MATH 5588 LECTURE LOG

Lecture 1, 1/16/2018

Laplace equation and gravitational potentials, Poisson equation

We discussed the Laplace equation

$$\Delta u = 0 \tag{1}$$

and the Poisson equation

$$\Delta u = f \tag{2}$$

in the three-dimensional space $\mathbf{R}^3$ in the historical context of gravitational potentials. We use the usual notation

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\,. \tag{3}$$

The equation was considered by P. S. Laplace in the three-dimensional space $\mathbf{R}^3$, corresponding to $n = 3$.

We first discuss the situation when the equation is considered in the whole space $\mathbf{R}^3$. (See Section 9.5.6 of the textbook.) This in some sense the simplest case, simpler than when the equation is considered in a domain with boundaries, with a boundary condition on the solution.

Having in mind the gravitational force, let us consider a point-mass M located at $a = (a_1, a_2, a_3) \in \mathbf{R}^3$. According to Newton, the force due to this mass on another mass m located at x is

$$F = -\kappa M m \frac{x - a}{|x - a|^3}\,, \tag{4}$$

where κ is the gravitational constant.

We can express the force in the following way: let

$$u(x) = -\frac{\kappa M}{|x - a|} \tag{5}$$

be the *gravitational potential* of the body at a. The function u has a physical meaning: $-u(x)$ represents the amount of work needed to move a particle of a unit mass from the point x to $x \sim \infty$ (assuming the particle is at rest before and after we move it, and the mass at a is fixed).

The force (4) can be expressed as

$$F = -m\nabla u\,, \tag{6}$$

where we denote by ∇u the vector with components $\frac{\partial u}{\partial x_i}$.

The fact that F can be expressed as a gradient of a function is of course very important in mechanics, and has a number of consequences for the equations of motion, but this will not be our focus here.

In the 1780s P. S. Laplace made the following observation:

$$\Delta u(x) = 0 \quad \text{in } \mathbf{R}^3 \setminus \{a\}\,. \tag{7}$$

This is easy to check by a direct calculation. For the calculation we can assume without loss of generality that $a = 0$, so we just need to verify

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) \frac{1}{|x|} \;=\; 0 \quad \text{in } \mathbf{R}^3 \setminus \{0\}\,, \tag{8}$$

which is left to the reader as an exercise.

All the above can be generalized to the situation when we have several masses $M_1, M_2, \ldots, M_r$ and the points $a^{(1)}, a^{(2)}, \ldots, a^{(r)}$, respectively. We set

$$u(x) = \sum_j -\frac{\kappa M_j}{|x - a^{(j)}|}\,, \tag{9}$$

and the force on a point-mass m located at x due to gravity from the masses $M_1, \ldots, M_r$ is again given by (6). Also, the function u satisfies

$$\Delta u(x) = 0 \qquad \text{in } \mathbf{R}^3 \setminus \{a^{(1)}, a^{(2)}, \ldots, a^{(r)}\}\,. \tag{10}$$

All this can also be applied to a continuous distribution of mass. Assume mass is not concentrated at finitely many points, but distributed with some density $\rho(x)$. This means that the mass contained in a region $\mathcal{O} \subset \mathbf{R}^3$ is $\int_{\mathcal{O}} \rho(x)\,dx$. Let us assume that ρ is a smooth function which vanishes outside a bounded set.

In this situation we set

$$u(x) = \int_{\mathbf{R}^3} \frac{-\kappa\rho(y)}{|x - y|}\,dy\,. \tag{11}$$

The force on a mass m at x due to gravity from the mass described by ρ is again given by (6).

In any open set where ρ vanishes we again have $\Delta u = 0$.

This can be used to prove the Shell Theorem, that the gravitational force due to a mass in a ball $B_R = \{x\,,\, |x| \le R\}$ which is spherically distributed is in the reagion outside of the ball exactly the same as of all the mass was concentrated at $x = 0$. This was already known to Newton (and you can find Newton's proof at the Wikipedia page linked above). Let us sketch a proof of the theorem using the Laplace equation $\Delta u = 0$. Let ρ be the distribution of the mass, and let u be the potential given by (11). If ρ is spherically symmetric, then $\rho = \rho(r)$, where $r = |x|$. One can check easily that then u is also depends only on r, or $u = u(r)$. Outside of the ball B_R we have $\Delta u = 0$, and by direct calculation we check that for functions depending on r this equation is

$$u'' + \frac{2u'}{r} = 0\,. \tag{12}$$

The general solution of this equation is

$$u(r) = \frac{A}{r} + B\,. \tag{13}$$

Since $u \to 0$ as $x \to \infty$, we see that for our potential u we must have $B = 0$. Far away from the mass, when $|x|$ is large, the integral (11) gives

$$u(x) = -\frac{\kappa M}{|x|} + \text{possible error of order } |x|^{-2}\,, \tag{14}$$

where $M = \int_{\mathbf{R}^3} \rho(y)\,dy$ is the total mass. A comparison with (13) now shows that $A = -\kappa M$ and hence

$$u(x) = \frac{-\kappa M}{|x|}\,, \qquad |x| > R\,, \qquad \text{(an exact expression)} \tag{15}$$

This proves the Shell Theorem.

In the early 1800s Poisson has made the following observation. Assume for simplicity that the density ρ is sufficiently regular and vanishes outside a bounded set. Then

$$\Delta u(x) = 4\pi\kappa\rho(x)\,, \qquad x \in \mathbf{R}^3\,. \tag{16}$$

This has far-reaching consequences. Let us reformulate the result somewhat. Set

$$G(x) = -\frac{1}{4\pi|x|}\,. \tag{17}$$

For a smooth function $f\colon \mathbf{R}^3 \to \mathbf{R}$ which vanishes outside of a bounded region set

$$u(x) = \int_{\mathbf{R}^3} G(x-y) f(y)\,dy\,. \tag{18}$$

The result of Poisson can be stated as

$$\Delta u = f\,. \tag{19}$$

One can think about it in the following way: consider the "operator" $\mathcal{G}$ (which maps functions on $\mathbf{R}^3$ to functions on $\mathbf{R}^3$) given by

$$f \to \mathcal{G}f\colon x \to \int_{\mathbf{R}^3} G(x-y) f(y)\,dy\,. \tag{20}$$

Then, at least on functions which are smooth and supported in a bounded set, we have

$$\Delta\ \mathcal{G} = \text{Identity}\,. \tag{21}$$

This can be interpreted as $\mathcal{G}$ being an inverse operator to Δ. (Or, more precisely, right-inverse. We will see that for functions u which vanish at ∞ one also has $\mathcal{G}\ \Delta u = u$.)

The operator $f \to \mathcal{G}f$ is a typical *integral operator*, and the above illustrates an important general point: *inverses to differential operators are often given by integral operators.* The simplest example of this is the fundamental theorem of calculus - the inverse of differentiation is given by integration.

Lecture 2, 1/18/2018

The discrete Laplacian and its relation to the standard Laplacian

There are similarities between the equation

$$\Delta u = f\,, \tag{22}$$

we discussed last time, and the equation

$$A\xi = b\,, \tag{23}$$

where A is an $n \times n$ matrix b is a given $n-$vector and ξ is an unknown $n-$vector, which you have studied in the linear algebra classes. If the matrix A is invertible, we can write the solution ξ of (23) as

$$\xi = A^{-1}b\,. \tag{24}$$

In coordinates this is

$$\xi_i = \sum_j (A^{-1})_{ij} b_j\,. \tag{25}$$

This can be compared with the formula

$$u(x) = \int_{\mathbf{R}^3} G(x-y) f(y)\, dy\,, \qquad G(x) = \frac{1}{4\pi|x|}\,, \tag{26}$$

which we discussed last time. In some sense, the function $G(x-y)$ represents the inverse operator to Δ, with the variables x, y playing the role of the indices i, j in (25).

To make this more specific, let us consider the discrete Laplacian. It is defined for functions on a discrete mesh

$$X^3_{\mathrm{discr},h} = \{x = (x_1, x_2, x_3)\,, x_1 = hk_1\,, x_2 = hk_2\,, x_3 = hk_3\,,\ k_1, k_2, k_3 \in \mathbf{Z}\}\,, \tag{27}$$

where, as usual, $\mathbf{Z}$ denotes the integers and $h > 0$ is a parameter. For a function u on the mesh $X^3_{\mathrm{discr},h}$ we define

$$\Delta_{\mathrm{discr}} u(x) = \frac{u(x+he_1) + u(x-he_1) + u(x+he_2) + u(x-he_2) + u(x+he_3) + u(x-he_3) - 6u(x)}{h^2}\,, \tag{28}$$

where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. This expression can be obtained from the expression for Δ if we replace the second partial derivatives by the corresponding difference quotients.[1]

In the class we discussed an interpretation of this expression in terms of electric currents. We can imagine a mesh of wires in the three direction of the axes x_1, x_2, x_3 passing through the points of $X^3_{\mathrm{discr},h}$ and connected at those points. If $u(x)$ represents the voltage at $x \in X^3_{\mathrm{discr},h}$, then $\Delta_{\mathrm{discr}} u(x)$ represents[2] the sum of currents emanating from that point.[3]

The equation $\Delta_{\mathrm{discr}} u = f$ on $X^3_{\mathrm{discr},h}$ has still infinitely many unknowns $u(x)$ (labeled by $x \in X^3_{\mathrm{discr},h}$), but we can choose some $L > 0$ (which we can think as large), restrict it to a finite subset

$$\Omega^L_{\mathrm{discr},h} = \{(hk_1, hk_2, hk_3)\,,\ |hk_j| < L\,,\ j = 1,2,3\} \tag{29}$$

and impose a "boundary condition" on the "discrete boundary" $\partial_{\mathrm{discr}}\Omega^L_{\mathrm{discr},h}$ of the (finite) set $\Omega^L_{\mathrm{discr},h}$ consisting of the points of $X^3_{\mathrm{discr},h}$ which are "just outside" of $\Omega^L_{\mathrm{discr},h}$.

In this situation the equation

$$\Delta_{\mathrm{discr}} u(x) = f(x)\,, \qquad x \in \Omega^L_{\mathrm{discr},h}\,, \tag{30}$$

taken with the understanding the $u(x) = 0$ at $\partial_{\mathrm{discr}}\Omega^L_{\mathrm{discr},h}$ (which is needed when calculating $\Delta_{\mathrm{discr}} u(x)$ when $x \in \Omega^L_{\mathrm{discr},h}$ is next to the boundary) is just a linear equation of the form (23), with the unknown ξ identified with $\{u(x)\,,\ x \in \Omega^L_{\mathrm{discr},h}\}$. The points $x \in \Omega^L_{\mathrm{discr},h}$ serve the same purpose as the indices i identifying the coordinates ξ_i of ξ. In the situation above, the "matrix" Δ_{discr} is invertible. This may not be obvious if you see it for the first time, but we will begin to see it as we become more familiar with the properties of the Laplacian.

We can express the solution $u(x)$ of (30) in a way analogous to (25):

$$u(x) = \sum_{y \in \Omega^L_{\mathrm{discr},h}} G^L_{\mathrm{discr}}(x,y) f(y)\,, \qquad x \in \Omega^L_{\mathrm{discr},h} \tag{31}$$

where $G^L_{\mathrm{discr}}(x,y)$ (with x, y running through $\Omega^L_{\mathrm{discr},h}$) representins the inverse matrix of Δ_{discr} (considered in $\Omega^L_{\mathrm{discr},h}$).

The function $G^L_{\mathrm{discr}}(x,y)$ is not described by a simple explicit formula. (An formula can be written down, but it is not quite simple.) It can be calculated by a computer as an inverse matrix of Δ_{discr}. (We should note that representing the matrix given by Δ_{discr} in a computer requires some thought - it is not as simple as in the case of space dimension one, when the discretization of the second derivative gives a simple matrix. However, at this point we will not focus on this detail.)

[1] On the real line the difference quotient is given by $\frac{g(x+h)+g(x-h)-2g(x)}{h^2}$.

[2] modulo a multiple, given by the resistance of the wires between two points

[3] If there is no "source" at x, the sum has to vanish by Kirchhoff laws.

The connection to the function $G = \frac{1}{4\pi|x|}$ is the following: if we fix x, y with $|x-y|$ much larger than h and take L much larger than $\max(|x|, |y|)$, then

$$G^L_{\text{discr}}(x,y) \sim h^3 G(x-y) \sim \frac{h^3}{4\pi|x-y|}\,. \tag{32}$$

We will not try to prove this, our purpose here is to explain in which sense the function $G(x-y)$ is similar to an inverse matrix. We see that, remarkably, in a suitable limit the function $G^L_{\text{discr}}(x,y)$ approaches a simple expression.

Just as the continuous Laplacian Δ is relevant for many phenomena (gravity, electricity, heat conduction, random walks, diffusion,...), the discrete Laplacian, besides being an approximation of the "true Laplacian", is relevant for many phenomena on a lattice, some of which we discussed in class.

We can also do all this in dimension two and dimension one. In that case the function $G^L_{\text{discr}}(x,y)$ will approach $+\infty$ as $L \to +\infty$, but we can still obtain a suitable limit considering $G^L_{\text{discr}}(x,y) - c_L$, where c_l is a suitable constant (depending on L).

Lecture 3, 1/23/2018

Integration by parts

Integration by parts (in the multi-dimensional case) is discussed in the textbook under the name Divergence theorem. The theorem is formulated on pages 21-22, and you can check the index of the book for other places where the topic comes up.

In the class we went through some of the standard formulae, with some explanations as to why such formulae hold.

The first observation is that for a smooth function $f\colon \mathbf{R}^m \to \mathbf{R}$ which vanishes outside a bounded region we have

$$\int_{\mathbf{R}^m} \frac{\partial f}{\partial x_i}(x)\,dx = 0\,, \qquad i = 1, \dots, m\,. \tag{33}$$

This is easy to understand by integrating first over x_1 and using the Fundamental Theorem of Calculus.

Let f be any smooth function on $\mathbf{R}^m$ and g a smooth function on $\mathbf{R}^m$ which vanishes outside a bounded set. Then we can apply (33) to fg and, applying the Leibnitz rule, obtaining

$$\int_{\mathbf{R}^m} \frac{\partial f}{\partial x_i}(x) g(x)\,dx = \int_{\mathbf{R}^m} -f(x)\frac{\partial g}{\partial x_i}(x)\,dx\,. \tag{34}$$

A more subtle formula is necessary when we integrate over a domain. Let $\Omega \subset \mathbf{R}^m$ be a smooth bounded domain. (You can assume that $m \in \{1,2,3\}$,

although we can work with any $m \geq 1$.) The boundary of Ω will be denoted by $\partial\Omega$, and for $x \in \partial\Omega$ we will denote by $n(x) = (n_1, \ldots, n_m)$ the outward unit normal to $\partial\Omega$ at x.

Let $f\colon \Omega \to \mathbf{R}$ be a function in Ω which is smooth up to the boundary. Then one has

$$\int_\Omega \frac{\partial f}{\partial x_i}(x)\, dx = \int_{\partial\Omega} f(x) n_i(x)\, dx\,. \tag{35}$$

For a vector field $u = (u_1, \ldots, u_m)$ in Ω we define

$$\operatorname{div} u(x) = \sum_{i=1}^{m} \frac{\partial u_i}{\partial x_i}(x)\,. \tag{36}$$

Applying (35) to each component u_i and summing over i, we obtain

$$\int_\Omega \operatorname{div} u(x)\, dx = \int_{\partial\Omega} u(x) n(x)\, dx\,, \tag{37}$$

where

$$u(x)n(x) = \sum_{i=1}^{m} u_i(x) n_i(x) \tag{38}$$

is the scalar product of u and n. Formula (37) is known under several names: Divergence theorem, Gauss theorem, Gauss-Ostrogradsky's theorem, and others.

For a smooth function $v\colon \mathbf{R}^m \to \mathbf{R}$ we denote the vector field with coordinates $\frac{\partial v}{\partial x_i}$ by ∇v. Note that

$$\operatorname{div} \nabla v = \Delta v\,. \tag{39}$$

Applying (37) with $u = \nabla v$, we obtain

$$\int \Delta v\, dx = \int_{\partial\Omega} (\nabla v) n\, dx\,. \tag{40}$$

The quantity $\nabla v(x) n(x)$ is the derivative of v in the direction $n(x)$, and is called the *normal derivative* of v.

Applying (35) with f replaced by fg and using the Leibnitz rule, we obtain

$$\int_\Omega \frac{\partial f}{\partial x_i} g\, dx = \int_{\partial\Omega} f g n_i\, dx - \int_\Omega f \frac{\partial g}{\partial x_i}\, dx\,. \tag{41}$$

A variant of this formula is that for a vector field u and a scalar function f we have

$$\int_\Omega f(x) \operatorname{div} u(x)\, dx = \int_{\partial\Omega} f(x)(u(x)n(x))\, dx = \int_\Omega -u(x)\nabla f(x)\, dx\,. \tag{42}$$

Taking the special case $f = v,\ u = \nabla v$, we obtain

$$\int_\Omega v(x)\Delta v(x)\, dx = \int_{\partial\Omega} v(x)\nabla v(x) n(x)\, dx - \int_\Omega |\nabla v(x)|^2\, dx\,. \tag{43}$$

The last formula can be used to prove the following statement: *If* $v\colon \Omega \to \mathbf{R}$ *satisfies* $\Delta v(x) = 0$ *in* Ω *and* v *is constant at* $\partial\Omega$*, then* v *is constant in* Ω*.*

To see this, we note that when $\Delta v = 0$ the integral on the left of (43) vanishes. If v is constant at $\partial\Omega$ and $\Delta v = 0$, then the boundary integral in (43) also vanishes, due to (40), and hence we are left with

$$\int_\Omega |\nabla v|^2 \, dx = 0 \,, \tag{44}$$

which means that $\nabla v = 0$ in Ω.

Overall, the integration by parts is one of the main tools in the theory of PDEs, and we will use is quite often.

Lecture 4, 1/25/2018

Convolution

For two functions $f, g\colon \mathbf{R}^m \to \mathbf{R}$ such that we define the convolution $f * g$ by the formula

$$f * g(x) = \int_{\mathbf{R}^m} f(x-y) g(y) \, dy \,, \tag{45}$$

assuming the integral is convergent. (Note, for example, that when $f \equiv 1$ and $g \equiv 1$, the integral is not finite.)

We can think about the definition in several ways. One of them is the following: for $f\colon \mathbf{R}^m \to \mathbf{R}$ and a fixed $y \in \mathbf{R}^m$, the function

$$x \to f(x-y) \tag{46}$$

is often called the *shift* of the function f. Its graph is obtained by shifting the graph of f by the vector y. If we have several vector $y^{(1)}, \dots, y^{(r)}$ and coefficients $c_1, dots, c_r$, we can combine the shifts $f(x0 - y^{(j)}$ using the coefficients c_j and form a function

$$c_1 f(x-y^{(1)}) + c_2 f(x-y^{(2)}) + \cdots + c_r f(x-y^{(r)}) \,. \tag{47}$$

When $c_j \geq 0$ and $\sum_j c_j = 1$, we can think of (47) as a certain average of the shifts $f(x-y^{(j)})$ (with "weights" c_j).

The convolution can be thought of as a version of (47), where the sum is replaced by an integral and the weights c_j are replaced by "infinitesimal weights" $g(y)\,dy$. However, this is just one of various ways to think about the convolution. Note if we set $x - y = y'$ in (45), we obtain

$$f * g(x) = \int f(y') g(x-y') \, dy' \,, \tag{48}$$

which is the same as saying that

$$f * g = g * f \,. \tag{49}$$

Hence instead of viewing $f * g$ as a suitable average of the shifts of the function f with "weights" $g(y)\,dy$ (which may be a good way to view $f * g$ when, say, f is bounded and g is localized in a bounded region), we can also view it as a suitable average of the shifts of g with "weights" $f(y)\,dy$.

In the textbook, the operation of convolution of two functions is discussed for example in section 10.4.3, in connection with Fourier transformation.

Convolution can be used for approximating general functions by smooth functions. Let $\phi\colon \mathbf{R}^m \to \mathbf{R}$ be a smooth function which vanishes outside of a unit ball in $\mathbf{R}^m$, and satisfies $\int_{\mathbf{R}^m} \phi(x)\,dx = 1$. (We could also demand that $\phi \geq 0$, but this is not necessary.) For $\varepsilon > 0$ we set

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^m}\phi\left(\frac{x}{\varepsilon}\right) \,. \tag{50}$$

The function ϕ_ε vanishes outside of the ball of radius ε, and still has the property

$$\int_{\mathbf{R}^m} \phi_\varepsilon(x)\,dx = 1 \,. \tag{51}$$

The functions

$$f_\varepsilon = f * \phi_\varepsilon \tag{52}$$

provide a very good approximation of f as $\varepsilon \to 0$. Heuristically, for small ε, the function $f * \phi_\varepsilon$ are combinations of small shifts of f, and a small shift of f should be close to f.

Under quite general assumption we have

$$\lim_{\varepsilon \to 0_+} f * \phi_\varepsilon = f \,. \tag{53}$$

We did not specify the sense in which the limit is taken. That depends on the situation. For example, when f is continuous and vanishes outside of a bounded set, then $\max_x |f(x) - f * \phi_\varepsilon(x)| \to 0$ as $\varepsilon \to 0_+$, but when f is discontinuous, we have to measure the proximity of f and $f * \phi_\varepsilon$ in a different way. At this point we do not have to worry about these details.

The functions ϕ_ε can also be viewed as an approximation of the Dirac function δ, in the sense that

$$\phi_\varepsilon \to \delta \,, \qquad \varepsilon \to 0_+ \,. \tag{54}$$

In the textbook the Dirac function is introduced at first in dimension $m = 1$ on pages 384–390, and later discussed also in higher dmensions, see for example page 500.

In the context of convolution, the Dirac function serves as a unity of the convolution operation:

$$f * \delta = f \,. \tag{55}$$

When f is continuous, we clearly have

$$\int_{\mathbf{R}^m} f(x-y)\delta(y)\,dy = f(x) \tag{56}$$

for each x. For functions which are not continuous (but still can be integrated) the last formula needs a more careful interpretation, which we do not need to worry about at the moment.

We will use the formula

$$\frac{\partial}{\partial x_i}(f * g) = (\frac{\partial}{\partial x_i} f) * g = f * (\frac{\partial}{\partial x_i} g)\,, \tag{57}$$

which holds, roughly speaking, when the corresponding expressions are well defined. The main point is that we can choose whether we put the derivative on f or on g, depending which is better for the situation under consideration. For example, in $f * \phi_\varepsilon$ the function ϕ_ε is smooth, so we can use

$$\frac{\partial}{\partial x_i}(f * \phi_\varepsilon) = f * \frac{\partial}{\partial x_i}\phi_\varepsilon\,, \qquad \frac{\partial^2}{\partial x_i \partial x_j}(f * \phi_\varepsilon) = f * (\frac{\partial^2}{\partial x_i \partial x_j}\phi_\varepsilon)\,, \tag{58}$$

and similarly for derivatives of any order, and we see that $f * \phi_\varepsilon$ should be a smooth function. This is indeed the case, under some minimal assumptions.

When we can take derivatives of both f and g, we can distribute higher derivatives between f and g. For example,

$$\Delta(f * g) = \sum_i \left(\frac{\partial}{\partial x_i} f\right) * \left(\frac{\partial}{\partial x_i} g\right)\,, \tag{59}$$

but we of course also have

$$\Delta(f * g) = (\Delta f) * g = f * (\Delta g)\,, \tag{60}$$

if f or g are sufficiently smooth so that we can put both derivatives on them.

Using some of the above formulae, it is easy to verify the observation of Poisson from the early 1800s that $u = G * f$ (with G given by (17) (from Lecture 1) satisfies

$$\Delta u = f\,. \tag{61}$$

To see it, consider any smooth functions K on $\mathbf{R}^3$ satisfying

$$K(x) = -\frac{1}{4\pi|x|}\,, \qquad |x| \geq 1\,. \tag{62}$$

For $\varepsilon > 0$ we set

$$K_\varepsilon(x) = \frac{1}{\varepsilon} K\left(\frac{x}{\varepsilon}\right)\,. \tag{63}$$

Claim: *ΔK is vanishes outside $B = \{x \in \mathbf{R}^3\,,\, |x| < 1\}$ and*

$$\int_{\mathbf{R}^m} \Delta K(x)\,dx = 1\,. \tag{64}$$

To see this, we recall that outside of the ball B we have $K(x) = \frac{1}{4\pi|x|}$, and hence $\Delta K = 0$ in $\mathbf{R}^3 \setminus B$. Thus we have

$$\int_{\mathbf{R}^3} \Delta K(x)\, dx = \int_B \Delta K(x)\, dx = \int_{\partial B} \sum_i \frac{\partial K}{\partial n}\, dx \tag{65}$$

where $n = n(x)$ is the outward unit normal to ∂B and

$$\frac{\partial K}{\partial n}(x) = \sum_i n_i(x) \frac{\partial K}{\partial x_i}(x) = \sum_i \frac{x_i}{|x|} \frac{\partial K}{\partial x_i}(x) = \frac{1}{4\pi|x|^2}\,. \tag{66}$$

Now we just have to check that

$$\int_{\partial B} \frac{1}{4\pi|x|^2}\, dx = 1\,. \tag{67}$$

Once we have the claim, the formula (61) can be easily seen for example as follows. First, we note that $\lim_{\varepsilon\to 0_+} K_\varepsilon(x) = G(x)$ for $x \neq 0$. Second, we note that

$$\Delta K_\varepsilon = \frac{1}{\varepsilon^3} (\Delta K) \left(\frac{x}{\varepsilon}\right) \tag{68}$$

$$\Delta\left(G * f\right) = \lim_{\varepsilon\to 0_+} \Delta\left(K_\varepsilon * f\right) = \lim_{\varepsilon\to 0_+} \left(\Delta K_\varepsilon\right) * f\,. \tag{69}$$

We see that we can think of the function ΔK_ε as playing the role of the function ϕ_ε in (52): it is of the form $\frac{1}{\varepsilon^3}\phi\left(\frac{x}{\varepsilon}\right)$ with $\phi = \Delta K$ being smooth, vanishing outside of the unit ball, and having integral 1. Therefore

$$\lim_{\varepsilon\to 0_+} \left(\Delta K_\varepsilon\right) * f = f\,, \tag{70}$$

under quite general assumptions on f.

Another way to think about it is

$$\Delta G = \lim_{\varepsilon\to 0_+} \Delta K_\varepsilon = \delta \qquad \text{(the Dirac function)}\,, \tag{71}$$

and therefore

$$\Delta(G * f) = (\Delta G) * f = \delta * f = f\,. \tag{72}$$

Lecture 5, 1/30/2018

Another argument that $\Delta(G * f) = f$

We have

$$\Delta(G * f) = \sum_i \left(\frac{\partial G}{\partial x_i}\right) * \left(\frac{\partial f}{\partial x_i}\right)\,, \tag{73}$$

at least when f vanishes outside a bounded set and is continuously differentiable. This needs some justification, as G is not differentiable at 0. The function $\frac{\partial G}{\partial x_i}$ is defined everywhere except at the origin by the usual differentiation:

$$\frac{\partial G}{\partial x_i}(x) = \frac{x_i}{4\pi|x|^3}\,. \tag{74}$$

The justification of (74) (under our assumptions on f) is not completely obvious. We briefly sketch it, but it is optional, you can just accept (73) and skip the justification.

To see that (73) should be correct, we first note that we have $\frac{\partial}{\partial x_i}(G*f) = G*\frac{\partial f}{\partial x_i}$, due to our assumptions on f. Next, we note that the integral defining $\frac{\partial G}{\partial x_i} * \frac{\partial f}{\partial x_i}$ (where we take the derivative of G as in (74) is convergent: we have

$$\left(\frac{\partial G}{\partial x_i} * \frac{\partial f}{\partial x_i}\right)(x) = \int_{\mathbf{R}^3} \frac{x_i - y_i}{4\pi|x-y|^3}\, f_i(y)\, dy\,, \qquad f_i = \frac{\partial f}{\partial x_i}\,. \tag{75}$$

The function f vanishes outside some ball B, so in the integral we can integrate only over B. The potentially dangerous contribution to the integral is from the neighborhood of x as the function $y \to \frac{x_i - y_i}{|x-y|^3}$ has a singularity there. The key point now is that

$$\left|\frac{\partial G}{\partial x_i}(x-y)\right| \le \frac{1}{4\pi|x-y|^2} \tag{76}$$

and the integral

$$\int_{\{y\,,\,|x-y|<1\}} \frac{1}{4\pi|x-y|^2}\, dy = \int_{\{y\,,\,|y|<1\}} \frac{1}{4\pi|y|^2}\, dy \tag{77}$$

is finite. To see this, we note that the last integral can be expressed as

$$\int_0^1 \frac{1}{4\pi r^2} 4\pi r^2\, dr = \int_0^1 dr = 1\,. \tag{78}$$

Note that a similar argument would not work for the second derivatives of G. In that case we get that $\int_{\{y\,,|y|<1\}} \left|\frac{\partial^2 G}{\partial x_i \partial x_j}(y)\right|\, dy = +\infty$.

Finally, to finish our justification of (73), we calculate the derivative of $G * f_i$ as the standard limit of difference quotients $D_i^h(G*f_i)$, defined as usual by $D_i^h g(x) = (g(x+he_i) - g(x))/h$, with the vector e_i having 1 in the $i-$th place and zeroes otherwise. We have

$$D_i^h\,(G*f_i) = \left(D_i^h G\right) * f_i\,. \tag{79}$$

We can now take h to zero to obtain (73). The behavior of $D_i^h G$ deteriorates as we get closer to the origin, but the key point is that this deterioration is not sufficiently strong so as to prevent passing to the limit in the integral defining the convolution in (79), by a calculation similar to (78). One can check that, under our assumptions, the functions $(D_i^h G) * f_i$ converge uniformly to $\left(\frac{\partial G}{\partial x_i}\right) * \left(\frac{\partial f}{\partial x_i}\right)$ as $h \to 0_+$.

Ler $B_\varepsilon = \{y \in \mathbf{R}^3\,,\, |y| \le \varepsilon\}$ and $\mathcal{O}_\varepsilon = \mathbf{R}^3 \setminus B_\varepsilon$. Using (73), we can write

$$\begin{aligned}
\Delta\,(G*f)\,(x) &= \int_{\mathbf{R}^3} \frac{\partial f}{\partial x_i}(x-y)\frac{\partial G}{\partial x_i}(y)\, dy \\
&= \lim_{\varepsilon\to 0_+} \int_{\mathcal{O}_\varepsilon} \frac{\partial f}{\partial x_i}(x-y)\frac{\partial G}{\partial x_i}(y)\, dy \\
&= \lim_{\varepsilon\to 0_+} \int_{\partial\mathcal{O}_\varepsilon} f(x-y)\frac{\partial G}{\partial x_i}(y) n_i(y)\, dy \;-\; \int_{\mathcal{O}_\varepsilon} f(x-y)\Delta G(y)\, dy \\
&= \lim_{\varepsilon\to 0_+} \int_{\partial B_\varepsilon} f(x-y)\frac{1}{4\pi|y|^2}\, dy = f(x)\,,
\end{aligned} \tag{80}$$

where we have used that $\Delta G = 0$ in $\mathcal{O}_\varepsilon$ and $\frac{\partial G}{\partial x_i}(y) n_i(y) = \frac{1}{4\pi|y|^2}$ for $y \neq 0$. Note the last integral in (80) is the average of f over the ball of radius ε around x, and this approaches f as $\varepsilon \to 0_+$ as f is continuous.

In the above calculation we assumed that f has continuous derivatives, but weaker assumptions are sufficient. One can ask about the minimal assumptions on f under which the function $G * f$ is twice differentiable. These are relatively subtle questions which will not be our focus here.

For most purposes one can assume that $u = G * f$ satisfies $\Delta u = f$ in a suitable (possibly generalized) sense, as long as the convolution $G * f$ is well-defined.

Lecture 6, 2/1/2018

Domains with boundary, the method of images

So far we have thought about the Laplace equation in the three dimensional space in the context of gravity. However, it arises in many other contexts, such as electrostatics and heat conduction (steady-state solutions). Gravity cannot be "screened" and therefore considerations in the whole space are very natural.

In electrostatics, parts of the space can be screened by conduction surfaces, and this leads to boundary-value problems. A typical example is the following. Let $\Omega \subset \mathbf{R}^3$ be a ball. Imagine that the boundary of Ω is made of of a conducting material and is "grounded", e. i. , the electric potential is always kept at 0. (One can think of connecting the boundary of Ω to the "ground wire".) If we put an electric change inside Ω, its potential will satisfy the equation $\Delta u(x) = f(x)$ inside Ω, where f describes the density of charges in suitable units. (If we use the units used in practice one will need to put some constant in front of f, such as $-\frac{1}{4\pi\varepsilon_0}$, but we can always choose units in which the constant is exactly 1.)

At he boundary the value of the potential u will constant (as the boundary is a conductor), and assuming the potential at the Earth's surface is set to 0 (we can choose such a normalization), given that our boundary is grounded, we will have to have $u|_{\partial\Omega} = 0$ (where we denote by $u|_{\partial\Omega}$ the restriction of u to the boundary of Ω). So to determine u inside Ω we have to solve the boundary value problem:

$$\begin{array}{rcll} \Delta u(x) & = & f(x)\,, & x \in \Omega\,, \\ u(x) & = & 0\,, & x \in \partial\Omega\,. \end{array} \tag{81}$$

In SI units, which are often used in the physic textbooks, the equation would read

$$\begin{array}{rcll} -\Delta u(x) & = & \frac{1}{4\pi\varepsilon_0}\rho(x)\,, & x \in \Omega\,, \\ u(x) & = & 0\,. & x \in \partial\Omega\,. \end{array} \tag{82}$$

where ρ is the change density and ε_0 is the vacuum permittivity. However, for the purposes of PDE analysis we can work with the normalization (81). An often used normalization is

$$\begin{array}{rcll} -\Delta u(x) & = & f(x)\,, & x \in \Omega\,, \\ u(x) & = & 0\,, & x \in \partial\Omega\,. \end{array} \tag{83}$$

which has some advantages, but for now we will work with (81) .

We will also introduce the following terminology: a *domain* is a subset of $\mathbf{R}^3$ (or, more generally, of $\mathbf{R}^m$) which is *open* and *connected*. Recall that a set $\Omega \subset \mathbf{R}^m$ is open if with every $x \in \Omega$ it also contains some ball of positive radius around x, and is connected if any two points in Ω can be joined by a smooth curve in Ω.

The problem of analyzing (81) for general domains is fairly difficult. However, one can solve it (or, more precisely, its approximate version) numerically on a computer without much theory. Writing some code which solves some suitable approximations

of (81) (such as those discussed in Lecture 2) is not too hard. The problem becomes more challenging (and mathematically interesting), even at the discrete approximation level, if we wish to find fast algorithms for solving it. That is not easy.

For now we will turn to situations which can be analyzed relatively easily - namely that of special domains, where symmetries can be used to write down solutions of (81) in terms of the functions $G = -\frac{1}{4\pi|x|}$ introduced in Lecture 1.

We will start with the simplest non-trivial domain with boundary, which arguably is the half-space

$$\Omega = \mathbf{R}^3_+ = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3\,,\ x_3 > 0\}\,. \tag{84}$$

Its boundary is given by the plane $x_3 = 0$:

$$\partial\Omega = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3\,,\ x_3 = 0\}\,. \tag{85}$$

For this domain one can write down solutions of (81) using the following trick, which can also be used in some other (special) situations. Assume a unit charge is located at $y \in \Omega$ its whole-space potential $x \to G(x-y) = -\frac{1}{4\pi|x-y|}$ does not vanish at $\partial\Omega$, but we can adjust $G(x-y)$ as follows: take $y* = (y_1, y_2, -y_3)$ and consider

$$G_\Omega(x,y) = -\frac{1}{4\pi|x-y|} + \frac{1}{4\pi|x-y^*|}\,. \tag{86}$$

Note that for $x \in \partial\Omega$ we have $x_3 = 0$ which implies

$$|x-y|^2 = (x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2 = |x-y^*|^2\,, \tag{87}$$

and hence $G_\Omega(x,y) = 0$ for $x \in \partial\Omega$. In Ω we have

$$\Delta_x G_\Omega(x,y) = \Delta_x\left(-\frac{1}{4\pi|x-y|}\right) = \delta(x-y)\,, \qquad \text{(Dirac mass)}\,, \tag{88}$$

as $\Delta_x \frac{1}{4\pi|x-y^*|} = 0$ in Ω.

The solution of (81) can then be given

$$u(x) = \int_\Omega G_\Omega(x,y) f(y)\, dy\,. \tag{89}$$

Note that the function $G_\Omega(x,y)$ is *not* of the form $F(x-y)$ (unlike in the case $\Omega = \mathbf{R}^3$).

The function $G_\Omega(x,y)$ can be compared to an inverse matrix for the problem

$$A\xi = b\,, \tag{90}$$

where $A = \{A_{ij}\}$ is an $n \times n$ matrix, $\xi = (\xi_1, \dots, \xi_n)$ is an unknown vector and $b = (b_1, \dots, b_n)$ is a given vector. If the matrix A is invertible, with the inverse $B = A^{-1}$, with $B = \{B_{ij}\}$, we can write

$$\xi_i = \sum_j B_{ij} b_j\,. \tag{91}$$

This can be compared with formula (89), with f playing the role of b, the variables x, y playing the role of the indices i, j the function G_Ω playing the role of B, and the integral replacing the summation.

We see that the formulae (89) and (91) are in many respects quite similar.

Lecture 7, 2/6/2018, by Dallas Albritton[4]

Introduction to Fourier Transformation, a calculation of the heat kernel

Today, we introduced the Fourier transform and used it to derive the heat kernel. Most of the calculations may be found in Chapter 10 of Haberman.

To begin, let us obtain the Fourier transform as a suitable limit of Fourier series. Recall Fourier series for a smooth function f on $[-L, L]$, where $L > 0$. For the moment, assume that $f \equiv 0$ outside $[-L/2, L/2]$. Then[5]

$$f(x) = \frac{1}{2L} \sum_{k \in \frac{\pi}{L}\mathbf{Z}} \hat{f}(k) e^{ikx} \text{ for all } x \in [-L, L], \tag{92}$$

where $\hat{f}$ is defined for all $k \in \mathbf{R}$ by

$$\hat{f}(k) = \int_{-L}^{L} f(x) e^{-ikx}\, dx = \int_{\mathbf{R}} f(x) e^{-ikx}\, dx. \tag{93}$$

Let $L \to \infty$ in (92) to obtain an integral from its Riemann sums:

$$f(x) = \lim_{L\to\infty} \frac{1}{2\pi} \Big(\frac{\pi}{L} \sum_{k \in \frac{\pi}{L}\mathbf{Z}} \hat{f}(k) e^{ikx} \Big) = \frac{1}{2\pi} \int_{\mathbf{R}} \hat{f}(\xi) e^{i\xi x}\, dx \tag{94}$$

for all $x \in \mathbf{R}$. The identity

$$f(x) = \frac{1}{2\pi} \int_{\mathbf{R}} \hat{f}(\xi) e^{i\xi x}\, dx \text{ for all } x \in \mathbf{R} \tag{95}$$

is known as the *Fourier inversion formula*, while the complex-valued function $\hat{f}$ is known as the *Fourier transform*. The variable x is known as the spatial (sometimes temporal) variable, while ξ is the frequency variable.[6] The two have inverse units, so that the quantity ξx in the exponential is dimensionless. For instance, x might be measured in meters, and ξ might be measured in meters^{-1}. The Fourier inversion formula states that a function (or signal) f can be written as a superposition of waves $e^{i\xi x}$, where $\hat{f}(\xi)$ represents (up to a constant) the amplitude of the wave $e^{i\xi x}$ in the superposition.

There are several different normalizations of the Fourier transform. Here are a few:

1. $\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-2\pi i x\xi}\, dx, \quad f(x) = \int_{\mathbf{R}} \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi,$
2. $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x) e^{-i\xi x}\, dx, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{f}(\xi) e^{i\xi x}\, d\xi,$
3. $\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-i\xi x}\, dx, \quad f(x) = \frac{1}{2\pi} \int_{\mathbf{R}} \hat{f}(\xi) e^{i\xi x}\, d\xi,$ and
4. $\hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbf{R}} (x) e^{-i\xi x}\, dx\,, \qquad f(x) = \int_{\mathbf{R}} \hat{f}(\xi) e^{i\xi x}\, d\xi\,.$

[4] Many thanks to Dallas for teaching the class (and preparing the notes) while V.S. was out of town.

[5] To make sure you have the right normalization of the Fourier transform, you should 1) make sure the basis function e^{ikx} have the correct period, and 2) make sure that the formula is correct for a constant function.

[6] The symbol ξ is often pronounced "ksee," and one should practice writing it.

The 2π must go *somewhere*, and it doesn't matter too much except that one must keep track of the constants (for instance, when differentiating). **In class I used #3, but now I will use #2,** since I prefer the symmetry. [7] The textbook uses #4. Another notation for $\hat{f}$ (resp. $\check{f}$) is $\mathcal{F}$ (resp. $\mathcal{F}^{-1}$). Note that Haberman's book also reverses the role of $\hat{f}$ and $\check{f}$.

To make sense of the integral in

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x) e^{-i\xi x}\, dx, \tag{96}$$

a natural condition on the function $f : \mathbf{R} \to \mathbf{R}$ is to be absolutely integrable:

$$\int_{\mathbf{R}} |f|\, dx < \infty. \tag{97}$$

If f is not continuous, the integral needs a suitable interpretation that is more general than the Riemann integral. That is, one uses the *Lebesgue integral*. This is roughly the point of *measure theory*, which is usually taught in graduate real analysis. Then (96) makes sense for functions $f \in L^1(\mathbf{R})$.[8] There is also a natural class of functions on which the Fourier transform is an isomorphism. These are the *Schwartz functions*, which we roughly think of as "like Gaussian" in their regularity and decay properties.

From now on, we will assume that all functions are "sufficiently regular" and decay "sufficiently fast" as $|x| \to \infty$.

Fact #0. *Fourier inversion*:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{f}(\xi) e^{i\xi x}\, dx \text{ for all } x \in \mathbf{R}, \tag{98}$$

or $(\hat{f})\check{} = f$.

Fact #1. *The Fourier transform diagonalizes differentiation*:

$$(f')\hat{}(\xi) = i\xi \hat{f}(\xi) \text{ for all } \xi \in \mathbf{R}. \tag{99}$$

Fact #2. *The Fourier transform turns convolution into multiplication, and vice versa*:

$$(f * g)\hat{}(\xi) = \sqrt{2\pi} \hat{f}(\xi) \hat{g}(\xi), \tag{100}$$

$$(fg)\hat{}(\xi) = \frac{1}{\sqrt{2\pi}} (\hat{f} * \hat{g})(\xi), \tag{101}$$

for all $\xi \in \mathbf{R}$. A good exercise is to find the analogue of this for Fourier series.

Fact #3. *The Fourier transform of a Gaussian is a Gaussian*: If $\alpha > 0$ and

$$g(x) = e^{-\frac{\alpha |x|^2}{2}}, \tag{102}$$

then

$$\hat{g}(\xi) = \frac{1}{\sqrt{\alpha}} e^{-\frac{|\xi|^2}{2\alpha}}. \tag{103}$$

We proved this using that $\hat{g}$ satisfies the ODE

$$\frac{d\hat{g}}{d\xi}(\xi) = -\frac{\xi}{\alpha} \hat{g}(\xi) \tag{104}$$

[7] In general dimension, this normalization is $\frac{1}{(2\pi)^{\frac{n}{2}}}$, and ξx becomes the dot product $\xi \cdot x$.

[8] The L is for "Lebesgue."

for all $\xi \in \mathbf{R}$. To compute $\hat{g}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} g(x)\,dx$, we used that

$$A = \int_{\mathbf{R}} e^{-\frac{|x|^2}{2}}\,dx = \sqrt{2\pi}. \tag{105}$$

This last fact is proven by evaluating $A^2 = \int_{\mathbf{R}} \int_{\mathbf{R}} e^{-\frac{|x|^2+|y|^2}{2}}\,dx\,dy$ in polar coordinates, see Chapter 10 of Haberman.

Fact #4. *The Fourier transform of the Dirac delta function is the constant function* $1/\sqrt{2\pi}$. This may be observed, for instance, by taking $\alpha \to 0^+$ in the identities from Fact #3. This fact is one instance of the *uncertainty principle*, which states that a function and its Fourier transform cannot both be highly concentrated.

Fact #5. *Plancherel theorem/Parseval's identity*:

$$\int_{\mathbf{R}} f\overline{g}\,dx = \int_{\mathbf{R}} \hat{f}\overline{\hat{g}}\,d\xi. \tag{106}$$

Here, the overline denotes complex conjugation. In particular,

$$\int_{\mathbf{R}} |f|^2\,dx = \int_{\mathbf{R}} |\hat{f}|^2\,d\xi. \tag{107}$$

I did not cover this in class today, but it is an important fact nonetheless!

I think it's a good idea to practice the computations involved in obtaining the above facts (except for Fact #0).

Heat equation in the whole space $\mathbf{R}^n$.

For $n \geq 1$, consider the heat equation

$$\partial_t u = \Delta u \text{ in } \mathbf{R}^n \times \mathbf{R}_+ \tag{108}$$

with initial condition $u(x,0) = u_0(x)$. Here, $\mathbf{R}_+$ denotes the half-line $(0,\infty)$. We supplement the problem with a boundary condition at spatial infinity:

$$|u(x,t)| \to 0 \text{ as } |x| \to \infty. \tag{109}$$

We will now derive a solution formula for (108). Taking the Fourier transform of (108) in the spatial variables gives

$$\partial_t \hat{u}(\xi,t) = -|\xi|^2 \hat{u}(\xi,t) \tag{110}$$

for all $\xi \in \mathbf{R}^n$ and $t \in \mathbf{R}_+$. Here, we have used Fact #1 (Fourier transform diagonalizes multiplication) and also that

$$\partial_t \hat{u}(\xi,t) = (\partial_t u)\hat{}\,(\xi,t). \tag{111}$$

Notice that for each $\xi \in \mathbf{R}^n$, (110) is an ODE in time for $\hat{u}(\xi,t)$. Solving the ODE gives

$$\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u_0}(\xi). \tag{112}$$

Observe that the high frequencies of u_0 are instantly damped after the initial time.

Now apply the inverse Fourier transform on each side of (112). As in Fact #2, the inverse Fourier transform turns multiplication into convolution (up to a factor

of $(2\pi)^{-\frac{n}{2}}$), so we only need to know the inverse Fourier transform of our Gaussian $e^{-|\xi|^2 t}$. According to Fact #3 with $\alpha = 2t$,[9]

$$e^{-|\xi|^2 t} \overset{\mathcal{F}^{-1}}{\to} (2t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}. \tag{113}$$

In conclusion, we have obtained the solution formula

$$u(x,t) = (\Gamma(\cdot,t) * u_0)(x), \tag{114}$$

where Γ is the *heat kernel*, defined for all $(x,t) \in \mathbf{R}^n \times \mathbf{R}_+$ by

$$\Gamma(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}. \tag{115}$$

The heat kernel solves the heat equation in $\mathbf{R}^n \times \mathbf{R}_+$ with inital data a Dirac mass.

Let $\mathbf{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_n > 0\}$. A good exercise is to use the method of images to obtain a solution formula for the heat equation in $\mathbf{R}^n_+ \times \mathbf{R}_+$ with (a) Dirichlet condition $u(x,t) = 0$, $x \in \partial\mathbf{R}^n_+$, and (b) Neumann condition $\frac{\partial u}{\partial x_n}(x,t) = 0$, $x \in \partial\mathbf{R}^n_+$.

Lecture 8, 2/8/2018

Green's functions - continuation , Poisson kernel

We saw that for a half-space $\Omega = \mathbf{R}^3_+$ one can construct the Green function by adjusting a whole-space Green function with a field of a change outside of the domain:

$$G_\Omega(x,y) = -\frac{1}{4\pi|x-y|} + \frac{1}{4\pi|x-y^*|}\,, \qquad y^* = (y_1, y_2, -y_3)\,. \tag{116}$$

The first term in (116) makes sure that $\Delta_x G_\Omega(x,y) = \delta(x-y)$. Note that y^* does not belong to Ω, and hence the function $x \to \frac{1}{4\pi|x-y^*|}$ satisfies $\Delta_x \frac{1}{4\pi|x-y^*|} = 0$ in Ω. The role of this term is to adjust the "leading term" $-\frac{1}{4\pi|x-y|}$ to achieve the boundary condition

$$G_\Omega(x,y) = 0 \qquad x \in \partial\Omega\,, \quad y \in \Omega\,. \tag{117}$$

One can try to generalize this construction to other domains. For example, when Ω is the first octant, $\Omega = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3\,,\, x_i > 0\,,\, i = 1,2,3\}$, one can check that the following formula can be used for the Green's function. For $y \in \mathbf{R}^3$ let us define

$$Q_1 y = (-y_1, y_2, y_3)\,, \quad Q_2 y = (y_1, -y_2, y_3)\,, \quad Q_3 y = (y_1, y_2, -y_3)\,, \tag{118}$$

and, using the notation $G(x) = -\frac{1}{4\pi|x|}$, set

$$\begin{aligned} G_\Omega(x,y) =& G(x-y) \\ & -G(x-Q_1y) - G(x-Q_2y) - G(x-Q_3y) \\ & +G(x-Q_1Q_2y) + G(x-Q_2Q_3y) + G(x-Q_1Q_3y) \\ & -G(x-Q_1Q_2Q_3y)\,. \end{aligned} \tag{119}$$

[9] Notice that you can obtain a version of Fact #3 in general dimensions from the version in one dimension. This is done by writing the integral for the Fourier transform of the Gaussian in n variables as a product of n integrals.

In this case we are adding 7 charges outside Ω to fix the boundary condition $G_\Omega(x,y) = 0$ for $x \in \Omega$. As an exercise, you can check that formula (119) produces the desired outcome.

An important special case where such constructions work is the case of a ball $B_R = \{x \in \mathbf{R}^3\,,\, |x| < R\}$. In this it is enough to place one charge outside of Ω (with a well-chosen coefficient) as follows: For $y \in B_R$ let y^* be defined by

$$y^* = y\frac{R^2}{|y|^2} \tag{120}$$

and set

$$G_{B_R}(x,y) = G(x-y) - \frac{R}{|y|}G(x-y^*) = -\frac{1}{4\pi|x-y|} + \frac{R}{4\pi|y||x-y^*|} \tag{121}$$

You can check again as an exercise that this formula works, and $G_{B_R}(x,y) = 0$ for $x \in \partial B_R\,,\ y \in B_R$. (As $y^* \notin B_R$, we clearly have $\Delta_x G_{B_R}(x,y) = \delta(x-y)$.)

You can also consider the following exercise: By general principles we know that we should have $G_{B_R}(x,y) = G_{B_R}(y,x)$. Hence expression (121) should be symmetric in x, y. The way we have written it, this symmetry is not transparent. Can you write the expression in a way which would make the symmetry transparent?

Poisson kernel

Once the Green's function of a domain is known, one can also use it for solving the following problem: given a (sufficiently regular) function $g\colon \partial\Omega \to R$, find a solution of $\Delta u = 0$ in Ω such that $u|_{\partial\Omega} = g$. The functions u satisfying $\Delta u = 0$ are called *harmonic functions* and the problem just mentioned is often called the Dirichlet problem. One can also think about it as the problem of finding an extension of a given function g at the boundary $\partial\Omega$ to a harmonic function u in Ω.

If we know the Green functions of the domain Ω, we can find the solution of the Dirichlet problem s follows. We recall the formula

$$\int_\Omega (u\Delta v - (\Delta u)v)\, dx = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n}v\right) dx\,, \tag{122}$$

where $\frac{\partial u}{\partial n}$ denotes the normal derivative at the boundary, given by $\sum_i \frac{\partial u(x)}{\partial x_i}n_i(x)$. Assume now that $\Delta u = 0$ and use (122) with $v(x) = G_\Omega(x,y)$, for some given $y \in \Omega$. Then $\Delta v(x) = \delta(x-y)$, and $v(x)$ vanishes when $x \in \partial\Omega$. Hence we obtain

$$u(y) = \int_{\partial\Omega} u(x)\frac{\partial G}{\partial n_x}(x,y)\, dx\,, \qquad \frac{\partial G}{\partial n_x}(x,y) = \sum_i n_i(x)\frac{\partial G}{\partial x_i}(x,y)\,. \tag{123}$$

Due to the symmetry $G_\Omega(x,y) = G_\Omega(y,x)$, this also can be written as

$$u(x) = \int_{\partial\Omega} u(y)\frac{\partial G}{\partial n_y}(x,y)\, dy\,, \quad \frac{\partial G}{\partial n_y}(x,y) = \sum_i n_i(y)\frac{\partial G}{\partial y_i}(x,y)\,. \tag{124}$$

The function $(x,y) \to \frac{\partial G_\Omega}{\partial n_y}(x,y)$ is called the Poisson kernel, and is often denoted by $P_\Omega(x,y)$. The solution of the Dirichlet problem can be expressed by it as

$$u(x) = \int_{\partial\Omega} P_\Omega(x,y)u(y)\, dy = \int_{\partial\Omega} P_\Omega(x,y)g(y)\, dy\,, \tag{125}$$

assuming, of course, $\Delta u = 0$ in Ω and $u(y) = g(y)$ at $\partial\Omega$. For simple domains, such as the half-space, or the ball, one can calculate $P_\Omega(x, y)$ explicitly. For example:

$$P_{\mathbf{R}^3_+}(x, y) = \frac{x_3}{2\pi|x - y|^3}\,, \qquad x \in \mathbf{R}^3_+,\ y \in \partial\mathbf{R}^3_+\,, \tag{126}$$

$$P_{B_R}(x, y) = \frac{R^2 - |x|^2}{4\pi R|x - y|^3}\,, \qquad x \in B_R\,,\ y \in \partial B_R\,. \tag{127}$$

Lecture 9, 2/13/2018

Green's function and Poisson kernel , 3d Fourier series

The Green's function and the Poisson kernel of a domain Ω we discussed last time have many interesting properties. Let us look at a few of them for the Poisson kernel, for example. (This part is optional.) One way to think about the Poisson kernel $P(x, y) = P_\Omega(x, y)$ (at a heuristic level) is that for a fixed $y \in \partial\Omega$ the function $x \to P(x, y)$ solves the Dirichlet problem $\Delta u = 0$ in Ω with the boundary condition $u|_{\partial\Omega} = \delta_{y,\partial\Omega}$, where $\delta_{y,\partial\Omega}$ is the Dirac function at the boundary which is concentrated at y. (In other words, $\int_{\partial\Omega} \varphi(x)\delta_{y,\partial\Omega}(x)\,dx = \varphi(y)$ for each smooth functions φ on the boundary $\partial\Omega$.) As we approach the boundary from the inside of the domain, this Dirac function is approached for example in the following sense. If we let $\Omega_\varepsilon = \{x \in \Omega\,,\, \mathrm{dist}\,(x, \partial\Omega > \varepsilon\}$, then $\lim_{\varepsilon\to 0_+} \int_{\partial\Omega_\varepsilon} P(x, y)\varphi(x)\,dx = \varphi(y)$ for any smooth function $\varphi\colon \mathbf{R}^3 \to \mathbf{R}$. You can do this calculation for the half-space $\mathbf{R}^3_+$ as an optional exercise.

One can also look at the function $y \to P(x, y)$ when $x \in \Omega$ is fixed. For each $x \in \Omega$ the function $y \to P(x, y)$ defined for $y \in \partial\Omega$ is well-defined on $\partial\Omega$. Note that $\int_{\partial\Omega} P(x, y)\,dy = 1$ for each $x \in \Omega$, because the function $u = 1$ in Ω solves $\Delta u = 0$ with the boundary condition $g(y) = 1$, and if we set $u = 1$ and $g = 1$ in the formula $u(x) = \int_{\partial\Omega} P(x, y)g(y)\,dy$, we obtain $\int_{\partial\Omega} P(x, y)\,dy = 1$. If we pick $\overline{x} \in \partial\Omega$ and consider points $x \in \Omega$ approaching $\overline{x}$, the functions $y \to P(x, y)$ will approach the Dirac function $\delta_{\overline{x},\partial\Omega}$. This can again be nicely illustrated in the half-space $\mathbf{R}^3_+$, where one can do the calculation explicitly.

Using the Green's function and the Poisson kernel, one can write express the solution of the more general Dirichlet problem $\Delta u = f$ in Ω and $u = g$ at $\partial\Omega$ as follows:

$$u(x) = \int_\Omega G_\Omega(x, y)f(y)\,dy + \int_{\partial\Omega} P_\Omega(x, y)g(y)\,dy\ . \tag{128}$$

Fourier series in 3d and the equation $\Delta u = f$ in rectangular boxes

So far we have been inverting the Laplacian Δ (i. e. solving the equation $\Delta u = f$, with suitable assumptions) using the observation due to Poisson that $\Delta(G*f) = f$. We now start discussing a different method, based on Fourier series. Let us recall some basic formulae for the Fourier series of one variable. In the textbook the following convention is used. If f is a periodic function with period $2L$, then its representation by the Fourier series is

$$f(x) = \sum_{k\in\mathbf{Z}} c_k e^{\pi k i \frac{x}{L}}\,, \qquad c_k = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx\,. \tag{129}$$

where $\mathbf{Z}$ denotes the integers. See Section 3.6. We will modify the convention as follows.

(i) We will use L (rather than $2L$) to denote the period of the function, i. e. we assume $f(x+L) = f(x)$ for $x \in \mathbf{R}$.

(ii) We will write the series as

$$f(x) = \frac{1}{L} \sum_{k \in \frac{2\pi}{L}\mathbf{Z}} \hat{f}(k) e^{ikx}\,, \qquad \hat{f}(k) = \int_0^L f(x) e^{-ikx}\,dx\,. \tag{130}$$

Note that the frequencies k now do not have to be integers, they run through the set given by the $\frac{2\pi}{L}$ multiples of integers.

Note that when f vanishes outside of $[0, b]$ for some $b > 0$ and we take L to $+\infty$ in (130), then in the limit k runs through the real numbers $\mathbf{R}$, the definition of $\hat{f}(k)$ does not change, and the Fourier series for f becomes the Fourier integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\,dk\,, \tag{131}$$

because

$$\frac{1}{L} \sum_{k \in \frac{2\pi}{L}\mathbf{Z}} g(k) = \frac{1}{2\pi} \frac{2\pi}{L} \sum_{k \in \frac{2\pi}{L}\mathbf{Z}} g(k) \quad \xrightarrow[L\to\infty]{} \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k)\,dk \tag{132}$$

for any continuous function g which approaches zero sufficiently fast as $k \to \pm\infty$.

An important point for our considerations will be the formula

$$\widehat{\left(\frac{\partial f}{\partial x}\right)}(k) = ik\hat{f}(k)\,, \tag{133}$$

which means that the operation of taking a derivative of f in the physical space is represented by the operation of multiplying $\hat{f}$ by ik in the "Fourier space".

All this generalizes to functions of several variables. For example, for three variables we have $x = (x_1, x_2, x_3)$ and for a function $f\colon \mathbf{R}^3 \to \mathbf{R}$ which is L_j-periodic in the variable x_j (where $j = 1, 2, 3$) we can write

$$f(x) = \frac{1}{L_1 L_2 L_3} \sum_{(k_1,k_2,k_3) \in \frac{2\pi}{L_1}\mathbf{Z} \times \frac{2\pi}{L_2}\mathbf{Z} \times \frac{2\pi}{L_3}\mathbf{Z}} \hat{f}(k_1, k_2, k_3) e^{i(k_1x_1 + k_2x_2 + k_3x_3)}\,, \tag{134}$$

with

$$\hat{f}(k_1, k_2, k_3) = \int_0^{L_1} \int_0^{L_2} \int_0^{L_3} f(x) e^{-i(k_1x_1 + k_2x_2 + k_3x_3)}\,dx_1 dx_2 dx_3\,. \tag{135}$$

We can use a more concise notation

$$f(x) = \frac{1}{|Q|} \sum_k \hat{f}(k) e^{ikx}\,, \qquad \hat{f}(k) = \int_Q f(x) e^{-ikx}\,dx\,, \tag{136}$$

where Q is the rectangular box $[0, L_1] \times [0, L_2] \times [0, L_3]$, with $|Q|$ denoting its volume, and k runs through the set $\frac{2\pi}{L_1}\mathbf{Z} \times \frac{2\pi}{L_2}\mathbf{Z} \times \frac{2\pi}{L_3}\mathbf{Z}$. The equation $\Delta u = f$ becomes very simple in the Fourier variables $\hat{u}$ and $\hat{f}$:

$$-|k|^2 \hat{u}(k) = \hat{f}(k)\,, \qquad |k|^2 = k_1^2 + k_2^2 + k_3^2\,. \tag{137}$$

Lecture 10, 2/15/2018

We continued to discuss the method of solving the equation $\Delta u = f$ via Fourier series. Problems where the Fourier series or the Fourier transformation comes up will not be on Midterm 1.

Lecture 11, 2/20/2018

Using Matlab for solving the Laplace equation in rectangles

Let Q be a rectangular domain (i. e. product of intervals) in $\mathbf{R}^m$. We discussed the problem of solving

$$\begin{array}{rcll} \Delta u & = & f & \text{in } \Omega \\ u & = & 0 & \text{at } \partial\Omega\,. \end{array} \tag{138}$$

with a computer, using Matlab. For simplicity we took $m = 2$, but a very similar the same algorithm works for $m = 3$, with only minor changes in the code.

The Matlab code (with fairly detailed comments) of the example discussed in class can be found here. This function, called `solvelap`, can be used for example as follows. Assume we wish to calculate the Green's function $G(x, y)$ on the unit square, on a grid 128 times 128, with y corresponding to $(0.3, 0.4)$. By commands
`clear f` (to be sure that a previous use of f does not interfere with our calculation)
`f(128,128)=0;` (this creates a 128 times 128 field of zeroes)
`f(30,40)=127*127;` (this is where the Dirac function is concentrated)
we create the discrete approximation of the Dirac mass. Note that the grid divides the square into 127 times 127 small squares, which is where the number 127^2 comes from (to mimic the condition $\int_\Omega f(x)\,dx = 1$).

Now we solve $\Delta u = f$ (or, more precisely, its approximate version on the grid) with the zero boundary condition by
`u=solvelap(f);`
We can look at the function using, for example, the commands
`[X,Y]=meshgrid(linspace(0,1,128),linspace(0,1,128))`
`clf` (to clear previous pictures)
`surf(X,Y,u)`

As an exercise, you can re-write the function `solvelap` so that it solves $\Delta u = f$ in a rectangle with sides L_1 and L_2.

Lecture 12, 2/27/2018

Eigenfunctions and eigenvalues of the Laplace operator

We have often made comparisons between the Laplace operator and a symmetric matrix. For example, in Lecture 2 we discussed the discrete Laplacian, which, in fact, is given by a symmetric matrix. One of the main results for symmetric matrices is that in a suitable orthonormal basis they are diagonal. More precisely, if A is an $n \times n$ symmetric matrix, then there exists an orthogonal basis $b^{(1)}, \dots, b^{(n)}$ of $\mathbf{R}^n$ and real numbers $\lambda_1, \dots, \lambda_n$ such that

$$Ab^{(k)} = \lambda_k b^{(k)}\,. \tag{139}$$

This means that in the basis $b^{(1)}, \dots, b^{(n)}$ the mapping given by A is represented by the diagonal matrix $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$. We can also say (somewhat loosely) that in the

basis $b^{(1)}, \dots, b^{(n)}$ the matrix A is diagonal. If we work in a basis in which A becomes diagonal, various operations with A become very easy.

We recall some useful formulae related to this. We will use the following notation: for two vectors $a, b \in \mathbf{R}^n$ we will denote by $a \otimes b$ the matrix with entries $a_i b_j$. Note that $a \otimes a$ is always a symmetric matrix.

Assume that A is an $n \times n$ symmetric matrix, and let $b^{(1)}, \dots, b^{(n)}$ be an orthogonal basis consisting of its eigenvectors, with the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$.

Let us denote by I the identify matrix (defined by $I_{kl} = 1$ for $k = l$ and $I_{kl} = 0$ for $k \neq l$).

We have

$$b^{(1)} \otimes b^{(1)} + b^{(2)} \otimes b^{(2)} + \cdots + b^{(n)} \otimes b^{(n)} = I \,. \tag{140}$$

This can be seen from the fact that the matrix on the left maps $b^{(j)}$ to $b^{(j)}$ for each $j = 1, 2, \dots, n$:

$$(b^{(i)} \otimes b^{(i)}) b^{(j)} = b^{(i)} (b^{(i)}, b^{(j)}) = \begin{cases} b^{(i)} & i = j \,, \\ 0 & i \neq j \,. \end{cases} \tag{141}$$

We also have

$$\lambda_1 b^{(1)} \otimes b^{(1)} + \lambda_2 b^{(2)} \otimes b^{(2)} + \cdots + \lambda_n b^{(n)} \otimes b^{(n)} = A \,, \tag{142}$$

because

$$\lambda_i (b^{(i)} \otimes b^{(i)}) b^{(j)} = \lambda_i b^{(i)} (b^{(i)}, b^{(j)}) = \begin{cases} \lambda_i b^{(i)} & i = j \,, \\ 0 & i \neq j \,. \end{cases} \tag{143}$$

When none of the eigenvalues λ_j vanish, we also have

$$\frac{1}{\lambda_1} b^{(1)} \otimes b^{(1)} + \frac{1}{\lambda_2} b^{(2)} \otimes b^{(2)} + \cdots + \frac{1}{\lambda_n} b^{(n)} \otimes b^{(n)} = A^{-1} \,, \tag{144}$$

as one can easily check by the same reasoning as above.

All this has analogies when we go from a symmetric matrix to the Laplace operator (and, in fact, more general operators).

Let us consider a bounded smooth domain $\Omega \subset \mathbf{R}^m$ and the Laplacian Δ acting on functions vanishing at the boundary. The eigenvalue problem associated with this situation is

$$\begin{aligned} -\Delta \phi &= \lambda \phi \qquad \text{in } \Omega \,, \\ \phi &= 0 \qquad \text{at } \partial\Omega \,. \end{aligned} \tag{145}$$

Here we write the Laplacian with the minus sign (as is the usual convention), so that the corresponding eigenvalues λ are positive.

The main point is that in this situation one has a result which is fully analogous to the finite-dimensional situation discussed above, with analogous formulae. There exists a sequence of functions $\phi_1, \phi_2, \phi_3, \dots$ in Ω which vanish at the boundary, are mutually orthogonal in the sense that

$$\int_\Omega \phi_i(x) \phi_j(x) \, dx = 0 \,, \qquad i \neq j \,, \tag{146}$$

have "unit length" in the sense

$$\int_\Omega |\phi_j(x)|^2 \, dx = 1 \,, \tag{147}$$

together with a sequence of real numbers $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \cdots \to \infty$ such that (146) is satisfied with $\phi = \phi_j$ and $\lambda = \lambda_j$, $j = 1, 2, 3, \ldots$ Moreover, the functions ϕ_j form a "basis"[10] of a suitable space of functions, in the sense that every sufficiently regular function $f\colon \Omega \to \mathbf{R}$ can be represented as

$$f(x) = \sum_{j=1}^{\infty} c_j \phi_j(x)\,, \qquad c_j = \int_\Omega f(x)\phi_j(x)\,dx\,, \tag{148}$$

and, moreover,

$$\int_\Omega |f(x)|^2\,dx = \sum_{j=1}^{\infty} |c_j|^2\,. \tag{149}$$

If G is the Green's function of Ω, then (145) implies

$$\int_\Omega G(x,y)\phi(y)\,dy = -\frac{1}{\lambda}\phi(x)\,. \tag{150}$$

In other words, ϕ is an eigenfunction of the operator

$$f \to Gf, \qquad Gf(x) = \int_\Omega G(x,y)f(y)\,dy\,. \tag{151}$$

In addition, in analogy with (144) one has

$$\frac{1}{\lambda_1}\phi_1(x)\phi_1(y) + \frac{1}{\lambda_2}\phi_2(x)\phi_2(y) + \cdots = G(x,y)\,, \tag{152}$$

if the convergence is taken in an appropriate sense. Similarly, with an appropriately defined notion of convergence (which is not point-wise convergence), one has (in analogy with (140))

$$\phi_1(x)\phi_1(y) + \phi_2(x)\phi_2(y) + \cdots = \delta(x-y)\,, \quad x, y \in \Omega\,, \tag{153}$$

where δ is the Dirac function.

Lecture 13, 3/1/2018

Eigenfunctions, eigenvalues, and resonances

For many *linear* systems (i. e. those governed by linear equations), the eigenvalues can are related to certain frequencies. A classical example is the eigenvalues of the Laplacian, which can be "observed" in connection with the wave equation.(This is also the case for other equations, such as the Schödinger equation).

We consider a smooth bounded domain $\Omega \subset \mathbf{R}^3$ and the problem

$$\begin{aligned} u_{tt} &= \Delta u \qquad \text{in } \Omega \times (t_1, t_2), \\ u|_{\partial\Omega} &= 0\,, \end{aligned} \tag{154}$$

where we use the notation

$$u_{tt} = \frac{\partial^2 u}{\partial t^2}\,. \tag{155}$$

[10]The technical term is *Hilbert basis*, it is not an algebraic basis, as we take infinite sums, not just finite linear combinations.

This can be viewed as an infinite-dimensional version of the finite-dimensional problem

$$\ddot{\xi} = -A\xi\,, \tag{156}$$

where $\xi = (\xi_1, \dots \xi_n)$ is an $n-$vector (depending on time) and A is a positive-definite symmetric matrix (i. e. $(A\xi, \xi) > 0$ if $\xi \neq 0$.) If we know the eigenvectors and eigenvalues of A, it is easy to obtain an expression for a general solution of (156): Assuming the eigenvectors are $b^{(1)}, \dots, b^{(n)}$, with the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, the general solution can be for example written as

$$\xi(t) = A^1 \sin(\sqrt{\lambda_1}\,(t - t_1))b^{(1)} + \dots + A_n \sin(\sqrt{\lambda_n}\,(t - t_n))\,b^{(n)}\,, \tag{157}$$

where $A_1, \dots, A_n$ and $t_1, \dots, t_n$ are parameters. (We have $2n$ parameters, as we should, as we are dealing with a system of n ordinary differential equations of the second order). Often it may be useful to write the solution in the complex form

$$\xi(t) = \sum_{j=1}^{n} \left(A_j\, e^{i\sqrt{\lambda_j}\,t} + B_j e^{-i\sqrt{\lambda_j}\,t}\right)\, b^{(j)}\,. \tag{158}$$

We again have $2n$ parameters, which now can be complex, and the real-valued solutions are obtained as the real part of the complex-valued solutions. (The relation between the expressions (158) and (157) can be obtained from the Euler formula $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$ and the trigonometric formula $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.)

Let $\phi_1, \phi_2, \dots$ be the eigenfunctions of $-\Delta$ (with the zero boundary condition), as discussed in the last lecture, with the corresponding eigenvalues $\lambda_1, \lambda_2, \dots$. Then, in analogy with (158) the "general solution" of (154) can be written as

$$u(x,t) = \sum_j \left(A_j\, e^{i\sqrt{\lambda_j}\,t} + B_j e^{-i\sqrt{\lambda_j}t}\right)\, \phi_j(x)\,. \tag{159}$$

We see that the eigenvalues λ_j are related to the frequencies $\sqrt{\lambda_j}$ at which the system can oscillate. The series can be finite or infinite, and in the latter case one must of course pay attention to the questions of convergence, which we will disregard for the moment.

To derive the last expression, we simply seek the solution as as

$$u(x,t) = \sum_j c_j(t)\,\phi_j(x)\,, \tag{160}$$

and after substituting this expression into (154), we get the simple ordinary differential equation

$$\ddot{c}_j = -\lambda_j c_j\,, \tag{161}$$

which we know how to solve.

The eigenvalues are important for *resonances*. To illustrate this notion, let us add a forcing and also some friction to the wave equation (154), and consider

$$\begin{aligned} u_{tt} + \varepsilon u_t &= \Delta u + f(x,t) \qquad \text{in } \Omega \times (t_1, t_2)\,, \\ u|_{\partial\Omega} &= 0\,. \end{aligned} \tag{162}$$

where $\varepsilon > 0$ and f represents the forcing. (One can do the same for our finite-dimensional system (156) and consider $\ddot{\xi} + \varepsilon\xi = -A\xi + b(t)$, where b is a time-dependent $n-$vector.) We will write f as

$$f(x,t) = \sum_j f_j(t)\phi_j(x)\,, \tag{163}$$

and again seek u in the form

$$u(x,t) = \sum_j c_j(t)\phi_j(x)\,. \tag{164}$$

This gives

$$\ddot{c}_j + \varepsilon\dot{c}_j = -\lambda_j c_j + f_j(t)\,, \qquad j = 1, 2, \ldots \tag{165}$$

This is again a simple ordinary differential equation which can be solved.

Let us consider a special important case and when the forcing term is of the form

$$f(x,t) = e^{i\omega t}g(x)\,, \tag{166}$$

with

$$g(x) = \sum_j g_j\phi_j(x)\,. \tag{167}$$

(For the physical interpretation we take the real parts, as usual.) The equation (165) becomes

$$\ddot{c}_j + \varepsilon\dot{c}_j = -\lambda_j c_j + e^{i\omega t}g_j\,. \tag{168}$$

To simplify notation, we drop the index j and write simply

$$\ddot{c} + \varepsilon\dot{c} = -\lambda c + e^{i\omega t}g\,. \tag{169}$$

We can seek a particular solution as

$$c(t) = Ae^{i\omega t}\,. \tag{170}$$

This gives

$$A = \frac{g}{\lambda - \omega^2 + i\varepsilon}\,. \tag{171}$$

When $\varepsilon > 0$ general solution will approach the solution given by (170) as $t \to \infty$, as can be easily checked. Therefore the particular solution (170) captures to a large degree what is going on. We see that when $\omega \sim \sqrt{\lambda}$ and $\varepsilon > 0$ is small, the amplitude A of the oscillations will be large, approaching ∞ as $\varepsilon \to 0$ and $\omega = \sqrt{\lambda}$. When $\varepsilon = 0$ and $\omega = \sqrt{\lambda}$ there will be no solution of the form (170). Instead, there will be a solution of the form $Ate^{i\omega t}$, which means that the amplitude will be growing indefinitely.

The above considerations have practical implications - if we force a system with a force oscillating at a resonant frequency, the amplitude will grow. This can be a desired effect (e. g. for musical instruments) or an undesirable effect (e. g. for stability of bridges and other constructions), and eigenvalues often play a prominent role in these considerations.

Lecture 14, 3/6/2018

The variational nature of the Laplace equation

If A is a positive-definite $n \times n$ symmetric matrix, then finding the solution ξ of

$$A\xi = b \tag{172}$$

is equivalent to minimizing the function

$$J(\xi) = \frac{1}{2}(A\xi, \xi) - (b, \xi)\,, \tag{173}$$

where we use the usual notation $(\xi,\eta)=\xi_1\eta_1+\xi_2\eta_2+\cdots+\xi_n\eta_n$ for the scalar product. As an exercise, you can check that the n equations

$$\frac{\partial J}{\partial \xi_k}(\xi)=0\,,\qquad k=1,2,\ldots,n \tag{174}$$

exactly give (172). For the problem

$$\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega\,, \\ u|_{\partial\Omega} & = & 0\,, & \end{array} \tag{175}$$

the situation is similar. The associated function (defined, roughly speaking, on the space of functions on Ω vanishing at the boundary $\partial\Omega$), which we usually call a *functional* is

$$J(u)=\int_\Omega \left(\frac{1}{2}|\nabla u(x)|^2-f(x)u(x)\right)\,dx\,. \tag{176}$$

The solution of (175) minimized the functional $J(u)$ on (sufficiently regular) functions vanishing at $\partial\Omega$. One can see that the minimizer of J (on functions vanishing at $\partial\Omega$) should give (175) as follows. (The calculation goes back to Euler and Lagrange.) Let us take a smooth function $\varphi\colon \partial\Omega\to\mathbf{R}$ with $\varphi|_{\partial\Omega}=0$. If u minimizes J (in our class of functions vanishing at the boundary) then the function of one variable ε defined by $\varepsilon\to J(u+\varepsilon\varphi)$ should attain its minimum at $\varepsilon=0$. Note that $\varepsilon\to J(u+\varepsilon\varphi)$ is just a simple function of one variable (which we denote ε) of the form $\frac{1}{2}a\varepsilon^2+b\varepsilon+c$:

$$\begin{aligned} J(u+\varepsilon\varphi)=&\int_\Omega\left(\frac{1}{2}|\nabla u+\varepsilon\varphi|^2-f(u+\varepsilon\varphi)\right)\,dx \\ =&\varepsilon^2\int_\Omega\frac{1}{2}|\nabla\varphi|^2\,dx+\varepsilon\int_\Omega(\nabla u\nabla\varphi-f\varphi)\,dx+J(u)\,. \end{aligned} \tag{177}$$

We see that

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}\ J(u+\varepsilon\varphi)=\int_\Omega(\nabla u\nabla\varphi-f\varphi)\,dx\,, \tag{178}$$

which can of course also be arrived at by simply differentiating the expression for J. So at the minimum u we should have

$$\int_\Omega(\nabla u\nabla\varphi-f\varphi)\,dx=0 \tag{179}$$

for each smooth φ vanishing at the boundary $\partial\Omega$. To get the equation of u, we now integrate by parts:

$$\int_\Omega\nabla u\nabla\varphi\,dx=\int_{\partial\Omega}\frac{\partial u}{\partial n}\,\varphi\,dx-\int_\Omega\Delta u\,\varphi\,dx\,. \tag{180}$$

Under our assumptions on φ the boundary integral vanishes, and we see that (179) can be replaced by

$$\int_\Omega(-\Delta u-f)\varphi\,dx=0\,. \tag{181}$$

As this should be true for each φ, we must have $-\Delta u-f=0$.

If our class of admissible functions contained functions not vanishing at the boundary, the boundary integral in (180) would have to taken into account and would have

an effect on the boundary condition. In class we calculated the equation one gets from the problem of minimizing the functional

$$I_\nu(u) \int_\Omega \frac{1}{2}\nu|\nabla u|^2 + \frac{1}{2}|f-u|^2\,dx \tag{182}$$

over all (sufficiently regular) functions, regardless of the boundary values. Here $\nu > 0$ is a fixed parameter. We again use the equation

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} I(u+\varepsilon\varphi) = 0\,, \tag{183}$$

which this time should hold for any smooth function φ, regardless of its boundary condition. By a similar calculation as above we obtain that the derivative (183) is equal to

$$\int_\Omega (\nu\nabla u\nabla\varphi + u\varphi - f\varphi)\,dx \tag{184}$$

and this expression has to vanish for each smooth φ (regardless of its boundary values). Integrating by parts, we see that the last expression is equal to

$$\int_{\partial\Omega} \nu\frac{\partial u}{\partial n}\varphi + \int_\Omega (-\Delta u + u - f)\varphi\,dx\,. \tag{185}$$

Taking first all φ which vanish at $\partial\Omega$, we see that we have to have $-\Delta u + u - f = 0$ in Ω. Then, taking φ not vanishing at $\partial\Omega$, we see that $\frac{\partial u}{\partial n} = 0$ at $\partial\Omega$. So the PDE problem associated with the minimization of I is

$$\begin{array}{rcll} -\nu\Delta u + u & = & f & \text{in } \Omega\,, \\ \frac{\partial u}{\partial n} & = & 0 & \text{at } \partial\Omega. \end{array} \tag{186}$$

So far we have assumed that there is no problem with the minimization procedure and that the functionals we have considered attain their minimum at some function. In the two particular cases above this is indeed the case, but it is not automatic. For example, if we try to minimize

$$\int_\Omega \left(\frac{1}{2}|\nabla u|^2 - u\right) dx \tag{187}$$

over all functions u, we see easily that integral can attain arbitrary negative value by taking $u = c$ for a suitable constant c. This is related to the fact that the corresponding problem

$$\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega \\ \frac{\partial u}{\partial n} & = & 0 & \text{at } \partial\Omega\,, \end{array} \tag{188}$$

only has a solution when $\int_\Omega f = 0$. If $f = 1$, the problem has no solution.

Remarks on solvability of elliptic equations

Ler A be an $n \times n$ matrix and consider the equation

$$A\xi = b\,, \tag{189}$$

where b is a given $n-$vector and ξ is the unknown $n-$vector (to be calculated). One of the important results in Linear Algebra concerns the solvability of (300):

The equation (300) is uniquely solvable for each b if and only if the homogeneous solution $A\xi = 0$ only has the trivial solution $\xi = 0$.

Note that this is the same as saying that (300) is uniquely solvable for each b if and only if it is uniquely solvable for $b = 0$.

In addition, the dimension of the space of solutions of the homogeneous equation $A\xi = 0$ is the same as the number of conditions b has to satisfy for $A\xi = b$ to be solvable. In the linear algebra language, the sum of the dimension of $A^{-1}(0)$ and the dimension of $A(\mathbf{R}^n)$ equals n.

Similar results also hold for elliptic problems. We will not formulate the exact theorem, as we do not wish to go into technicalities, but the guiding principle which has a good chance of being true (and can often be proved rigorously under appropriate assumptions) is that a (scalar) non-homogeneous elliptic equation is uniquely solvable in appropriate classes of functions if and only if the homogeneous homogeneous equation only has the trivial solution $u = 0$.

Let us illustrate this by two examples:

Example 1
Let us consider a bounded domain $\Omega \subset \mathbf{R}^m$ and the problem

$$\begin{array}{rcll} -\Delta u & = & f(x) & \text{in } \Omega\,, \\ u|_{\partial\Omega} & = & 0\,. & \end{array} \tag{190}$$

As an exercise you can derive from the maximum principle or by integration by parts that the only solution of (190) with $f = 0$ is $u = 0$. By the above analogy we would expect that this means that the problem (190) is always solvable. This turns out to be the case, in a sense which has to be made precise by specifying the classes of admissible functions f. For example, if Ω is smooth, than for each smooth f we have a unique smooth solution u.

Example 2
With Ω as above, let us consider

$$\begin{array}{rcll} -\Delta u & = & f(x) & \text{in } \Omega\,, \\ \frac{\partial u}{\partial n} & = & 0 & \text{at } \partial\Omega\,. \end{array} \tag{191}$$

In this case the homogeneous problem (corresponding to $f = 0$) has non-trivial solutions $u = \text{const}$. As an exercise (which we did in class) you can show that every solution of the homogeneous problem is of this form, so the dimension of the space of solutions is one. Hence we expect that (191) will not be always solvable, and there will be one linear condition which f has to satisfy to ensure solvability. This is indeed the case: it can be shown that (191) is solvable (for a smooth f, say) if and only if $\int_\Omega f(x)\,dx = 0$.

The above examples are just a "tip of the iceberg" for theorems of this type. There are several levels of theorems in this direction, with the most well-known among the advanced ones being the Atiyah-Singer Index Theorem.

Lecture 15, 3/8/2018
More on variational principles and various boundary conditions

Let us consider a finite-dimensional situation relevant for our variational principles. Assume we have an electric circuit consisting of n "nodes" connected with resistors. We denote R_{ij} be the resistance of the resistor directly connecting the nodes i and j.

Given a subset B of the set X of all nodes, we impose the following conditions:
(i) (A "boundary condition"): The voltage u_j at $j \in B$ is prescribed by to be g_j, and can be kept at that level by connecting the node to an outside source.
(ii) (An analogue of the "right-hand side" for the Laplace equation): The nodes in $Y = X \setminus B$ are also connected to an outside source of electricity which can be adjusted so that the current flowing from the source to the node $k \in X \setminus B$ is equal to a given value I_k.

So our data are the values $g_j\,,\ j \in B$ and $I_k\,,\ k \in Y$. Our task is to find the voltages u_k for $k \in Y$.

In the context of the boundary value problem

$$\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega\,, \\ u|_{\partial\Omega} & = & g & \text{at } \partial\Omega\,, \end{array} \tag{192}$$

the set B would correspond to $\partial\Omega$ and the set Y would correspond to Ω. (Note that the sets Ω and $\partial\Omega$ are disjoint, with $\Omega \cup \partial\Omega = \overline{\Omega}$, the closure of Ω.)

The discrete problem can be solved by the following variational principle: consider

$$J(u) = \sum_{j,k \in X,\, j<k} \frac{(u_j - u_k)^2}{2R_{jk}} - \sum_{k \in Y} I_k u_k\,. \tag{193}$$

This is an analogy of the functional

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu \right) dx \tag{194}$$

in the continuous problem (192), in which we consider u satisfying $u|_{\partial\Omega} = g$, just as in (193) we only restrict our attention to u_j with $u_j = g_j$ for $j \in B$.

Assuming (193) attains a minimum, the equations for the minimizing u_k can be obtained from

$$\frac{\partial J(u)}{\partial u_k} = 0\,, \qquad k \in Y\,. \tag{195}$$

Denoting by m the number of elements in Y, we see that we get m equations for m unknowns. You can check as an exercise that the equations give the Kirchhoff laws for our electric circuit: the total sum of currents coming to each node in the set Y is zero (when counted with appropriate signs).

The functional (194) can be considered as a continuum version of (193), and the first equation in (192) is in some sense the continuum version of the Kirchhoff law. In fact, various finite-dimensional approximations of (192), such as the one given by (28) in Lecture 2 can be interpreted in terms of electrical circuits and the energy (193).

Boundary conditions arising from the variational formulation.
Let us consider the following variant of (194)

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu \right) dx + \int_{\partial\Omega} \left(\frac{1}{2} a|u|^2 - bu \right) dx\,, \tag{196}$$

where a, b are functions. Let us assume that we can minimize over all smooth functions on Ω, which is the case when $a > 0$, for example. (We could also minimize it over

various subclasses of all functions, such as the functions vanishing on a part of the boundary $\Gamma_1 \subset \partial\Omega$. The reader can consider work out such cases as an exercise.)

As usual, we evaluate the derivative $\frac{d}{d\varepsilon}|_{\varepsilon=0} J(u+\varepsilon\varphi)$ which we will also denote by $J'(u)\varphi$.

$$\begin{aligned} J'(u)\varphi &= \int_\Omega (\nabla u \nabla\varphi - f\varphi)\, dx + \int_{\partial\Omega} (au\varphi - b\varphi)\, dx \\ &= \int_\Omega (-\Delta u - f)\,\varphi\, dx + \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} + au - b\right)\varphi\, dx\,. \end{aligned} \tag{197}$$

If u is a minimizer of J, then $J'(u)\varphi$ has to vanish for each φ, leading to

$$\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega \\ \frac{\partial u}{\partial n} + au & = & b & \text{at } \partial\Omega\,. \end{array} \tag{198}$$

Lecture 16, 3/20/2018

Lower order terms

Let us consider the equation

$$u_t + b\nabla u + cu - \Delta u = 0\,, \tag{199}$$

in $\mathbf{R}^m \times (t_1, t_2)$. For now we will make an assumption that b is a constant vector (independent of x, t and c is a constant number (also independent of x, t). The notation $b\nabla u$ is a shorthand for $\sum_{j=1}^m b_j \frac{\partial u}{\partial x_j}$

We can think of this equation as consisting of three processes going on at the same time, with the first one described by

$$u_t - \Delta u = 0\,, \tag{200}$$

the second one described by

$$u_t + b\nabla u = 0\,, \tag{201}$$

and the third one described by

$$u_t + cu = 0\,. \tag{202}$$

We understand (200) in some detail: the solution is given by

$$u(x,t) = \int_{\mathbf{R}^m} \Gamma(x-y,t) u_0(y)\, dy\,, \tag{203}$$

where Γ is the heat kernel and $u_0(x) = u(x,0)$. We can also write this as

$$u(t) = \Gamma(t) * u_0\,. \tag{204}$$

Equation (201) is also easy to understand: its solution is given by

$$u(x,t) = u_0(x - bt)\,, \tag{205}$$

where, again, $u_0(x) = u(x,0)$. It describes a translation of the initial condition u_0 at a uniform speed b.

Finally, equation (202) is just an ODE, there are no partial $x-$derivatived in it, and the solution is simply

$$u(x,t) = u_0(x)e^{-ct}\,. \tag{206}$$

So we understand each part of (199) separately, but how do the three parts act together? Let us look at an analogy in ordinary differential equations. Consider

$$\dot{\xi} = (A+B+C)\xi\,, \tag{207}$$

where ξ is an $n-$vector and A, B, C are $n \times n$ matrices. The solution of (207) can be written as

$$\xi(t) = e^{(A+B+C)t}\xi(0)\,, \tag{208}$$

where the matrix exponential is given by the usual formula

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}\,, \tag{209}$$

where we use the standard conventions $X^0 = I$ (the identity matrix) and $0! = 1$. The solutions of the equations $\dot{\xi} = A\xi, \dot{\xi} = B\xi, \dot{\xi} = C\xi$ are given respectively by $e^{At}\xi(0), e^{Bt}\xi(0)$ and $e^{Ct}\xi(0)$. If A, B, C were just numbers a, b, c, we would have

$$e^{(a+b+c)t} = e^{at}e^{bt}e^{ct}\,. \tag{210}$$

This could be interpreted as solving $\dot{\xi} = (a+b+c)\xi$ in three stages: we first apply the evolution induced by c then the evolution induced by b and finally the evolution induced by a. Note that the order could be permuted in any way, it does not matter in which order we apply the exponentials.

For general matrices A, B, C is this not the case. For simplicity, let us think of the case of two matrices A, B and the equation

$$\dot{\xi} = (A+B)\xi\,. \tag{211}$$

At an infinitesimal level, we can still think about the evolution as a superposition of the motions given by A and B, and this is expressed in Trotter's formula:

$$e^{(A+B)t} = \lim_{n\to\infty} \left(e^{At/n}e^{Bt/n}\right)^n\,. \tag{212}$$

In general, we cannot take this at a "finite level", in the sense that typically it is *not true* that $e^{(A+B)t} = e^{At}e^{Bt}$. However, the formula is true under the additional assumption that the matrices commute, i. e. $AB = BA$. Similarly for three matrices: if $AB = BA, AC = CA, BC = CA$, then $e^{(A+B+C)t} = e^{At}e^{Bt}e^{Ct}$.

Going back to our PDE (199), we can apply the same rules. We let

$$L_1 u = \Delta u\,, \quad L_2 u = -b\nabla u\,, \quad L_3 u = -cu\,, \quad L = L_1 + L_2 + L_3 \tag{213}$$

We can write, in analogy with the notation e^{At} used for matrices

$$e^{L_1 t}u_0 = \Gamma(t) * u_0\,, \qquad (e^{L_2 t}u_0)(x) = u_0(x-bt)\,, \quad e^{L_3 t} = e^{-ct}u_0\,. \tag{214}$$

This notation is somewhat "formal". For example, the formula

$$u(t) = e^{\Delta t}u_0 = (I + \frac{\Delta t}{1!} + \frac{\Delta^2 t^2}{2!} + \frac{\Delta^3 t^3}{3!} + \dots)u_0 \tag{215}$$

does not imply that the series will converge. It is just a way of saying that $u(t)$ is obtained from u_0 by evolving it for time t by the heat equation. In some cases, such that when u_0 is a polynomial, we can take the formula literally (and in this particular case $u(x,t)$ will be a polynomial which solves the heat equation), but mostly we should think about such formulae as just indicating the time evolution by the operator at hand.

We note that all the operators L_1, L_2, L_3 have constant coefficients and hence $L_1L_2 = L_2L_1$, $L_1L_3 = L_3L_1$, $L_2L_3 = L_3L_2$ and we can write

$$e^{Lt} = e^{(L_1+L_2+L_3)t} = e^{L_1t}e^{L_2t}e^{L_3t}\,. \tag{216}$$

Therefore the solution of (199) can be written for example as

$$u(x,t) = \int_{\mathbf{R}^m} \Gamma(x-y,t)u_0(y-bt)e^{-ct}\,dy\,. \tag{217}$$

All this can be also seen just by a simple change of variables, without the analogies with the matrices. We simply seek the solution of (199) as

$$u(x,t) = v(x-bt,t)e^{-ct}\,, \tag{218}$$

and substituting this into (199) we obtain

$$v_t - \Delta v = 0\,. \tag{219}$$

The purpose of the above discussion essentially was that the substitution (218) is not just an *ad hoc* trick.

When the lower order terms have variable coefficients, i. e. $b = b(x,t)$ and $c = c(x,t)$ are functions, then the operators L_1, L_2, L_3 above no longer commute, and (216) is no longer valid. When $b = b(x), c = c(x)$, we still have the Trotter's formula

$$e^{Lt} = \lim_{n\to\infty} \left(e^{L_1t/n}e^{L_2t/n}e^{L_3t/n}\right)^n\,, \tag{220}$$

which can be useful when thinking about the equation (199), and also in its numerical simulation. In fact, the formula can be generalized to the general case $b = b(x,t), c = c(x,t)$ and it often provides a good heuristics. In the context of the numerical algorithms, it is good to know about Strang splitting, but here we will not discuss it.

Lecture 17, 3/22/2018 ,

The fundamental solution of the solution of the wave equation; more on the Dirac function

We will consider the wave equation

$$u_{tt} = \Delta u \tag{221}$$

and also its inhomogeneous form

$$u_{tt} = \Delta u + f(x,t) \tag{222}$$

in space-time $R^m \times (t_1, t_2)$, with $m = 1, 2, 3$.

We note that the more general version $u_{tt} = c^2\Delta u$ can be reduced to (221) by changing the time variable t to ct.

One form of the fundamental solution for $m = 3$ is

$$G(x,t) = \begin{cases} \frac{\delta(r-t)}{4\pi r}, & r = |x|\,,\ t \geq 0 \\ 0\,, & t < 0, \end{cases} \tag{223}$$

where δ is the Dirac function.

Before starting calculations with (223), we recall a few formulae concerning the Dirac function.

Some calculations with the Dirac function

Before starting calculations with (223), we recall a few formulae concerning the Dirac function.

Let $\phi\colon \mathbf{R} \to \mathbf{R}$ be a function which vanishes outside the interval $(-1,1)$ and $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$. As before, for $\varepsilon > 0$ we set

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right)\,. \tag{224}$$

One can also add the assumption that $\phi \geq 0$, but for the most part we will not need it here.

The Dirac function (which is not really a function in the traditional sense) can be though of as a limit of the functions ϕ_ε as $\varepsilon \to 0_+$.

In formula (223) the function $\delta(r-t)$ can be interpreted as a function on $\mathbf{R}^3$ which is the limit of the functions $\phi_\varepsilon(r-t)$ for $\varepsilon \to 0_+$, with $r = |x|$.

An important property of the Dirac function is

$$\int_{-\infty}^{\infty} \delta(x)\varphi(x)\,dx = \varphi(0)\,, \tag{225}$$

for every smooth function φ. This can be used to calculate the three-dimensional integral

$$\int_{\mathbf{R}^3} \delta(r-t)\varphi(x)\,dx\,. \tag{226}$$

One way to see what the integral should be is to consider it as the limit

$$\lim_{\varepsilon\to 0} \int_{\mathbf{R}^3} \phi_\varepsilon(x)\varphi(x)\,dx \tag{227}$$

Assume for a moment that $\phi = \frac{1}{2}\chi_{(-1,1)}$, where $\chi_{(-1,1)}$ is the characteristic function of the interval $(-1,1)$. In this special case it is not hard to see that the limit (227) is $\int_{S_t} \varphi(x)\,dx$, where S_t is the sphere of radius t centered at the origin. The same result is true for general ϕ as above. One can see it for example by considering the integral in the polar coordinates

$$x_1 = r\sin\theta\cos\alpha, \quad x_2 = r\sin\theta\sin\alpha, \quad x_3 = r\cos\theta. \tag{228}$$

Then

$$\begin{aligned} \int_{\mathbf{R}^3} \delta(r-t)\varphi(x)\,dx &= \int_0^\pi \int_0^{2\pi} \int_0^\infty \delta(r-t)\varphi(x_1,x_2,x_3)\ r^2\sin\theta\,dr\,d\alpha\,d\theta \\ &= \int_0^\pi \int_0^{2\pi} \varphi(t\sin\theta\cos\alpha, t\sin\theta\sin\alpha, t\cos\theta)\,t^2\ \sin\theta\,d\theta\,d\alpha \\ &= \int_{S_t} \varphi(x)\,dx\,. \end{aligned} \tag{229}$$

The spherical wave

The fundamental solution $\frac{\delta(r-t)}{4\pi r}$ is quite singular, so a-priori it is not clear in what sense it should solve some PDE.

It is useful to look at the approximation $\frac{\phi_\varepsilon(r-t)}{4\pi r}$ For times $t > \varepsilon$ this is a nice smooth function supported in an $\varepsilon-$nieghborhood of the sphere $\{|x| = t\}$.

Note that *for any smooth function* $f\colon \mathbf{R} \to \mathbf{R}$ *the function*

$$u(x,t) = \frac{f(r-t)}{r} \tag{230}$$

solves the wave equation $u_{tt} = \Delta u$ *in the region* $\{r \neq 0\}$. This calculation can be done in many ways. In class we did a calculation using the formula for the Laplacian of radial function (= functions depending on $r = |x|$):

$$\Delta g(r) = \frac{\partial}{r^2 \partial r} r^2 \frac{\partial}{\partial r}\, g\,. \tag{231}$$

The wave equation is reversible (we can run it backwards in time), just like the equation for a single harmonic oscillator $\ddot{x} + \omega^2 x = 0$. What happens with the solution (230) if we go backward in time? If f is smooth and vanishes outside of a bounded set, the solution (230) is smooth for large t, and hence if we go backward it should stay smooth. (In class we outlines the heuristics behind this non-trivial statement, based on energy conservation.)

Setting $g(s) = f(-s)$, we can write the spherical wave also as

$$u(x,t) = \frac{g(t-r)}{r}\,. \tag{232}$$

Let us now consider

$$v(x,t) = \frac{g(t-r) - g(t+r)}{r}\,, \quad r = |x|\,. \tag{233}$$

If g is a smooth function, then $v(x,t)$ is also a smooth function, the dangerous-looking division by r in the formula does not produce a singularity at $r = 0$, due to suitable cancellations. For example, one can see that

$$\lim_{x\to 0} v(x,t) = -2g'(t)\,. \tag{234}$$

When g vanishes outside of a bounded interval, the contribution of the term $g(t+r)$ will be zero for large positive times t, as we always have $r \geq 0$. In a similar way, the contribution of the term $g(t-r)$ will vanish for large negative times t. Therefore the function v given by (233) represents an incoming spherical waves (going towards the origin) for large negative times, and outgoing spherical waves (going towards the spacial infinity) for large positive times. When g is a smooth function, the function v is smooth. At the same time, we see that some "focusing" is possible. For example, we have seen that $v(0,t) = -2g'(t)$ and by considering a function g which is small, but whose derivative g' is large (or even unbounded), we will see large values of $v(0,t)$ for the times when the wave passes thought the origin, whereas v will remain small most for large negative or large positive times.

Let us now express the solution of the problem

$$\begin{aligned} u_{tt} &= \Delta u + f(x,t) \quad \text{in } \mathbf{R}^3 \times (0,\infty)\,, \\ u(x,0) &= u_0(x)\,, \qquad x \in \mathbf{R}^3\,, \\ u_t(x,0) &= u_1(x)\,, \qquad x \in \mathbf{R}^3 \end{aligned} \tag{235}$$

in terms of the fundamental solution. As the problem is linear, it is enough to consider the three cases when only one of the functions f, u_0, u_1 does not vanish. (When all functions f, u_0, u_1 vanish, we expect the solution u to vanish, too. This can be proved mathematically, and it is not difficult, but the proof is bases on a different idea than our discussion at the moment.)

Let us consider the case $f \neq 0$, with $u_0 = 0$ and $u_1 = 0$. Denoting $G(x,t)$ the fundamental soluton $\frac{\delta(r-t)}{4\pi r}$ (with $r = |x|$), we have

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbf{R}^3} G(y,s) f(x-y, t-s) f(y,s)\, dy\, ds \\ &= \int_0^t \int_{\mathbf{R}^3} \frac{\delta(|y|-s)}{4\pi|y|} f(x-y,t-s)\, dy\, ds \\ &= \int_0^t \frac{1}{4\pi s} \int_{S_{x,s}} f(y, t-s)\, dy\, ds \end{aligned} \tag{236}$$

The general case can be turned into the case just considered by formally setting

$$\tilde{f}(x,t) = f(x,t) + u_0(x)\delta'(t) + u_1(x)\delta(t)\,. \tag{237}$$

(This can be considered as one of the versions of the Duhamel's principle.) For example, when $u_0(x) = \delta(x)$ and $f = 0, u_1 = 0$, we obtain

$$u(x,t) = \frac{\partial}{\partial t} G(x,t) = \frac{\delta'(t-r)}{4\pi r}\,. \tag{238}$$

Lecture 18, 3/27/2009

More on the Duhamel's principle and the fundametal solution of the wave equation

Let us start with an ordinary differential equation (ODE) which is related to the wave equation, in the sense that both the wave equation and the ODE describe oscillations (except that in the case of the ODE the oscillating system only has one degree of freedom).

The ODE is

$$\ddot{x} + \omega^2 x = 0\,, \tag{239}$$

where $x = x(t)$ is a function of one variable. In the context of various physical system this equation may describe (such as an oscillator), the function $x(t)$ will typically be real-valued, but often it is advantageous to consider complex-valued solutions and take their real part.

The general solution is, of course, $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$, or, alternatively $x(t) = A\cos\omega t + B\sin\omega t$.

Let us now consider two problems for this ODE:

The first one is

$$\begin{aligned} \ddot{x} + \omega^2 x &= 0\,, \qquad t \geq 0\,, \\ x(0) &= 0\,, \\ \dot{x}(0) &= 1\,. \end{aligned} \tag{240}$$

The second one is

$$\begin{aligned} \ddot{x} + \omega^2 x &= \delta(t) \qquad t \in \mathbf{R}\,, \\ x(t) &= 0\,, \quad t < 0\,,\,, \end{aligned} \tag{241}$$

where $\delta(t)$ is the Dirac function. The solution of the first problem is, of course,

$$x(t) = \frac{1}{\omega}\,\sin \omega t\,, \qquad t \geq 0\,. \tag{242}$$

Consider now the function on the whole real line defined by

$$X(t) = \begin{cases} \frac{1}{\omega}\,\sin \omega t\,, & t \geq 0\,, \\ 0 & t \leq 0\,. \end{cases} \tag{243}$$

This turns out to be a solution of (241), as one can easily check. The Duhamel's principle for (241) says, roughly speaking, that this is no coincidence and problems of the type (240) and (241) are equivalent for much more general classes of equations.

The function (243) can be considered as a fundamental solution of for the equation

$$\ddot{x} + \omega^2 x = f(t)\,, \tag{244}$$

on the real line: if f vanishes for $t \to -\infty$, then the solution of (244) which vanishes for $t \to -\infty$ is given by

$$x(t) = \int_{\mathbf{R}} X(t-s) f(s)\, ds = \int_{\mathbf{R}} X(s) f(t-s)\, ds\,. \tag{245}$$

The fundamental solution of the wave equation $G(x,t)$ we discussed in the last lecture can be thought about in similar terms. One can think of it either as the solution of

$$\begin{aligned} u_{tt} - \Delta u &= \delta(x,t)\,, \qquad (x,t) \in \mathbf{R}^3 \times \mathbf{R}\,, \\ u(x,t) &= 0\,, \qquad\qquad t < 0\,, \end{aligned} \tag{246}$$

or as a solution of the initial-value problem

$$\begin{aligned} u_{tt} - \Delta u &= 0\,, \qquad t > 0\,, \\ u(x,0) &= 0\,, \\ u_t(x,0) &= \delta(x)\,. \end{aligned} \tag{247}$$

extended by 0 for $t < 0$.

Going back to (239), how should one think about the initial-value problem

$$\begin{aligned} \ddot{x} + \omega^2 x &= 0\,, \qquad t > 0 \\ x(0) &= 1\,, \\ \dot{x}(0) &= 0\,, \end{aligned} \tag{248}$$

in the context of the Duhamel's principle? The solution can, of course, be written down explicitly:

$$x(t) = \cos \omega t \,, \qquad t \geq 0 \,. \tag{249}$$

If we extend this function by 0 to the negative times, still denoting the extended function by $x(t)$, one has (in $\mathbf{R}$)

$$\ddot{x} + \omega^2 x = \delta'(t) \,. \tag{250}$$

In a similar way, one can think of the solution of the initial value problem

$$\begin{aligned} u_{tt} - \Delta u &= 0 \qquad t > 0 \\ u(x,0) &= u_0(x) \,, \\ u_t(x,0) &= 0 \,, \end{aligned} \tag{251}$$

extended by zero to the negative times as the solution of

$$u_{tt} - \Delta u = \delta'(t) u_0(x) \,, \tag{252}$$

which vanishes for negative times.

As an exercise, let us calculate the solution of

$$\begin{aligned} u_{tt} - \Delta u &= \delta(x) \,, \qquad t > 0 \,, \\ u(x,0) &= 0 \,, \\ u_t(x,0) &= 0 \,. \end{aligned} \tag{253}$$

Note that the function $\tilde{u}(x) = \frac{1}{4\pi|x|}$ satisfies the first and the third equations of (253), but does not satisfy the second equation. Denoting by $G(x,t)$ the fundamental solution (223), we have

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbf{R}^3} G(y,s) \delta(x-y) \, dy \, ds \\ &= \int_0^t G(x,s) \, ds = \int_0^t \frac{\delta(s-|x|)}{4\pi|x|} \, ds \\ &= \begin{cases} 0 \,, & t < |x| \,, \\ \frac{1}{4\pi|x|} \,, & t > |x| \,. \end{cases} \end{aligned} \tag{254}$$

Lecture 19, 3/29/2018 **Some calculations with the Dirac function; fundamental solution of the wave equation in dimensions two and one**

For calculations with the fundamental solution (223), it is useful to go deeper into some properties of the Dirac function. One important expression which often comes up in calculations with the fundamental solution is $\delta(ax)$ where $a > 0$.

If $\delta(x)$ is the Dirac function on the real line, we have

$$\delta(ax) = \frac{1}{a} \delta(x) \,. \tag{255}$$

One way to see it is via the approximation (224):

$$\delta(x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right) = \lim_{\varepsilon\to 0} \frac{1}{a}\frac{a}{\varepsilon}\phi\left(\frac{ax}{\varepsilon}\right) = \frac{1}{a}\lim_{\varepsilon'\to 0} \frac{1}{\varepsilon'}\phi\left(\frac{x}{\varepsilon'}\right) = \frac{1}{a}\delta(x), \tag{256}$$

where we have set $\varepsilon' = \frac{\varepsilon}{a}$. The same conclusion can be reached by the following calculation

$$\int_{\mathbf{R}} \delta(ax)\varphi(x)\,dx = \int_{\mathbf{R}} \delta(y)\varphi\left(\frac{y}{a}\right)\,\frac{dy}{a} = \frac{1}{a}\varphi(0)\,. \tag{257}$$

For a Dirac function in $\mathbf{R}^3$ we will have

$$\delta_{\mathbf{R}^3}(ax) = \delta(ax_1)\delta(ax_2)\delta(ax_3) = \frac{1}{a^3}\delta(x_1)\delta(x_2)\delta(x_3) = \frac{1}{a^3}\delta_{\mathbf{R}^3}(x)\,. \tag{258}$$

One can also just repeat a reasoning similar to (256) directly in $\mathbf{R}^3$.

An useful generalization of formula (255) is the following. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function with $f'(x) > 0$ and $b \in \text{range}(f)$. Let $x_b = f^{-1}(b)$, i. e. $f(x_b) = b$. Then

$$\delta(f(x) - b) = \frac{1}{f'(x_b)}\delta(x - x_b). \tag{259}$$

One way to see this is the following

$$\begin{aligned}\int_{\mathbf{R}} \delta(f(x) - b)\varphi(x)\,dx &= \int_{\text{range}f} \delta(y - b)\varphi(f^{-1}(y))\,d(f^{-1}(y)) \\ &= \varphi(f^{-1}(b))(f^{-1}(y))'|_{y=b} = \varphi(x_b)\frac{1}{f'(x_b)}\,,\end{aligned} \tag{260}$$

where we have used the formula $(f^{-1}(y))' = \frac{1}{f'(f^{-1}(y))}$ for the derivative of the inverse function.

Lecture 20, 3/4/2018

Fundamental solution of the wave equation in spatial dimension two

Let us now calculate the fundamental solution of the two-dimensional wave equation. We will derive it from the three-dimensional solution, by considering functions independent of x_3. (This method is sometimes called the method of descent.) We will calculate the solution of

$$u_{tt} - \Delta u = \delta(x_1)\delta(x_2)\delta(t)\,, \qquad (x,t) \in \mathbf{R}^3 \times \mathbf{R}\,, \tag{261}$$

specified by the condition that u vanishes for negative times.

For this calculation we note that

$$\int_{R^3} \varphi(x_1, x_2, x_3)\delta(x_1)\delta(x_2)\,dx = \int_{-\infty}^{\infty} \varphi(0, 0, x_3)\,dx_3\,. \tag{262}$$

Let us calculate u given by (261).

$$\begin{aligned}u(x,t) &= \int_{-\infty^t}\int_{\mathbf{R}^3} G(x - y, t - s)\delta(y_1)\delta(y_2)\delta(s)\,dy_1\,dy_2\,dy_3\,ds \\ &= \int_{-\infty}^{\infty} G(x_1, x_2, y_3, t)\,dy_3 = \int_{-\infty}^{\infty} \frac{\delta(\sqrt{x_1^2 + x_2^2 + y_3^2} - t)}{4\pi\sqrt{x_1^2 + x_2^2 + y_3^2}}\,dy_3 \\ &= \int_0^{\infty} \frac{\delta(\sqrt{x_1^2 + x_2^2 + y_3^2} - t)}{2\pi\sqrt{x_1^2 + x_2^2 + y_3^2}}\,dy_3\,.\end{aligned} \tag{263}$$

We now use formula (260) with $f(y_3) = \sqrt{x_1^2 + x_2^2 + y_3^2}$ and $b = t$. For $\sqrt{x_1^2 + x_2^2} < t$ have

$$\begin{aligned} (y_3)_t = f^{-1}(t) &= \sqrt{t^2 - x_1^2 - x_2^2}\,, \\ (f^{-1})'(t) &= \frac{t}{\sqrt{t^2 - x_1^2 - x_2^2}}\,, \\ \sqrt{x_1^2 + x_2^2 + (y_3)_t^2} &= t\,. \end{aligned} \tag{264}$$

From (260)and (264) we conclude that for $\sqrt{x_1^2 + x_2^2} < t$ we have

$$u(x,t) = \frac{1}{2\pi t} \frac{t}{\sqrt{t^2 - x_1^2 - x_2^2}} = \frac{1}{2\pi\sqrt{t^2 - x_1^2 - x_2^2}}\,. \tag{265}$$

For $\sqrt{x_1^2 + x_2^2} > t$ one sees from (263) that $u(x,t) = 0$. Hence the fundamental solution of the wave equation in dimension 2 is

$$G_2(x_1, x_2, t) = \begin{cases} \frac{1}{2\pi\sqrt{t^2 - x_1^2 - x_2^2}}\,, & \sqrt{x_1^2 + x_2^2} < t\,, \\ 0\,, & \sqrt{x_1^2 + x_2^2} > t\,. \end{cases} \tag{266}$$

A similar calculation also gives the fundamental solution of the wave equation is the spacial dimension one (which can be also obtained in a much simple way):

$$G_1(x_1, t) = \begin{cases} \frac{1}{2}\,, & |x_1| < t\,, \\ 0 & |x_1| > t\,. \end{cases} \tag{267}$$

It is interesting to compare these solutions with the one in dimension three,

$$G_3(x_1, x_2, x_3, t) = G(x_1, x_2, x_3, t) = \begin{cases} \frac{\delta(r-t)}{4\pi r}\,, & t > 0\,, \\ 0\,, & t < 0\,. \end{cases} \tag{268}$$

In dimnesion three the fundamental solution is concentrated on the "light cone" $\{t = |x|\}$, $(t > 0)$, whereas in dimensions two and one the fundamental solution does not vanish inside the light-cone. This means, for example, that a person speaking in lower dimension would hear a strong echo (the sound would would "linger on" in the light cone), and the sound a listener is hearing would be quite fuzzy. The propagation of waves (governed by the wave equation) in dimension three has the special property that the signal is propagated without distortion, and the signal leaves any bounded region in a finite time (assuming there are no obstacles). This is related to the Huygens' principle.

Lecture 20, 4/10/2019

Some calculations with fundamental solutions

In this lecture we continued with some calculations with fundamental solutions. For example, we considered the following problem: For $n \in \{1, 2, 3\}$ consider

$$\begin{aligned} u_t - \Delta u = \delta(x)\,, \qquad & x \in \mathbf{R}^n\,, t > 0\,, \\ u(x,0) = 0\,, \qquad & x \in \mathbf{R}^n\,, \end{aligned} \tag{269}$$

and calculate $u_\infty(x) = \lim_{t\to\infty} u(x,t)$. Let

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \tag{270}$$

be the fundamental solution of the heat equation in $\mathbf{R}^n$. Then $u(x,t)$ is given by

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbf{R}^3} \Gamma(x-y, t-s)\delta(y)\, dy\, ds \\ &= \int_0^t \Gamma(x, t-s)\, ds = \int_0^t \Gamma(x,s)\, ds\,. \end{aligned} \tag{271}$$

We recall that the integral $\int_1^\infty s^{-\alpha}\, ds$ is infinite for $\alpha \le 1$ and from this it is easy to see that

$$u_\infty(x) = \infty\,, \qquad n = 1, 2\,. \tag{272}$$

This is related to the fact that the equation $-\Delta u = \delta$ does not have non-negative solution in dimensions 1 and 2. In dimension $n = 3$ we have

$$u_\infty(x) = \int_0^\infty \Gamma(x,s)\, ds = \int_0^\infty \frac{1}{(4\pi s)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4s}}\, ds \tag{273}$$

One way to evaluate this integral is to use the substitution $\frac{|x|^2}{4s} = \sigma$. Then $s = \frac{|x|^2}{4\sigma}$, $ds = -\frac{|x|^2}{4\sigma^2}\, d\sigma$, and the integral becomes $\frac{1}{4(\pi)^{\frac{3}{2}}|x|} \int_0^\infty e^{-\sigma}\sigma^{-\frac{1}{2}}\, d\sigma$. The integral appearing in the last expression is known to be $\sqrt{\pi}$ from the theory of Euler's Gamma-function. Hence we conclude

$$u_\infty(x) = \frac{1}{4\pi|x|}\,, \qquad n = 3\,. \tag{274}$$

This is the fundamental solution of the Laplace equation, and a solution of the first equation of (269) when u is independent of t. We see that the solution of (269) approaches the fundamental solution of the Laplace equation, as one could heuristically expect.

This can be also seen from the following calculation:

$$\begin{aligned} \Delta_x \int_0^\infty \Gamma(x,s)\, ds &= \lim_{\tau\to 0_+} \int_\tau^\infty \Delta_x \Gamma(x,s)\, ds = \lim_{\tau\to 0_+} \int_\tau^\infty \Gamma_s(x,s)\, ds \\ &= \lim_{\tau\to 0_+} -\Gamma(x,\tau) = -\delta(x)\,. \end{aligned} \tag{275}$$

(The reason this calculation does not quite work in dimensions ≤ 2 is that the first integral is ∞ in those dimensions.)

Lecture 21, 4/12/2019

Symmetries of PDEs, the physical significance of the values of the coefficients

An important information about the phenomena PDEs are decribing comes from the symmetries of the equations. Understanding the symmetries can in many cases be crucial, and can sometimes be more important than calculations of solutions.

The heat equations $u_t = \Delta u$ offers a good example. It has a symmetry

$$u(x,t) \to u_\lambda(x,t) = u(\lambda x, \lambda^2 t)\,. \tag{276}$$

More precisely, let us assume assume that $u(x,t)$ solves the hear equation in $\Omega \times (t_1, t_2)$, with the boundary condition $u(x,t) = 0$ for $x \in \partial\Omega$ for all $t \in (t_1, t_2)$. Then the function u_λ is defined in the space-time domain $\lambda^{-1}\Omega \times (\lambda^{-2}t_1, \lambda^{-2}t_2)$ and solves the heat equation with the boundary condition $u_\lambda(x,t) = 0$ for $x \in \lambda^{-1}\partial\Omega$.

From this we see that a $1:10$ scale model of an object cools down 100 times faster (if it is dome using the same material), an information which may be more important than some precise calculation of the solution.

The coefficients appearing in PDE often have an important physical meaning, and can be often determined by observations of the solutions. For example, the coefficient c in the wave equation $u_{tt} = c^2\Delta u$ determines the velocity with which the waves described by the equation travel (and this can be sometimes measured directly, such as when in the case of the speed of sound). At the same time, from the derivation of the wave equation, the coefficient c^2 can be expressed from properties of the material in which the waves are observed, so measuring c can give us important information about the material. For example, for the vibration of a string, there is the relation

$$c = \sqrt{\frac{T}{\rho}} \tag{277}$$

where T is the tension in the string (= the force pulling on the string) and ρ is the (linear) density of the string.

In class we derived the relation

$$c^2 = \frac{dp(\rho)}{d\rho} \tag{278}$$

for the speed of sound in the air, where $p = p(\rho)$ is the dependence of the pressure on the density (assuming adiabatic compressions, i. e. no diffusion of the heat energy). This derivation can be found in many physic textbooks. See also the Wikipedia article on the speed of sound.

A well-known example concerns the speed of light. In the 1860s, J. C. Maxwell showed that his famous equations of electro-magnetism (which he obtained from earlier equations by adding an extra term) imply that the electric and magnetic fields should satisfy the wave equation in which the speed of propagation c is linked to the electric permittivity ε_0 and magnetic permeability μ_0 through the formula $c^2 = \frac{1}{\varepsilon_0\mu_0}$. The quantities ε_0 and μ_0 were known at the time, and the c obtained from the formula was in reasonable agreement with the speed of light (which was also known), giving strong support to his famous prediction that light should be identified with the electromagnetic waves.

The study of the symmetries of the Maxwell equations eventually lead to additional profound conclusions, including the special theory of relativity.

Lecture 22, 4/17/2019 (delivered by Dallas Albritton, who also kindly prepared the notes)

d'Alembert's paradox

Today, we discussed some topics in fluid mechanics. In particular, we considered the drag force F on a sphere of radius $R > 0$ moving uniformly with velocity $U \in \mathbf{R}^3$ in an incompressible fluid of constant mass density $\rho > 0$. In 1687, Newton concluded that, if the drag force F depends only on the quantities R, U, and ρ, then we must have

$$F = \text{const.} \times \rho R^2 U^2. \tag{279}$$

Indeed, the dimension of each quantity is

$$\begin{aligned} [R] &= L \\ [U] &= L/T \\ [\rho] &= M/L^3 \\ [F] &= ML/T^2. \end{aligned} \tag{280}$$

The dimensionally correct formula for F must contain factors ρ and U^2 in order to accomodate the dimensions M and $1/T^2$ in F. Finally, one completes the formula by filling in the right number of 'R's. Newton's formula works well for small velocities U, but we will revise it in a couple pages.

In 1752, d'Alembert investigated the drag force on a body $B \subset \mathbf{R}^3$ moving with uniform velocity in a potential flow. Let $\Omega = \mathbf{R}^3 \setminus \overline{B}$ be the region outside the body.[11] The motion of the fluid is modeled by the incompressible Euler equations

$$\left.\begin{aligned} \partial_t u + u \cdot \nabla u &= -\frac{1}{\rho} \nabla p \\ \operatorname{div} u &= 0 \end{aligned}\right\} \qquad \text{in } \Omega \tag{281}$$

$$u \cdot n = 0 \qquad \text{on } \partial\Omega \tag{282}$$

with the additional condition

$$u \to -U \qquad \text{as } |x| \to \infty. \tag{283}$$

We will normalize $\rho = 1$. In this frame of reference, we view the body B as "stationary" with the boundary condition (283). A *potential flow* is a solution of the Euler equations given by $u = \nabla h$, $p = -\frac{1}{2}|\nabla h|^2$. With this ansatz, the Euler equations become

$$\begin{aligned} \Delta h &= 0 && \text{in } \Omega \\ \frac{dh}{dn} &= 0 && \text{on } \partial\Omega \\ \nabla h &\to -U && \text{as } |x| \to \infty. \end{aligned}$$

Here, n is the outward unit normal of Ω, so n points "into" the body. Since h is harmonic, it is smooth, and the equations are satisfied in a classical sense.[12]

We let F denote the drag force on the body B. That is,

$$F = \int_{\partial\Omega} pn \, dx. \tag{284}$$

Proposition (d'Alembert's paradox). In the above notation, $F = 0$.

[11] B is assumed to be open and bounded with smooth boundary. We also assume $0 \in B$.

[12] Here, I introduced the notation $u_{i,j} = \frac{\partial u_i}{\partial x_j}$ with summation over repeated indices. For example, $(u \cdot \nabla u)_i = u_j u_{i,j}$.

Proof. [13]

If you let yourself freely integrate by parts, the "proof" is as follows:

$$F_i = \int_{\partial\Omega} pn_i \, dS = \int_\Omega p_{,i} \, dx = -\int_\Omega u_j u_{i,j} \, dx. \tag{285}$$

By the divergence-free condition, $u_j u_{i,j} = (u_i u_j)_{,j}$. This gives

$$F_i = -\int_\Omega (u_i u_j)_{,j} \, dx = -\int_{\partial\Omega} u_i u_j n_j \, dS = 0, \tag{286}$$

since $u_j n_j = u \cdot n = 0$ on $\partial\Omega$. However, this "proof" neglects the boundary condition as $|x| \to \infty$! Instead, we define[14]

$$T_{ij} = u_i u_j + p\delta_{ij}, \tag{287}$$

where δ_{ij} is the *Kronecker delta* symbol. The Euler equations may be rewritten as

$$\operatorname{div} T = 0, \text{ or } T_{ij,j} = 0 \text{ for all } 1 \le i \le 3. \tag{288}$$

For $R \gg 1$, we define $\Omega_R = B_R \setminus \overline{B}$. This is simply a truncated version of Ω. Then

$$0 = \int_{\Omega_R} T_{ij,j} \, dx = \int_{\partial B_R} T_{ij} n_j \, dS + \int_{\partial\Omega} T_{ij} n_j \, dS = \int_{\partial B_R} T_{ij} n_j \, dS + F_i, \tag{289}$$

where in the last equality we used that $\int_{\partial\Omega} u_i u_j n_j \, dS = 0$. If we can demonstrate that

$$\lim_{R\to\infty} \int_{\partial\Omega_R} T_{ij} n_j \, dS = 0, \tag{290}$$

then we will have $F = 0$, as desired. Therefore, our new goal is to examine the limit as $R \to \infty$.

So far, we have not used that $u = \nabla h$ is a potential flow. Recall that $\Delta h = 0$ and $\nabla h \to -U$ as $|x| \to \infty$. We will now use an expansion[15] for h when $|x| \gg 1$:

$$h = -U \cdot x + \text{const.} + \frac{a_0}{|x|} + a_j \frac{x_j}{|x|^3} + O(R^{-3}). \tag{292}$$

Here, $O(R^{-3})$ denotes a term that is bounded by const. $\times R^{-3}$ for $R \gg 1$ with constant independent of R. Taking the gradient gives

$$u = \nabla h = -U - a_0 \frac{x}{|x|^3} + O(R^{-3}). \tag{293}$$

[13] This proof is a combination of the proof in Landau-Lifschitz and one in Professor Šverák's mathematical fluid mechanics notes.

[14] the *energy-momentum tensor*

[15] The general idea of the expansion is as follows. Recall the fundamental solution

$$\Gamma = -\frac{1}{4\pi|x|}, \quad \Delta\Gamma = \delta. \tag{291}$$

Notice that Γ *and all its derivatives* satisfy Laplace's equation outside of a small ball B_ϵ. One might hope that such functions form a basis for the harmonic functions away from the origin which are $O(|x|^{-1})$ at infinity. Indeed, we are expanding h in terms of Γ and its derivatives (except for the terms $U \cdot x$ and const., which are also easily seen to be harmonic). By the Kelvin transformation, this expansion at infinity corresponds to an expansion in terms of *harmonic polynomials* for a harmonic function at the origin, and this can be made into a proof.

One can also use the above expansion to find an exact potential flow when B is a ball.

In fact, we may obtain $a_0 = 0$ by using that $\operatorname{div} u = 0$ in Ω_R and $n(x) = x/|x|$ on ∂B_R:

$$\int_{\Omega_R} \operatorname{div} u \, dx = \int_{\partial B_R} u \cdot n \, dS =$$
$$= \underbrace{\int_{\partial B_R} -U \cdot \frac{x}{|x|} \, dS}_{=0} + \underbrace{\int_{\partial B_R} -a_0 \frac{x}{|x|^3} \cdot \frac{x}{|x|} \, dS}_{=4\pi a_0} + \underbrace{\int_{\partial B_R} O(R^{-3}) \, dS}_{=O(R^{-1})}. \tag{294}$$

Taking $R \to \infty$ gives that $4\pi a_0 = 0$. Hence, we have shown that

$$u = \nabla h = -U + O(R^{-3}). \tag{295}$$

To conclude the proof, we must substitute the expansion for u back into (290). Since the surface area of ∂B_R is $4\pi R^2$, we will only care about terms which are $O(R^{-2})$ in the integrand. For $R \gg 1$,

$$\int_{\partial B_R} T_{ij} n_j \, dS = \int_{\partial B_R} (u_i u_j - \frac{1}{2}|u|^2 \delta_{ij}) n_j \, dS =$$
$$= \underbrace{\int_{\partial B_R} (U_i U_j - \frac{1}{2}|U|^2 \delta_{ij}) \frac{x_j}{|x|} dS}_{=0} + O(R^{-1}). \tag{296}$$

Let $R \to \infty$ to complete the proof. □

D'Alembert's paradox shows that potential flows are not sufficient to realistically describe a body moving uniformly with constant velocity. In reality, all fluids[16] have some viscosity, which cannot be ignored. The (kinematic) *viscosity* $\nu > 0$ has units

$$[\nu] = L^2/T. \tag{297}$$

We define a dimensionless quantity $\mathrm{Re} = RU/\nu$, called the *Reynolds number*, which enters into the formula for the drag force via the relation

$$F = \frac{1}{2} c_d(\mathrm{Re}) \times \rho R^2 U^2. \tag{298}$$

Here, $c_d > 0$ is the *drag coefficient*, which depends only on the Reynolds number. Compare with Newton's formula from 1687!

Near the end of class, we talked briefly about Stokes' contribution to the story, watched some videos of flows around a sphere at various Reynolds numbers, and discussed some computational hazards associated with stable and unstable fluid flows.

Lecture 23, 4/19/2018

More on symmetries of PDEs

In some cases the solutions of PDEs are almost completely determined by symmetries. Let us consider a few examples.

[16] Not superfluids!

Example 1: *Fundamental solution of the Laplace equation in dimensions $n \geq 3$.*

Let us consider the (already familiar) problem

$$\begin{aligned} -\Delta u &= \delta(x) \qquad \text{in } \mathbf{R}^n\,, \\ u(x) &\to 0\,, \qquad x \to \infty\,. \end{aligned} \tag{299}$$

It is clear that the solution of the problem, if it exists, is unique: if we have two solutions u_1, u_2, their difference $v = u_2 - u_1$ satisfies $-\Delta v = 0$ in $\mathbf{R}^n$ with $v(x) \to 0$ as $x \to \infty$. Due to the maximum principle for harmonic functions we see that $v = 0$ in $\mathbf{R}^n$.

Let now Q be an orthogonal $n \times n$ matrix. As the mapping $x \to Qx$ is volume preserving, we have $\delta(Qx) = \delta(x)$. We also have

$$\Delta(u(Qx)) = (\Delta u)(Qx)\,. \tag{300}$$

In other words, if $\Delta u(x) = f$, $u_Q(x) = u(Qx)$ and $f_Q(x) = f(Qx)$, then $\Delta u_Q(x) = f(Qx)$.

If u is a solution of (299) and Q is as above, then, by the above considerations, u_Q is again a solution of (299) and by uniqueness we have $u_Q = u$. Since Q is an arbitrary orthogonal matrix, we conclude that u depends only on $r = |x|$. We will write, with a slight abuse of notation, $u = u(r)$.

There is an additional symmetry which, together with the orthogonal symmetries, helps to determine the function u up to a constant: For $\lambda > 0$ set $u_\lambda(x) = \lambda^{n-2} u(\lambda x)$. Then

$$-\Delta u_\lambda(x) = \lambda^{n-2}(\Delta(u(\lambda x))) = \lambda^{n-2}\lambda^2(\Delta u)(\lambda x) = \lambda^n \delta(\lambda x) = \delta(x)\,. \tag{301}$$

By uniqueness we conclude $u_\lambda(x) = u(x)$, or, equivalently $u(\lambda x) = \lambda^{-n+2} u(x)$. Together with the fact that $u = u(r)$, this implies that

$$u(x) = \frac{c_n}{r^{n-2}}\,, \qquad n \geq 3. \tag{302}$$

The method also shows that there are no solutions of which would satisfy both conditions of (299). We see that the symmetries (and the uniqueness) determine the solution up to the constant c_n. The constant can be determined from the conditon $\int_{\mathbf{R}^n} -\Delta u(x)\,dx = \int_{\mathbf{R}^n} \delta(x)\,dx = 1$. Letting $B_R = \{x \in \mathbf{R}^n\,,\ |x| < R\}$, we have

$$1 = \int_{B_R} -\Delta u(x)\,dx = \int_{\partial B_R} -\frac{\partial u}{\partial n} = \int_{\partial B_R} -\frac{\partial u(r)}{\partial r} = \frac{(n-2)c_n||}{R^{n-1}}\,|\partial B_R|, \tag{303}$$

where $|\partial B_R|$ is the $(n-1)-$dimensional area of $|B_R|$. Taking $R = 1$, we obtain

$$c_n = \frac{1}{(n-2)|\partial B_R|}\,. \tag{304}$$

In particular, when $n = 3$, we have

$$u(x) = \frac{1}{4\pi|x|}\,. \tag{305}$$

When $n \leq 2$, the first equation of (299) still has a solution, of course, which for $n = 2$ is given for example by $u(x) = \frac{1}{2\pi}\log(\frac{R}{|x|})$ (where $R > 0$ is a parameter).

Example 2: *Fundamental solution of the wave equations in space-time dimensions* $n+1 \leq 3$.

We now apply the same method to the wave equation. Our space-time coordinates are (t, x_1, x_2) and we will use the notation $x_0 = t$, so that we can write the coordinates as $x \sim (x_0, x_1, x_2)$ in the space-time dimension $n+1 = 3$ and $x \sim (x_0, x_1)$ in the space-time dimension $n+1 = 2$. We will use the notation

$$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\,, \qquad n+1 = 3\,, \tag{306}$$

and similarly in other dimensions.

Let us now consider

$$\Box u = \delta(x)\,. \tag{307}$$

If we do not impose additional conditions, this problem has many solutions, which might be an issue when we wish to show that the solution has to be symmetric. However, we can remedy this by demanding that the solution vanishes outside of the "positive light cone" $C_+ = \{x \in \mathbf{R}^{n+1}\,,\, x_0 \geq \sqrt{x_1^2 + \cdots + x_n^2}\}$. This vanishing condition together with the equation (307) determine the solution uniquely. We can now apply practically the same method as we used for the Laplace equation, except that we have to replace the rotations by another set of transformations, the so-called *Lorentz transformations*. These can be defined as follows: Let us consider the quadratic form

$$q(x) = x_0^2 - x_1^2 \cdots - x_n^2\,. \tag{308}$$

Let G be the subset of $(n+1) \times (n+1)$ matrices L which preserve the positive light-cone C_+ (in the sense that $L(C_+) \subset C_+$) and also the quadratic for q, in the sense that

$$q(Lx) = q(x)\,,\, x \in \mathbf{R}^{n+1}\,. \tag{309}$$

It is instructive to consider the simplest case $n = 1$. In this case one can check that the matrices L (which, we recall, are 2×2 matrices when $n = 1$) are given by

$$L = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}\,, \tag{310}$$

where $\alpha \in \mathbf{R}$ is a parameter. Then

$$Lx = L \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_0 \cosh\alpha + x_1 \sinh\alpha \\ x_0 \sinh\alpha + x_1 \cosh\alpha \end{pmatrix}\,. \tag{311}$$

Note that for any two vectors $x, y \in \mathbf{R}^{1+1}$ which satisfy $x_0^2 - x_1^2 = y_0^2 - y_1^2 > 0$ and $x_0, y_0 > 0$ one can find α such that $Lx = y$. A similar statement is true in higher dimensions: if $x_0^2 - x_1^2 \cdots - x_n^2 = y_0^2 - y_1^2 \cdots - y_n^2 > 0$, then there exists L preserving (308) and the light cone C_+ such that $Lx = y$. The proof of this statement can be reduced to the case $n = 1$ by first using suitable rotations in the spatial variables $x_1, \ldots, x_n$, and then transformations (310).

A standard application of the chain rule shows that for L as above we have

$$\Box(u(Lx)) = (\Box u)(Lx)\,, \tag{312}$$

similarly to (300). Note

$$\delta(Lx) = \delta(x)\,, \tag{313}$$

for every L of the form (310), as $\det L = 1$. If u is a solution of (307) which vanishes outside of C, then $u(Lx) = u(x)$. This implies that $u(x) = u(\rho)$, where $\rho = \sqrt{x_0^2 - x_1^2 \cdots - x_n^2}$ when $x \in C$.

We now set $u_\lambda(x) = \lambda^{n-1} u(\lambda x)$ (where we assume that u solves (307) and u vanishes outside of the light-cone C). Then, similarly to the case with the Laplacian, one concludes that $u_\lambda(x) = u(x)$. Together with $u(Lx) = u(x)$, this implies that

$$u(x) = \frac{c_n}{\rho^{n-1}}\,, \qquad n = 1, 2\,. \tag{314}$$

The constant c_n can be determined from the condition

$$\int_{R^n} u(x,t)\, dx = t\,, \qquad t > 0\,. \tag{315}$$

Recalling that $t = x_0$, we see that we should have

$$\int_{B_t} \frac{c_n}{\rho^{n-1}}\, dx_1 \dots dx_n \, = t\,. \qquad n \le 2\,. \tag{316}$$

It is not hard to see that this implies $c_1 = \frac{1}{2}$ in the case $n = 1$. The calculation for $n = 2$ is can also be done quite easily:

$$\int_{B_t} \frac{1}{\sqrt{t^2 - x_1^2 - x_2^2}}\, dx_1\, dx_2 = \int_0^t \frac{2\pi r}{\sqrt{t^2 - r^2}}\, dr = 2\pi \int_0^t -\frac{\partial}{\partial r}\sqrt{t^2 - r^2}\, dr = 2\pi t\,, \tag{317}$$

which implies that we should take $c_2 = \frac{1}{2\pi}$. Hence we reproduced the result of the calculation in Lecture 20 just by using the symmetries of the equation.

One can also calculate by (a modification of) this method the fundamental solution in the space-time dimension 3+1, although that is somewhat less straightforward. Let us briefly explain what happens in that case, without going to details. Formula (314) suggests that we for $n = 3$ should take $u(x) = \frac{c_3}{\rho^2}$, but the difficulty is that $\int_{B_t} \frac{1}{\rho^2}\, dx_1 dx_2 dx_3 = +\infty$. One way to overcome this difficulty is replace $\frac{c_3}{\rho^2}$ by $\frac{c_{3-\varepsilon}}{\rho^{2-\varepsilon}}$ for a small $\varepsilon > 0$, determine $c_{3-\varepsilon}$ from the condition (316), and then take $\varepsilon \to 0_+$. This procedure recovers the 3d fundamental solution which we have calculated before. The calculation is more difficult than the previous examples, but still doable via basic calculus methods, without any advanced techniques. As an (optional and non-trivial) exercise you can try to complete that calculation.

Example 3: *The heat equation* We wish to derive formula for the solution of

$$u_t - \Delta u = \delta(x,t) = \delta(x)\delta(t)\,, \tag{318}$$

(which we derived previously via the Fourier transformation) using symmetries. The symmetries of the solutions we will use are the following

(i) $u(Qx, t) = u(x, t)$ for each orthogonal matrix Q.

(ii) $\lambda^n u(\lambda x, \lambda^2 t) = u(x, t)\,.$

The condition (i) and (ii) imply that

$$u(x,t) = \frac{c_n}{t^{\frac{n}{2}}}\, F\left(\frac{r}{\sqrt{t}}\right)\,. \tag{319}$$

In this case the group of symmetries which we have used is not as rich as in the previous calculations, and our formula (319) still contains an unknown function $F(\xi)$, although only of one variable, which is, of course, a significant simplification. The function F can be determined in several ways, the most straightforward one being a simple substitution of (319) into the heat equation $u_t = \Delta u$ which has to be satisfied by u for $t > 0$. This gives

$$-\frac{n}{2}F - \frac{1}{2}\xi F' = F'' + \frac{n-1}{\xi}F' \,, \tag{320}$$

where we think of F as a function of ξ, with $\xi = \frac{r}{\sqrt{t}}$, and we use the notation $F' = \frac{dF}{d\xi}$. We note that (320) can be written as

$$(F' + \frac{\xi}{2}F)' + \frac{n-1}{\xi}(F' + \frac{\xi}{2}F) = 0 \,, \tag{321}$$

from which we see a solution

$$F(\xi) = e^{-\frac{\xi^2}{4}} \,, \tag{322}$$

which gives the fundamental solution. Equation (320) is of the second order, and therefore it must have other solutions than (322). However, the solutions of the heat equation which the other solutions will generate do not satisfy $u(x,t) \to \delta(x)$ as $t \to 0_+$. The function F given by (322) give the usual fundamental solution of the hear equation via (319), after we normalize it to $F(\xi) = \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{\xi^2}{4}}$, so that $\int_{\mathbf{R}^n} F(\xi)\, dx = 1$.

Lecture 24, 4/24/2018

Dimensional analysis, non-dimensionalisation

Many physical laws expressed as a mathematical equation have a symmetry in it which is somewhat hidden - namely the independence on units. Consider for example formula (277) for the speed of propagation of waves in a string. At the first glance the formula may look somewhat peculiar. Let us change notation and denote the force pulling on the string F, and keep the notation ρ for the (linear) density of the string. In this notation the formula is

$$c = \sqrt{\frac{F}{\rho}} \,. \tag{323}$$

Based on our physical intuition we might agree that c should be increasing with F and decreasing with ρ (although even this may not be an automatic guess if one does not have some experience in mechanics), but how does the square root appear? The square root is inevitable from the point of view of dimensional analysis.

The key point is that the formula should be true regardless of a particular system of units units one uses, as long as the units are used consistently. This may look obvious, but it actually imposes very strong requirements on the formula, as we now show.

Let L be a unit of length, M be a unit of mass, and T be a unit of time. (We emphasize that the force is now denoted by F, not T as in (277), so that using T for the unit of time is ok.) The units of velocity is than L/T, the units of linear density is M/L and the unit of force is ML/T^2. Assume that we have

$$c = \phi(F, \rho). \tag{324}$$

This really means the following: let us assume that in some particular choice of units the force is given by F units of force, the density is given by ρ units of density, and the velocity is given by c units of velocity. Then one has the relation (324) between the numbers c, F, ρ. Assume now that somebody uses a different set of units of length, time, and mass, let us call them L', T', M'. In these using the situation is described by different numbers c', F', ρ'. The assumption now is that the numbers c', F', ρ' must satisfy the same formula:

$$c' = \phi(F', \rho')\,, \tag{325}$$

where ϕ is *the same* as in (324). This clearly puts very strong requirements on the function ϕ. To express these requirements explicitly let us assume $L = \lambda L'$, $M = \mu M'$ and $T = \tau T'$, where λ, μ, τ are positive numbers. Then $c' = \frac{\lambda}{\tau} c$, $F' = \frac{\mu\lambda}{\tau^2} F$ and $\rho' = \frac{\mu}{\lambda}\rho$. By (325),

$$\frac{\lambda}{\tau} c = \phi(\frac{\mu\lambda}{\tau^2} F, \frac{\mu}{\lambda}\rho) \tag{326}$$

and by (325)

$$c = \phi(F, \rho). \tag{327}$$

Hence

$$\frac{\lambda}{\tau}\phi(F, \rho) = \phi(\frac{\mu\lambda}{\tau^2} F, \frac{\mu}{\lambda}\rho)\,, \tag{328}$$

which is the same as

$$\phi(F, \rho) = \frac{\tau}{\lambda}\phi(\frac{\mu\lambda}{\tau^2} F, \frac{\mu}{\lambda}\rho)\,. \tag{329}$$

This should be true for any $\lambda, \mu, \tau > 0$. This is a very strong requirement on the function ϕ. In fact, setting $\mu = 1, \lambda = \rho, \tau^2 = \rho F$, we obtain

$$\phi(F, \rho) = \sqrt{\frac{F}{\rho}}\ \phi(1, 1)\,. \tag{330}$$

We see that the requirement that formula (324) be independent of the units (which can be thought of as a sort of symmetry requirement) determines ϕ up to a constant, and the square root in the formula is an inevitable consequence of the formula being unit-independent. Every formula in physics can (and should) be analysed in this way.

Let us consider a PDE from this point of view. We will use the heat equation as an example. Assume that a heat equation

$$u_t = \kappa u_{xx} \tag{331}$$

is to be solved to be on an interval $(0, a)$ and a time interval $(0, b)$. Let us look at how a change of units will change our equation and the relevant parameters. Set $t = Tt'$ and $x = Lx'$, where we think about x', t' as dimension-less numbers, which just give the number of units of time contained in the time period t, and, similarly, the number of the units of length L contains in the segment of length x. We can also write $u = Uu'$, where U is a unit of temperature and u' is the number of the units of temperature U described by the temperature u. We note that

$$\frac{\partial}{\partial t} = \frac{\partial}{T\partial t'}\,, \qquad \frac{\partial}{\partial x} = \frac{\partial}{L\partial x'}. \tag{332}$$

The equation for u' in the coordinates x', t' becomes

$$\frac{U\partial u'}{T\partial t'} = \kappa\frac{U\partial^2 u'}{L^2\partial x'^2}\,, \tag{333}$$

which is the same as

$$\frac{\partial u'}{\partial t'} = \kappa \frac{T}{L^2} \frac{\partial^2 u'}{\partial x'^2}. \tag{334}$$

All quantities with primes are just dimension-less numbers, and hence the quantity $\kappa \frac{T}{L^2}$ also must be dimension-less. When we have two situations described by the heat equation above with parameters L_1, T_1, κ_1 and L_2, T_2, κ_2, respectively, the two situations can be turned to one another by a simple change of coordinates if $\kappa_1 \frac{T_1}{L_1^2} = \kappa_2 \frac{T_2}{L_2^2}$. This is important when we wish to use measurement on a scale model (possibly with a different coefficient κ) to make conclusions about the actual situation.

Lecture 25, 4/26/2018

well-posedness, stability

The initial-value problem

Let us consider the initial value problem for the wave equation

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(x,0) &= u_0(x) \\ u_t(x,0) &= u_1(x) \end{aligned} \tag{335}$$

where $u(x,t)$ is a function 1-periodic in x.

Expressing $u(x,t)$ through its Fourier series

$$u(x,t) = \sum_k \hat{u}(k,t) e^{2\pi i k x}\,, \tag{336}$$

we get the following ODEs for the coefficients $\hat{u}(k,t)$:

$$\begin{aligned} \frac{d^2\hat{u}(k,t)}{dt^2} + (2\pi k)^2 \hat{u}(k,t) &= 0 \\ \hat{u}(0) &= \hat{u}_0(k) \\ \frac{d\hat{u}}{dt}(0) &= \hat{u}_1(k)\,, \end{aligned} \tag{337}$$

where $\hat{u}_0(k)$ and $\hat{u}_1(k)$ are the coefficients for the initial data. We know how to solve this ODE:

$$\hat{u}(k,t) = \hat{u}_0(k) \cos 2\pi k t + \hat{u}_1(k) \frac{\sin 2\pi k t}{2\pi k}\,. \tag{338}$$

This formula determined $\hat{u}(k,t)$ is a nice way from $\hat{u}_0(k)$ and $\hat{u}_1(k)$. Note that the functions $\cos 2\pi k t$ and $\sin 2\pi k t$ only take values between -1 and 1, so there are no "dangerous" large number in the formula (338).

What happens if we try to think about the equation $u_{yy} = -u_{xx}$ in the same way? Let us write again

$$u(x,y) = \sum_k \hat{u}(k,y) e^{2\pi i k x}. \tag{339}$$

This time we obtain the following ODE for the coefficients:

$$\begin{aligned} \frac{d^2\hat{u}(k,y)}{dy^2} - (2\pi k)^2\hat{u}(k,y) &= 0 \\ \hat{u}(k,0) &= \hat{u}_0(k) \\ \frac{d\hat{u}(k,0)}{dy} &= \hat{u}_1(k)\,. \end{aligned} \tag{340}$$

The solution can again be written down explicitly

$$\hat{u}(k,y) = \hat{u}_0(y)\cosh 2\pi k y + \hat{u}_1(y)\frac{\sinh 2\pi k y}{2\pi k}\,. \tag{341}$$

There is a serious danger in expressions (341): for positive $y > 0$ the expressions $\hat{u}(k,y)$ may grow exponentially in y and the series (339) may fail to converge to any function for $y > 0$. Also, even when it converges, the mapping $\hat{u}(k,0) \to \hat{u}(k,y)$ becomes extremely sensitive to the values of $\hat{u}_0(k)$ and $\hat{u}_1(k)$ once k is not small. For example for $k = 3$ and $y = 10$ the value of $\cosh 2\pi k y$ is of order 10^{81}, so even very small errors the coefficients $\hat{u}(k,0)$ will be enormously amplified. We can say that the determination of $\hat{u}(k,y)$ from $\hat{u}(k,0)$ is increasingly unstable as k grows, and at the level of the whole function $u(x,y)$ the "initial conditions" $u(x,0)$ and $u_y(x,0)$ may not even define a $u(x,y)$ as a function. In this situation we can say that the problem of determining $u(x,y)$ based on the equation $u_{xx} = -u_{yy}$ and the initial conditions $u(x,0)$ and $u_y(x,0)$ is *ill-posed*.

Note that for (338) these difficulties do not arise. The determination of $\hat{u}(k,t)$ from $\hat{u}(k)$ is "stable" and we say that the problem (335) is *well-posed*.

The boundary-value problem

Let us now change the problem for the Laplace equation $u_{xx} = -u_{yy}$ as follws: instead for determining u based on $u(x,0)$ and $u_y(x,0)$, let us try to determine it based on $u(x,0)$ and $u(x,b)$ for some $b > 0$:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x,0) &= u_0(x) \\ u(x,b) &= u_b(x)\,, \end{aligned} \tag{342}$$

where u is $1-$periodic in x. We apply a similar procedure. We write the general solution of the ODE in (340) as

$$\hat{u}(k,y) = A(k)\cosh 2\pi k y + B(k)\sinh 2\pi k y\,, \tag{343}$$

and determine $A(k), B(k)$ from the condition $u(x,0) = u_0(x)$ and $u(x,b) = u_b(x)$. After some calculation we obtain

$$\hat{u}(k,y) = \hat{u}_0(k)\frac{\sinh 2\pi k(b-y)}{\sinh 2\pi k b} + \hat{u}_b(k)\frac{\sinh 2\pi k y}{\sinh 2\pi k b}\,. \tag{344}$$

We note that the functions $\frac{\sinh 2\pi k(b-y)}{\sinh 2\pi k b}$ and $\frac{\sinh 2\pi k y}{\sinh 2\pi k b}$ both take values only between 0 and 1 when $y \in [0,b]$, and therefore the determination of $\hat{u}(k,y)$ from $\hat{u}_0(k)$ and $\hat{u}_b(k)$ does not involve any dangerous operations which would make it unstable.

If we replace the Laplace in (342) with the wave equation $u_{tt} - u_{yy} = 0$ and repeat the procedure proceeding formally with the claculation, we obtain instead of (344) the formula

$$\hat{u}(k,t) = \hat{u}_0(k)\frac{\sin 2\pi k(b-t)}{\sin 2\pi kb} + \hat{u}_b(k)\frac{\sin 2\pi kt}{\sin 2\pi kb}\,. \tag{345}$$

This formula has results in serious problem when $\sin 2\pi kb$ is very small, or even vanishes. In the latter case our problem simply may not have a solution. Even when $\sin 2\pi kb$ does not vanish for any $k \in \mathbf{Z}$, it can get very small for many k, and the determination of $\hat{u}(k,t)$ from u_0 and u_b will be unstable. We conclude that the boundary condition in (342) are not good for the wave equation, while the boundary conditions in (335) are not good for the Laplace equation.