1. *Assume that $u_0 = u_0(x_1,x_2)$ is a smooth function on $\mathbf{R}^2$ which is $1-$periodic in each direction, i. e. $u(x_1+1,x_2) = u(x_1,x_2) = u(x_1,x_2+1)$. Let us consider the problem*

$$u_t = \frac{\partial^2 u}{\partial x_1^2} + 3\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial u}{\partial x_1}\,, \quad x\in\mathbf{R}^2\,,\, t\ge 0 \tag{1}$$

$$u|_{t=0} = u_0\,. \tag{2}$$

If we express the solution u as

$$u(x,t) = \sum_{k\in\mathbf{Z}^2} c(k,t)e^{2\pi i(k_1x_1+k_2x_2)}\,, \tag{3}$$

what are the formulae for $c(k,t)$ (assuming we know the Fourier coefficients of u_0)?

Solution: Differentiating (3), we see that the Fourier coefficients of the right-hand side of(1) is given by $A(k)c(k,t)$ with $A(k) = -(2\pi)^2k_1^2 - 3(2\pi)^2k_2^2 + 2\pi i k_1$. Hence $c(k,t) = e^{tA(k)}c(k,0)$, where $c(k,0) = c_0(k)$, the Fourier coefficients of u_0.

2. *If you have a computer program which can calculate the Fourier coefficient of a functions and also sum a given Fourier series on the square $[0,1]\times[0,1]$, how would you use it to solve the problem*

$$-\Delta u = f \quad in\ \Omega \tag{4}$$

$$u|_{\partial\Omega} = 0 \quad in\ \Omega \tag{5}$$

when Ω is the rectangle $\{(x_1,x_2)\,,\, 0<x_1<2\,,\, 0<x_2<3\}$?

Solution: We first need to adapt the problem to the setting of periodic functions. For that we extend f to a function $f_{\text{per}}\colon \mathbf{R}^2\to\mathbf{R}$, which is odd and periodic with period 4 in x_1, and odd and periodic with period 6 in x_2, in a similar way as in Problem 2 of Homework 2. In particular, when $2<x_2<4$ and $0<x_2<3$, we set $f_{\text{per}}(x_1,x_2) = -f(4-x_1,x_2)$; when $0<x_1<2$ and $3<x_2<6$, we set $f_{\text{per}}(x_1,x_2) = -f(x_1,6-x_2)$; and when $2<x_1<4$ and $3<x_2<6$ we set $f_{\text{per}}(x_1,x_2) = f(4-x_1,6-x_2)$. It is now enough to find a function u_{per} which is odd and periodic with period 4 in x_1, and odd and periodic with period 6 in x_2 such that $-\Delta u_{\text{per}} = f_{\text{per}}$. Such a function will solve (4), and the boundary condition $u|_{\partial\Omega} = 0$ will be satisfied due to the symmetries. As our program can only work with periodic functions of period 1 in each direction, we need to change variables to tranfer the equation to $[0,1]\times[0,1]$. For this we set $F(y_1,y_2) = f_{\text{per}}(4y_1,6y_2)$ and $U(y_1,y_2) = u_{\text{per}}(4y_1,6y_2)$. In the y variables, the equation becomes $\frac{\partial^2 U}{16\partial y_1^2} + \frac{\partial^2 U}{36\partial y_2^2} = F$. We use the program to calculate the Fourier coefficients $\hat{F}(k)$ of F. The Fourier coefficients of U will be given by $\hat{U}(k) = \frac{\hat{F}(k)}{\frac{4\pi^2k_1^2}{16}+\frac{4\pi^2k_2^2}{36}}$. We then use the program to sum the Fourier series for U to obtain the function $U(y_1,y_2)$. The solution u is then given by $u(x_1,x_2) = U(\frac{x_1}{4},\frac{x_2}{6})$.

3. *Let $\Omega\subset\mathbf{R}^3$ be a bounded smooth domain. Assume that $\phi_1,\phi_2,\phi_3\dots$ are the eigenfunctions of the Laplacian $-\Delta$ in Ω with the zero boundary condition and eigenvalues $\lambda_1<\lambda_2\le\lambda_3\le\dots$. (In other words, $-\Delta\phi_j = \lambda_j\phi_j$ and $\phi_j|_{\partial\Omega} = 0$.) Find the solution of the initial-value problem for the Schödinger equation*

$$iu_t + \Delta u - u = 0 \quad in\ \Omega\times(0,\infty)\,, \tag{6}$$

$$u|_{\partial\Omega} = 0\,, \tag{7}$$

$$u(x,0) = u_0(x) \quad in\ \Omega\,, \tag{8}$$

where $i=\sqrt{-1}$ and $u_0(x) = \sum_{k=1}^n c_k\phi_k(x)$.

Solution: We seek the solution in the form $u(x,t) = \sum_k c(k,t)\phi_k(x)$. Substituting this expression into the PDE, we obtain $i\dot{c}(k,t) - (\lambda_k+1)c(k,t) = 0$ for each k. Solving the ode for each k, we obtain $u(x,t) = \sum_{k=1}^n e^{-i(\lambda_k+1)t}c_k\phi_k(x)$.

4. *Let $\Omega\subset\mathbf{R}^3$ be a smooth bounded domain and let g be a smooth function on $\partial\Omega$. What will be the PDE and the boundary conditions corresponding to the minimization of the functional*

$$J(u) = \int_\Omega \frac{1}{2}\left(|\nabla u|^2 + u^2\right)\,dx - \int_{\partial\Omega} gu\,dx \tag{9}$$

over all smooth functions on Ω?

Solution: If the minimum is attained at u, then the derivative of the function $\varepsilon\to J(u+\varepsilon\varphi)$ at $\varepsilon=0$ must vanish for each smooth function φ on Ω. The derivative is given by $\int_\Omega(\nabla u\nabla\varphi + u\varphi)\,dx - \int_{\partial\Omega} g\varphi\,dx$. Integrating by parts in the first integral, we see that the last expression can also be written as $\int_\Omega(-\Delta u+u)\varphi\,dx + \int_{\partial\Omega}\left(\frac{\partial u}{\partial n} - g\right)\varphi\,dx$. This can only vanish for all smooth φ if $-\Delta u + u = 0$ in Ω and $\frac{\partial u}{\partial n} = g$ at $\partial\Omega$.

5. *Find the solution $u(x,t)$ of the equation*

$$u_t + \frac{\partial u}{\partial x_1} - 2\frac{\partial u}{\partial x_2} + 5u = 0 \qquad in\ \mathbf{R}^3\times(-\infty,\infty) \tag{10}$$

with $u(x,0) = u_0(x)$.

Solution: The solution of the equation $u_t + \frac{\partial u}{\partial x_1} - 2\frac{\partial u}{\partial x_2} = 0$ corresponds to the transport of the original function with speed $(1,-2,0)$, and hence is given by $u_0(x_1-t,x_2+2t,x_3)$. The solution of $u_t+5u=0$ corresponds is given by $u_0(x)e^{-5t}$. The two "processes" are going on at the same time, but they commute. Hence the solution of (10) is given by $u(x,t) = u_0(x_1-t,x_2+2t,x_3)e^{-5t}$.

6. *Let $\Omega\subset\mathbf{R}^3$ be the complement of the ball of radius 3, i. e. $\Omega = \{x\in\mathbf{R}^3\,,|x|>3\}$. Find the Green's function of the domain Ω for the equation $\Delta u = f$ with the boundary conditions $u|\partial\Omega = 0$ and $u(x)\to 0$ as $x\to\infty$.*

Solution: The Green's function of the ball of radius R is given by $G(x,y) = -\frac{1}{4\pi|x-y|} + \frac{R}{4\pi|y||x-y^*|}$, with $y^* = \frac{R^2}{|y|^2}y$, and we can think of it as the field of the unit charge at y minus the field of a "fictitious" charge $R/|y|$ at y^*. For the ball we think of y being inside the ball, i. e. $|y|<R$. When the domain is the outside of the ball, we can just exchange the role of y and y^* and normalize the charge outside of the ball to 1. The resulting formula is again $G(x,y) = -\frac{1}{4\pi|x-y|} + \frac{R}{4\pi|y||x-y^*|}$, with $y^* = \frac{R^2}{|y|^2}y$, except that this time y is outside of the ball.