1. On the final exam there will be 6 problems, any 4 of them will count as 100%. You can use any books or notes you like, as well as a calculator. However, no electronic devices with wireless communication capabilities will be allowed.

2. The problems on the exam will be similar to the problems which appeared during the semester in the two midterms, the two posted practice tests, the three homework assignments, and the two problems below.

Problem 1

Let f be a smooth function on the three-dimensional space $\mathbf{R}^3$ which vanishes outside the unit ball $B_1 = \{x \in \mathbf{R}^3\,,\, |x| < 1\}$, let u_0 be a smooth function in $\mathbf{R}^3$ which also vanishes outside of B_1, and let c, ρ, K be strictly positive numbers. Assume $u(x,t)$ solves the problem

$$\begin{aligned} c\rho u_t &= K\Delta u + f\,, \qquad && \text{in } \mathbf{R}^3 \times (0,\infty)\,, \\ u(x,0) &= u_0(x)\,, && x \in \mathbf{R}^3\,, \end{aligned} \tag{1}$$

and that

$$\lim_{t\to\infty} u(x,t) = u_\infty(x)\,. \tag{2}$$

(i) How does the limit u_∞ change if we change f to $\tilde{f} \equiv 0$, while leaving all the other parameters the same?

(ii) How does the limit u_∞ change of we replace u_0 by $\tilde{u}_0 \equiv 0$, while leaving all the other parameters the same?

(iii) How does the limit u_∞ change if we change K to $2K$ while leaving all the other parameters the same?

(iv) How does the limit u_∞ change if we change c to $2c$ and ρ to 2ρ while leaving all the other parameters the same?

Problem 2

Consider the heat equation

$$u_t = \kappa\Delta u \tag{3}$$

in a 3-dimensional box $\Omega = [0,L]^3$, with the boundary condition $u|_{\partial\Omega} = 0$. Assume all temperatures are non-negative and that L and κ are strictly positive.

(i) If one measures time in seconds and length in meters, what should one take as the unit of κ?

(ii) Let us define the *cooling time* of the box as the minimal time $T \geq 0$ such that

$$\max_{x\in\Omega} u(x,T) \leq 0.1 \max_{x\in\Omega} u(x,0)$$

for any non-negative solution of (3). Assuming that $T = \phi(L,\kappa)$ for some function $\phi = \phi(L,\kappa)$ of two variables and that $\phi(L_0,\kappa_0) = T_0$, where L_0, κ_0, T_0 are known, determine the function ϕ.