prove that the ring $\mathbb{Z}[x,y]/(ax+by, cx+dy)$ is isomorphic to $\mathbb{Z}$ if and only if the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has determinant ± 1.
4. Find all prime ideals in the ring $\mathbb{Z}[x]/(x^3-1)$ that contain the element 3.
5. Suppose R is an integral domain.
   (a) Prove that if $r \in R$ is a root of a polynomial $f(x) \in R[x]$, and $f(x) = g(x)h(x)$, then r is either a root of g or h.
   (b) Prove that a degree n monic polymonial $f(x) \in R[x]$ has at most n distinct roots.