Do at least four of the following six problems.[1]

1. sl Assume that $\gamma \geq -1$ is a real number. Find a formula for the solution $u(x,t)$ of the following problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} - \gamma u\,, \qquad x \in (0,\pi) \\ u(0,t) &= 0\,, \\ u(\pi,t) &= 0\,, \\ u(x,0) &= \sin x + 0.1\sin 2x\,, \\ \frac{\partial u(x,0)}{\partial t} &= 0\,. \end{aligned}$$

Solution: We seek the solution as $u(x,t) = \sum_k B_k(t)\sin(kx)$. Substituting this expression into the equation, we obtain $\ddot{B}_k = -(k^2+\gamma)B_k$. We need to solve this with the initial conditions for $B_1, B_2, \ldots$ for each k given by the Fourier series of $u(x,0)$ and $\frac{\partial u(x,0)}{\partial t}$. We have $B_1(0) = 1, \dot{B}_1(0) = 0\,,\ B_2(0) = 0.1\,,\ \dot{B}_2(0) = 0$ and $B_k(0) = 0\,,\ \dot{B}_k(0) = 0$ for $k \geq 3$. We obtain $B_1(t) = \sin(\sqrt{1+\gamma}\ t)\,,\ B_2(t) = 0.1\sin(\sqrt{4+\gamma}\ t)$, and the rest of the coefficients vanish. Hence $u(x,t) = \sin x\ \sin(\sqrt{1+\gamma}\,t) + 0.1\sin 2x\ \sin(\sqrt{4+\gamma}\,t)$.

2. Consider the heat equation for an inhomogeneous rod

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right)\,, \qquad x \in (0,L)\,, \tag{1}$$

with the boundary conditions at the endpoints given by

$$u(0,t) = u(L,t) = 0\,. \tag{2}$$

We assume that $c(x), \rho(x)$ and $K_0(x)$ are smooth and strictly positive functions on the closed interval $[0,L]$. Let $\phi_1(x), \phi_2(x), \ldots$ be the eigenfunctions of the Sturm-Liouville problem $-\frac{d}{dx}\left(K_0(x)\frac{d\phi(x)}{dx}\right) = \lambda c(x)\rho(x)\phi(x)\,,\ \phi(0) = \phi(L) = 0$, with the corresponding eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots$. Find a formula for the solution of (1) with the boundary conditions (2) satisfying the initial condition

$$u(x,0) \quad = \quad a_1\phi_1(x) + a_2\phi_2(x) + \cdots + a_m\phi_m(x)\,,$$

where $m > 0$ is an integer and $a_1, a_2, \ldots, a_m$ are given real numbers.

Solution: We seek the solution as $u(x,t) = c_1(t)\phi_1(x) + \ldots c_m(t)\phi_m(x)$. Substituting this into the equation and using that ϕ_k are eigenfunctions of our Sturm-Liouville problem, we see that the equation for c_k is $\dot{c}_k = -\lambda_k c_k$. The general solution for a given k is $C_k e^{-\lambda_k t}$ and from the expression for $u(x,0)$ we see that $C_k = a_k$ for $k \leq m$ and $C_k = 0$ for $k > m$. This gives $u(x,t) = a_1\phi_1(x)e^{-\lambda_1 t} + a_2\phi_2(x)e^{-\lambda_2 t} + \cdots + a_m\phi_m(x)e^{-\lambda_m t}$.

3. Let us consider two inhomogeneous strings, both fixed at the ends and under tension. Both strings have length L, and their density is given by the same function $\rho(x)$.[2] The tension in first string is T_1 and we know from a measurement that the lowest possible frequency of its oscillations (under tension T_1) is ω_1. We do not know the tension T_2 in the second string, but we know from a measurement that the lowest possible frequency of oscillations of the string is ω_2. Express the unknown tension T_2 in terms of the quantities T_1, ω_1 and ω_2.

Solution: The equation of motion for the strings is $\rho(x)\frac{\partial^2 u_j(x,t)}{\partial t^2} = T_j\frac{\partial^2 u_j(x,t)}{\partial x^2}$ for $x \in (0,L)$ and $t \in \mathbf{R}$, with $u_j(0,t) = 0$ and $u_j(L,t) = 0$. We note that the transformation $u_2(x,t) = u_1(x,\kappa t)$ maps the solutions of string with tention T_1 onto the

[1] For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more. You can use the textbook, any notes, and a calculator, as long as it does not have wireless capabilities. Devices with wireless communication capabilities are not allowed. Hints for solutions such as the ones above might not be included on the real exam.

[2] Here we interpret density as mass per unit length.

solutions of the string with tension $\kappa^2 T_1$. The condition $\kappa^2 T_1 = T_2$ then can be used to conclude that in our case we should take $\kappa = \sqrt{\frac{T_2}{T_1}}$. With the change of variables above, the lowest frequency ω_1 of the first string corresponds to the frequency $\kappa\omega_1$ of the second string, but this quantity is known to be ω_2 from a measurement. Hence $\kappa = \frac{\omega_2}{\omega_1}$. Comparing the two expressions for κ, we obtain $T_2 = T_1 \frac{\omega_2^2}{\omega_1^2}$.
Alteratively, one could use dimensional analysis, arguing for example in the following way. The dimensionally correct expression for ω_2 must be of the form $T_2 = T_1\, f\left(\frac{\omega_2}{\omega_1}\right)$ for some function f, because one needs to keep the units of mass and length in T_1, T_2 intact to keep the right dimensionality. When ρ is constant, we have an explicit formula $\omega = \frac{1}{L}\sqrt{\frac{T}{\rho}}$ and in this special case we conclude $f(\xi) = \xi^2$. Therefore there is no other choice in the general case than $T_2 = T_1 \frac{\omega_2^2}{\omega_1^2}$.

4. Let γ, ω be strictly positive numbers satisfying $\gamma < 2\omega$. Let us write the second order differential equation (describing damped oscillations) $\ddot{x} = -\gamma\dot{x} - \omega^2 x$ for a scalar function $x = x(t)$ (where $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$) as a first-order system

$$\begin{aligned} \dot{x} &= y\ , \\ \dot{y} &= -\gamma y - \omega^2 x\,, \end{aligned} \tag{3}$$

and consider the following numerical scheme (forward Euler method) for system (3):

$$\begin{aligned} \tfrac{x(t+\tau)-x(t)}{\tau} &= y(t)\ , \\ \tfrac{y(t+\tau)-y(t)}{\tau} &= -\gamma y(t) - \omega^2 x(t)\,. \end{aligned} \tag{4}$$

Here $\tau > 0$ is the time-step (and we think of it as a small number), and the scheme is used to start from $t = 0$ and successively evaluate the solution at the times $t = \tau, 2\tau, 3\tau, \dots$.
(a) Show that under our assumptions any solution $x(t)$ of the original equation satisfies $\lim_{t\to\infty} x(t) = 0$. (Hint: The general solution of the equation is $A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$ where λ_1, λ_2 are the roots of the characteristic equation $\lambda^2 = -\gamma\lambda - \omega^2$.)
(b) What condition on τ has to be satisfied so that for any initial vector with coordinates $x(0), y(0)$ the sequence of vectors with coordinates $x(k\tau)\,,\ y(k\tau)\,,\ k = 1, 2, 3, \dots$ obtained from the numerical approximation (4) converges to zero as $k \to \infty$?
Solution: (a) Under our assumptions there are two distinct roots of the characteristic polynomial $\lambda^2 + \gamma\lambda + \omega^2$ with the real part $-\frac{\gamma}{2} < 0$, and hence the general solution must approach zero as $t \to \infty$.
(b) Denoting the vector with the coordinates $x(t), y(t)$ by $X(t)$, the iteration scheme can be written we $X(t+\tau) = BX(t)$, where $B = \begin{pmatrix} 1 & \tau \\ -\tau\omega^2 & 1-\tau\gamma \end{pmatrix}$. We need to calculate the eigenvalues of B. The characteristic polynomial of B is $\det(B - \lambda I) = \lambda^2 + (-2+\tau\gamma)\lambda + 1 - \tau\gamma + \tau^2\omega^2$. Its roots are $\lambda_{1,2} = \frac{2-\tau\gamma\pm\sqrt{\tau^2\gamma^2-4\tau^2\omega^2}}{2}$ and with our assumptions we have $|\lambda_{1,2}|^2 = 1 - \tau\gamma + \tau^2\omega^2$. To have $X(k\tau) \to 0$ as $k \to \infty$ for any initial data, we need $|\lambda_{1,2}| < 1$, which gives the condition $\tau < \frac{\gamma}{\omega^2}$.

5. Let us consider the equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{5}$$

for functions $u(x,t)$ which are $2\pi-$periodic, i. e. $u(x+2\pi, t) = u(x,t)$. Consider the following approximation of equation (5)

$$\frac{u(x,t+\tau) - u(x,t)}{\tau} + a\frac{u(x+h,t) - u(x,t)}{h} = 0\,, \tag{6}$$

where $\tau > 0$. Assume we implement (6) as a numerical method for calculating (approximate) solutions for functions which are $2\pi-$peridic in x: for a positive integer n we set $h = \frac{2\pi}{n}$ and consider a grid $x_0 = 0, x_1 = h, x_2 = 2h, \dots, x_{n-1} = (n-1)h$.
(a) Let l be any integer. Show that when $a > 0$ and we use (6) to calculate the function $u(x,t)$ at $x = x_0, x_1, \dots x_{n-1}$ and $t = \tau, 2\tau, 3\tau, \dots$ starting from the (complex-valued) function $u^{(l)}(x_j, 0) = e^{ilx_j}$, then

$$\lim_{k\to\infty} |u(x_j, k\tau)| = \infty\,, \tag{7}$$

for each $j = 0, 1, 2, \dots, n-1$.

(b) Use the result in (a) to find at least one real-valued function $u(x,0)$ on the grid for which the functions $u(x,k\tau)$ obtained by (6) will not be bounded when $a>0$. (We say that the function $u(x,k\tau)$ is bounded if there exists a constant C such that $|u(x,k\tau)|\le C$ for all $k=1,2,\dots$ and all points x on the grid.)

Hint: Look at the real part or the imaginary part of the complex-valued solutions or, alternatively, use that the functions of the form e^{ilx} form a basis of the functions on the grid.

(c) Show that when $u(x,0)$ is a smooth function, then the solution of (5) with the initial value $u(x,0)$ always stays bounded.

Solution: (a) We can re-write the scheme as $u(x,t+\tau)=u(x,t)-\frac{a\tau}{h}(u(x+h,t)-u(x,t))$. If $u(x,t)=e^{ilx}$, then $u(x,t+\tau)=(1-\frac{a\tau}{h}(e^{ilh}-1))u(x,t)$. Letting $\lambda_l=1+\frac{a\tau}{h}(1-e^{ilh})$, we see that for $u(x,0)=e^{ilx}$ we will have $u(x,k\tau)=\lambda_l^k u(x,0)$. When l is an integer satisfying $0<l<n$, then it is easy to see that $\mathrm{Re}(1-e^{ilh})>0$, and hence $\mathrm{Re}\,\lambda_l>1$. Hence $u(x_j,k\tau)=\lambda_l^k u(x,0)$ will be unbounded as $k\to\infty$.
(b) Consider the solution $u(x,k\tau)$ of our scheme constructed in part (a). As the equation (6) is linear and has real coefficients, the functions $\mathrm{Re}\,u(x,\kappa t)$ and $\mathrm{Im}\,u(x,\kappa\tau)$ will again satisfy (6). Clearly, at least one of them is unbounded. When e^{ilh} is not real, it is not hard to see that in fact neither the real nor the imaginary parts can be bounded. When $e^{ilh}=-1$ our solution starts real-valued and remains real-valued.
(c) The solution an be written as $u(x,t)=u(x-ct,0)$, and will clearly be bounded if $u(x,0)$ is bounded.

6. Assume that the equations for small oscillations of a physical system with n degrees of freedom described by some coordinates $x_1,x_2,\dots,x_n$ are

$$m_i\ddot{x}_i=-\sum_{j=1}^{n}a_{ij}x_j\,,\quad i=1,2,\dots,n\,, \tag{8}$$

where $\{a_{ij}\}_{i,j=1}^{n}$ is a symmetric positive-definite matrix and the numbers $m_i>0\,,\ i=1,2,\dots,n$ represent masses. Assume that in a situation when m_i are exactly equal to some specific $\mu>0$, the lowest frequency of the oscillations of the system is known, and is equal to some specific number $\Omega>0$. Given this information, find the best estimate of the lowest possible frequency of the oscillations of the system (with the same matrix A) in a situation when we only know about the masses m_i that $\frac{1}{2}\mu\le m_i\le 2\mu\,,\ i=1,2,\dots,n$. A correct justification of the answer should be a part of the solution.

Solution: Let M be the diagonal matrix $\mathrm{diag}(m_1,m_2,\dots,m_n)$. The lowest frequency will be given by $\sqrt{\lambda_{\min}}$, where $\lambda_{\min}=\min_{x\ne0}\frac{(Ax,x)}{(Mx,x)}$, which is the same as the lowest (generalized) eigenvalue λ of the problem $Ax=\lambda Mx$. From the properties of the Rayleigh quotient we have $\frac{\Omega^2}{2}=\min_{x\ne0}\frac{(Ax,x)}{2\mu(x,x)}\le\min_{x\ne0}\frac{(Ax,x)}{(Mx,x)}\le\min_{x\ne0}\frac{(Ax,x)}{\frac{\mu}{2}(x,x)}=2\Omega^2$, and $\lambda_{\min}=\min_{x\ne0}\frac{(Ax,x)}{(Mx,x)}$. We see that the lowest frequency Ω' for the system with the variable masses $\frac{1}{2}\mu\le m_i\le 2\mu$ will satisfy $\frac{1}{\sqrt{2}}\Omega\le\Omega'\le\sqrt{2}\Omega$. Taking $m_i=\frac{1}{2}\mu$ we see that the upper bound cannot be improved and taking $m_i=2\mu$ we see that the lower bound cannot be improved.