Appendix B.
Criterion of Riemann-Stieltjes Integrability

This note is complementary to [R, Ch. 6] and [T, Sec. 3.5]. The main result of this note is Theorem B.3, which provides the necessary and sufficient conditions for Riemann-Stieltjes integrability of f with respect to α in terms of sets of point of discontinuity of these functions. In an equivalent form, this result is contained in [H, Theorem C]. Here we give a more direct proof, which does not use explicitly the Lebesgue measure.

Let $\alpha = \alpha(x)$ be a monotonically non-decreasing function on a finite interval $[a, b]$, and let $f = f(x)$ be a bounded real function on $[a, b]$. For an arbitrary **partition**

$$P := \{a = x_0 \le x_1 \le \ldots \le x_{n-1} \le x_n = b\} \quad \text{of} \quad [a, b],$$

we define the **upper** and **lower sums** as follows:

$$(1) \qquad U(P, f, \alpha) := \sum_{i=1}^{n} M_i \, \Delta\alpha_i, \qquad L(P, f, \alpha) := \sum_{i=1}^{n} m_i \, \Delta\alpha_i,$$

where

$$(2) \qquad M_i := \sup_{[x_{i-1}, x_i]} f \ge m_i := \inf_{[x_{i-1}, x_i]} f, \qquad \Delta\alpha_i := \alpha(x_i) - \alpha(x_{i-1}) \quad \text{for} \quad i = 1, 2, \ldots, n.$$

For any two partitions P_1 and P_2, their **common refinement** $P^* := P_1 \cup P_2$ satisfies (see [R, Theorem 6.4])

$$(3) \qquad U(P_1, f, \alpha) \ge U(P^*, f, \alpha) \ge L(P^*, f, \alpha) \ge L(P_2, f, \alpha).$$

Therefore, we always have

$$(4) \qquad \inf_P U(P, f, \alpha) \ge \sup_P L(P, f, \alpha).$$

Definition B.1. The function f is **Riemann-Stieltjes integrable** with respect to α on $[a, b]$ if both sides of (4) are equal. In this case, we write $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and define the **Riemann-Stieltjes integral**

$$(5) \qquad \int_a^b f \, d\alpha := \inf_P U(P, f, \alpha) = \sup_P L(P, f, \alpha).$$

Theorem B.2 ([R], Theorem 6.6)**.** *$f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that*

$$(6) \qquad U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Proof. For every $\varepsilon > 0$ there are partitions P_1 and P_2 such that

$$U(P_1, f, \alpha) < \inf_P U(P, f, \alpha) + \frac{\varepsilon}{2}, \qquad L(P_2, f, \alpha) > \sup_P U(P, f, \alpha) - \frac{\varepsilon}{2}.$$

If $f \in \mathcal{R}(\alpha)$ on $[a, b]$, then we have equality in (4), which implies

$$0 \le U(P_1, f, \alpha) - L(P_2, f, \alpha) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and (6) follows from (3) with $P = P^* := P_1 \cup P_2$.

On the other hand, if we have (6), then the difference between inf and sup in (4) is less than ε. Since $\varepsilon > 0$ is arbitrary, we must have the equality, i.e. $f \in \mathcal{R}(\alpha)$ on $[a, b]$. □

Further, since $\alpha(x)$ is non-decreasing on $[a,b]$, there are one-sided limits

$$\alpha(p-) := \lim_{y\to p-} \alpha(y), \quad a < p \le b; \qquad \alpha(p+) := \lim_{y\to p+} \alpha(y), \quad a \le p < b,$$

and $\alpha(p-) \le \alpha(p) \le \alpha(p+)$.

Theorem B.3. *Let f be a bounded real function on $[a,b]$. Then $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if f and α satisfy both properties (I) and (II) below.*

(I) **(i)** *If $\alpha(p-) < \alpha(p)$, $a < p \le b$, then $\exists f(p-) = f(p)$.*
(ii) *If $\alpha(p) < \alpha(p+)$, $a \le p < b$, then $\exists f(p+) = f(p)$.*

(II) *Let S_f and S_α denote the sets of points of discontinuity of f and α correspondingly. Then for every $\varepsilon > 0$ there exists a (finite or countable) sequence of intervals (a_j, b_j), $j \ge 1$, such that*

$$S := (S_f \setminus S_\alpha) \subset \bigcup_j (a_j, b_j), \quad \textit{and} \quad \sum_j \big(\alpha(b_j) - \alpha(a_j)\big) < \varepsilon. \tag{7}$$

Here the intervals (a_j, b_j) are not necessarily contained in $[a,b]$. We extend $f \equiv f(a)$, $\alpha \equiv \alpha(a)$ on $(-\infty, a)$ and $f \equiv f(b)$, $\alpha \equiv \alpha(b)$ on $(b, +\infty)$, so that the last expression, and also the expression in (9) below, are well defined in any case.

Remark B.4. The property (I) simply says that if $f \in \mathcal{R}(\alpha)$ on $[a,b]$, then f and α cannot be both left-discontinuous, or both right-discontinuous at same point. Of course, this property is redundant if α is continuous on $[a,b]$. By change of variable ([R, Theorem 6.19]), this case can be reduced to $\alpha(x) \equiv x$. In this particular case, our theorem is contained in [T, Theorem 3.5.6].

Definition B.5. The **oscillation** of f **on a set** A,

$$\operatorname*{osc}_A f := \sup_A f - \inf_A f = \sup_{x,y\in A} |f(x) - f(y)|. \tag{8}$$

If f is defined on $[a,b]$, then the **oscillation** of f **at a point** $p \in [a,b]$,

$$\omega_f(p) := \lim_{h\to 0+} \operatorname*{osc}_{[p-h,p+h]} f. \tag{9}$$

Lemma B.6. **(i)** *f is continuous at p if and only if $\omega_f(p) = 0$;*
(ii) *$f(p-) = f(p)$ if and only if $\operatorname*{osc}_{[p-h,p]} f \to 0$ as $h \to 0+$;*
(iii) *$f(p+) = f(p)$ if and only if $\operatorname*{osc}_{[p,p+h]} f \to 0$ as $h \to 0+$.*

We skip the proof, because it is very elementary (see [T, Theorem 3.5.2]).

Lemma B.7. *If $f \in \mathcal{R}(\alpha)$ on $[a,b]$, then f and α satisfy the properties (I) in Theorem B.3.*

Proof. Let p be a point such that $\alpha(p-) < \alpha(p)$, $a < p \le b$. By Theorem B.2, for every $\varepsilon > 0$ there is a partition $P := \{a = x_0 \le x_1 \le \ldots \le x_n = b\}$ (depending on α) such that

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^n (M_i - m_i) \cdot \Delta\alpha_i < \varepsilon. \tag{10}$$

Next, for small $h \in (0, p-a)$, the interval $(p-h, p)$ does not contain point $x_i \in P$. From (3) (with $P_1 = P_2 = P$) it follows that the refined partition $P^* := P \cup \{p-h, p\}$ satisfies

$$U(P^*,f,\alpha) - L(P^*,f,\alpha) \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Therefore, replacing P by P^* if necessary, we can assume that $p-h, p \in P$, i.e.

$$p-h = x_{i_0-1} < p = x_{i_0} \quad \text{for some} \quad i_0 \in \{1, 2, \ldots, n\}.$$

Then from (10) it follows

$$\operatorname*{osc}_{[p-h,p]} f \cdot \Delta\alpha_{i_0} = \big(M_{i_0} - m_{i_0}\big) \cdot \Delta\alpha_{i_0} < \varepsilon.$$

Since $\Delta\alpha_{i_0} = \alpha(p) - \alpha(p-h) \ge \alpha(p) - \alpha(p-) > 0$, and $\varepsilon > 0$ can be chosen arbitrarily small, we conclude that $\operatorname*{osc}_{[p-h,p]} f \to 0$ as $h \to 0+$. By Lemma B.6(ii), we have $f(p-) = f(p)$.

The proof of part (i) in (I) is complete. Part (ii) can be proved quite similarly. □

Lemma B.8. *If $f \in \mathcal{R}(\alpha)$ on $[a,b]$, then f and α satisfy the property (II) in Theorem B.3.*

Proof. By Lemma B.6(i), the set of points of discontinuity of f,

$$\text{(11)} \qquad S_f = \big\{p \in [a,b] : \ \omega_f(p) > 0\big\} = \bigcup_{k=1}^{\infty} F_k, \quad \text{where} \quad F_k := \big\{p \in [a,b] : \ \omega_f(p) \ge 2^{-k}\big\}.$$

Fix $\varepsilon > 0$. By Theorem B.2, for every $k = 1, 2, \ldots$, there exists a partition $P := \{a = x_0 \le x_1 \le \ldots \le x_n = b\}$ (depending on k) such that

$$\text{(12)} \qquad U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i) \cdot \Delta\alpha_i < \varepsilon_k := 4^{-k}\varepsilon.$$

Note that if $p \in F_k \setminus P$, then for some $i \in \{1, 2, \ldots, n\}$ we have $p \in (x_{i-1}, x_i)$, and $M_i - m_i \ge \omega_f(p) \ge 2^{-k}$. Let A_k denote the set of all such indices i. Then

$$\text{(13)} \qquad (F_k \setminus P) \subset \bigcup_{i \in A_k} (x_i - x_{i-1}), \quad \text{and} \quad \sum_{i \in A_k} \Delta\alpha_i \le 2^k \sum_{i \in A_k} (M_i - m_i) \cdot \Delta\alpha_i < 2^{-k}\varepsilon.$$

Further, $F_k \setminus S_\alpha$ is contained in $(F_k \setminus P) \cup (P \setminus S_\alpha)$. Since $\alpha(x)$ is continuous at every point $p \in P \setminus S_\alpha$, one can cover such point by intervals $(p-h, p+h)$ with arbitrarily small $\alpha(p+h) - \alpha(p-h)$. Together with (x_{i-1}, x_i), $i \in A_k$, these intervals compose a finite family of intervals $(a_{k,i}, b_{k,i})$ such that

$$(F_k \setminus S_\alpha) \subset \bigcup_i (a_{k,i}, b_{k,i}), \quad \text{and} \quad \sum_i \big(\alpha(b_{k,i}) - \alpha(a_{k,i})\big) < 2^{-k}\varepsilon.$$

Finally, by virtue of (11),

$$(S_f \setminus S_\alpha) = \bigcup_{k=1}^{\infty} (F_k \setminus S_\alpha) \subset \bigcup_{k=1}^{\infty} \bigcup_i (a_{k,i}, b_{k,i}), \quad \text{and} \quad \sum_{k=1}^{\infty} \sum_i \big(\alpha(b_{k,i}) - \alpha(a_{k,i})\big) < \sum_{k=1}^{\infty} 2^{-k}\varepsilon = \varepsilon.$$

Since the countable set of intervals $\{(a_{k,i}, b_{k,i})\}$ can be renumbered as $\{(a_j, b_j)\}$, we get the desired property (7). □

The following lemma, together with the previous Lemmas B.7 and B.8, completes the proof of Theorem B.3.

Lemma B.9. *Let f be a bounded function on $[a,b]$ satisfying the properties (I) and (II) in Theorem B.3. Then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.*

Proof. Step 1. We have $|f| \le M = \text{const} < \infty$ on $[a,b]$. By Theorem B.2, it suffices to show that for an arbitrary $\varepsilon > 0$, there exists a partition $P := \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$ of $[a,b]$ satisfying the inequality (6) for given f and α. This inequality can be written in the form

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} \operatorname*{osc}_{I_i} f \cdot \operatorname*{osc}_{I_i} \alpha < \varepsilon, \quad \text{where} \quad I_i := [x_{i-1}, x_i]. \tag{14}$$

Step 2. Fix a constant $\varepsilon_1 > 0$. Note that since $\alpha(x)$ is a monotone function, its set of points of discontinuity S_α is at most countable: $S_\alpha := \{c_1, c_2, \ldots\}$. From the assumption (I) in Theorem B.3 it follows that for each $j = 1, 2, \ldots$, one can choose a small constant $h_j > 0$ such that

$$\operatorname*{osc}_{I^-_{1,j}} f \cdot \operatorname*{osc}_{I^-_{1,j}} \alpha < 2^{-j}\varepsilon_1, \quad \operatorname*{osc}_{I^+_{1,j}} f \cdot \operatorname*{osc}_{I^+_{1,j}} \alpha < 2^{-j}\varepsilon_1, \quad \text{for} \quad j = 1, 2, \ldots, \tag{15}$$

where $I^-_{1,j} := [c_j - h_j, c_j]$, $I^+_{1,j} := [c_j, c_j + h_j]$. Obviously, we also have

$$S_\alpha := \{c_1, c_2, \ldots\} \subset V_1 := \bigcup_{j \ge 1} I_{1,j}, \quad \text{where} \quad I_{1,j} := (a_{1,j}, b_{1,j}) := (c_j - h_j, c_j + h_j). \tag{16}$$

Step 3. Based on the constant $\varepsilon_1 > 0$, define the set

$$F := \{p \in [a,b] : \ \omega_f(p) \ge \varepsilon_1 > 0\}. \tag{17}$$

We claim (as in [T, Lemma 3.5.4]) that F is **compact**. Indeed, if $p_j \in F$ and $p_j \to p_0 \in [a,b]$ as $j \to \infty$, then for an arbitrary $h > 0$ there is j such that $|p_j - p_0| < h/2$. For such j, we have $(p_j - h/2, p_j + h/2) \subset (p_0 - h, p_0 + h)$, hence by (8) and (9), the oscillation of f,

$$\operatorname*{osc}_{[p_0-h, p_0+h]} f \ge \operatorname*{osc}_{[p_j - h/2,\, p_j + h/2]} f \ge \omega_f(p_j) \ge \varepsilon_1,$$

and

$$\omega_f(p_0) := \lim_{h \to 0+} \operatorname*{osc}_{[p_0-h, p_0+h]} f \ge \varepsilon_1 > 0, \quad \text{i.e.} \quad p_0 \in F.$$

This argument proves the compactness of F.

Step 4. Further, note that $F \subset S_f$ – the set of points of discontinuity of f. Therefore, by our assumption (II), for the given constant $\varepsilon_1 > 0$, there exists a sequence of intervals $I_{2,j} := (a_{2,j}, b_{2,j})$ such that

$$(F \setminus S_\alpha) \subset (S_f \setminus S_\alpha) \subset V_2 := \bigcup_j I_{2,j}, \quad \text{and} \quad \sum_j \big(\alpha(b_{2,j}) - \alpha(a_{2,j})\big) < \varepsilon_1. \tag{18}$$

Step 5. From (16) and (18) it follows $F \subset (V_1 \cup V_2)$, so that the compact set F is covered by the union of two families of open intervals $\{I_{1,j}\}$ and $\{I_{2,j}\}$. Therefore, one can choose finite subfamilies $\{I'_{1,j}\} \subset \{I_{1,j}\}$ and $\{I'_{2,j}\} \subset \{I_{2,j}\}$ such that

$$F \subset (V'_1 \cup V'_2), \quad \text{where} \quad V'_1 := \bigcup_j I'_{1,j}, \quad V'_2 := \bigcup_j I'_{2,j}. \tag{19}$$

Consider another compact set $F' := [a,b] \setminus (V'_1 \cup V'_2)$. Since F' does not intersect F, we have $\omega_f(p) < \varepsilon_1$ for every $p \in F'$. By definition of $\omega_f(p)$ in (9),

$$\operatorname*{osc}_{[p-h, p+h]} f < \varepsilon_1 \quad \text{for every} \quad p \in F' \quad \text{with some} \quad h = h(p) > 0. \tag{20}$$

The family of the corresponding open intervals $\{(p-h,p+h),\ p\in F'\}$ covers the compact F'. Therefore, this family contains a finite subfamily $\{I'_{3,j} := (a_{3,j},b_{3,j})\}$ such that

$$F' \subset V'_3 := \bigcup_j I'_{3,j}, \quad \text{and} \quad \operatorname*{osc}_{[a_{3,j},b_{3,j}]} f < \varepsilon_1 \quad \text{for each} \quad j. \tag{21}$$

Step 6. It is easy to see that (19) and (21) imply $[a,b]\subset (V'_1\cup V'_2\cup V'_3)$, so that $[a,b]$ is covered by the union of three finite families of open intervals $\{I'_{1,j}\}$, $\{I'_{2,j}\}$, and $\{I'_{3,j}\}$. Let $P := \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$ be a partition of $[a,b]$, which includes the point a,b, all the endpoints of intervals $I'_{1,j}, I'_{2,j}, I'_{3,j}$, and also the centers c_j of the intervals $I'_{1,j} := (c_j-h_j, c_j+h_j)$, which belong to (a,b).

Denote $I_i := [x_{i-1},x_i]$ for $i=1,2,\ldots,n$. Note that I_i are closed intervals, whereas $I'_{1,j}, I'_{2,j}, I'_{3,j}$ are open. However, all the estimates (15), (18), and (21), hold true for closed intervals.

Let A_1 denote the set of all indices $i\in\{1,2,\ldots,\}$ such that $I_i\subset V'_1$, A_2 – the set of all $i\notin A_1$ such that $I_2\subset V'_2$, and A_3 – the set of all the remaining i, for which we automatically have $I_i\subset V'_3$, because $[a,b]\subset (V'_1\cup V'_2\cup V'_3)$.

For each $i\in A_1$, we have either $I_i\subset I^-_{1,j}$ or $I_i\subset I^+_{1,j}$ for some j, hence by virtue of (15),

$$\sum_{i\in A_1} \operatorname*{osc}_{I_i} f\cdot \operatorname*{osc}_{I_i}\alpha < 2\sum_{j=1}^{\infty} 2^{-j}\varepsilon_1 = 2\varepsilon_1. \tag{22}$$

Similarly, since $|f|\le M$, we have $\operatorname{osc} f\le 2M$, and the last inequality in (18) implies

$$\sum_{i\in A_2} \operatorname*{osc}_{I_i} f\cdot \operatorname*{osc}_{I_i}\alpha \le 2M\sum_{i\in A_2} \operatorname*{osc}_{I_i}\alpha < 2M\cdot\varepsilon_1. \tag{23}$$

Finally, from (21) and monotonicity of α it follows

$$\sum_{i\in A_3} \operatorname*{osc}_{I_i} f\cdot \operatorname*{osc}_{I_i}\alpha \le \varepsilon_1\sum_{i\in A_3} \operatorname*{osc}_{I_i}\alpha \le \big(\alpha(b)-\alpha(a)\big)\cdot\varepsilon_1. \tag{24}$$

Since $A_1\cup A_2\cup A_3 = \{1,2,\ldots,n\}$, the estimates (22)–(24) yield

$$\sum_{i=1}^{n} \operatorname*{osc}_{I_i} f\cdot \operatorname*{osc}_{I_i}\alpha \le \big(2+2M+\alpha(b)-\alpha(a)\big)\cdot\varepsilon_1 < \varepsilon,$$

provided $0<\varepsilon_1<\big(2+2M+\alpha(b)-\alpha(a)\big)^{-1}\varepsilon$. Thus we have the desired estimate (14) and lemma is proved. □

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