Math 5286H

Final Exam

No collaboration is allowed. This test is open-book and open-library but no electronic sources may be consulted.

This test is due in-class on **Friday, May 7**.

Part A: Short answer.

Answers only, proofs are not required. (3 points each)

1. How many units are there in the ring $\mathbb{Z}/192$?
2. Give generators for the kernel of the map $\mathbb{C}[x, y] \to \mathbb{C}[t]$ sending x to t^2 and y to t^5.
3. The nilradical of a ring R is the set of elements x such that $x^k = 0$ for some $k \in \mathbb{N}$. The nilradical is an ideal. If we have $n = p^e$ for p prime, find the nilradical of the ring $\mathbb{Z}/n$.
4. How many elements are in the ring $\mathbb{Z}[x]/(4x - 3, 2x + 3)$?
5. How many maximal ideals are there in the ring $\mathbb{C}[x, y]/(y^2 - x^3 - 1, y - x)$?
6. What is the greatest common divisor of $x^{20} - 1$ and $x^{12} + 1$ in $\mathbb{R}[x]$?
7. Factor the polynomial $x^5 + x + 1$ into irreducibles in $\mathbb{Z}/2[x]$.
8. Express the abelian group A with presentation $\langle x, y, z | 2x + 4y = 0, 2y + 12z = 0\rangle$ as a product of cyclic groups.
9. For which values of a and b in $\mathbb{Z}$ is the ring $\mathbb{Z}[x, y]/(x^5 - ax^3 + by^2)$ Noetherian?
10. How many possibilities are there for the Jordan form of a matrix with characteristic polynomial $(x - 2)^4(x - 3)^3$? (Note that we call two Jordan forms the same if they differ by permuting the order of the blocks.)
11. What is the minimal polynomial of $\sqrt{-3} + \sqrt{3}$ over $\mathbb{Q}$?
12. Give a formula for the real root of the polynomial $x^3 = 3x + 4$ using Cardano's method.

Part B: Longer answer.

Prove what you assert. (10 points each.)

1. Use the Frobenius homomorphism $\sigma : x \mapsto x^p$ to show that, if $q = p^n$ for a prime p, that any field $\mathbb{F}_q$ of order q is a Galois extension of $\mathbb{Z}/p$ with Galois group generated by σ.

2. Prove or disprove: The real roots of $x^4 - x^2 - 1$ are constructible (in the sense of compass and straightedge constructions).

3. Find the splitting field of $x^8 - 2$ over $\mathbb{Q}$, including its degree. (Hint: The degree is not 32.)

4. Let K be the splitting field of $x^3 - 5$ over $\mathbb{Q}$. Describe all subfields of K and the containments between them.

5. Suppose $F \to K$ is a Galois extension and $f(x) \in F[x]$ is an irreducible polynomial that has a root in K. Show that $f(x)$ factors into a product of linear factors in $K[x]$.