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Duality in the Combinatorics of Posets

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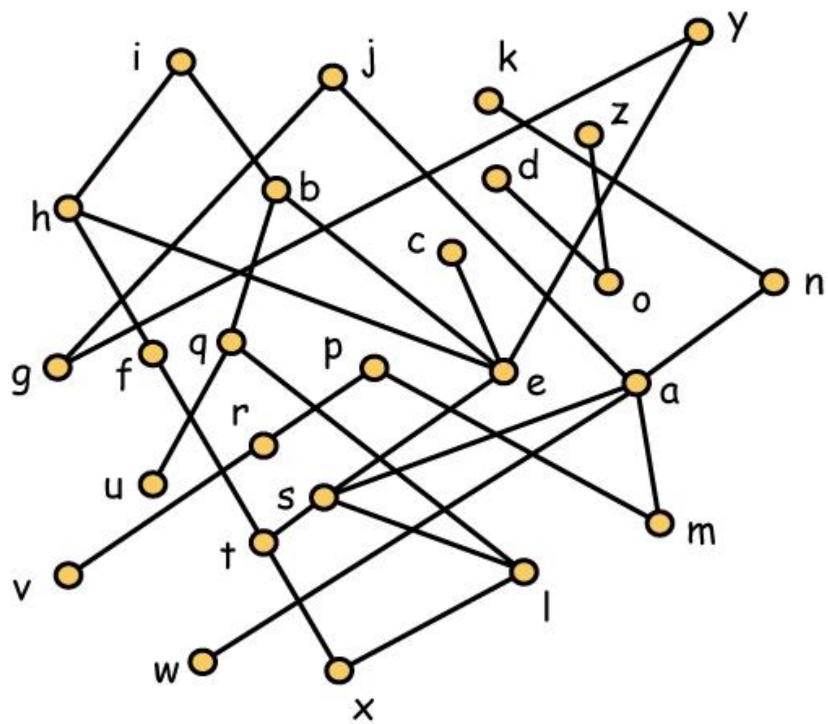
Diagram for a Poset on 26 points

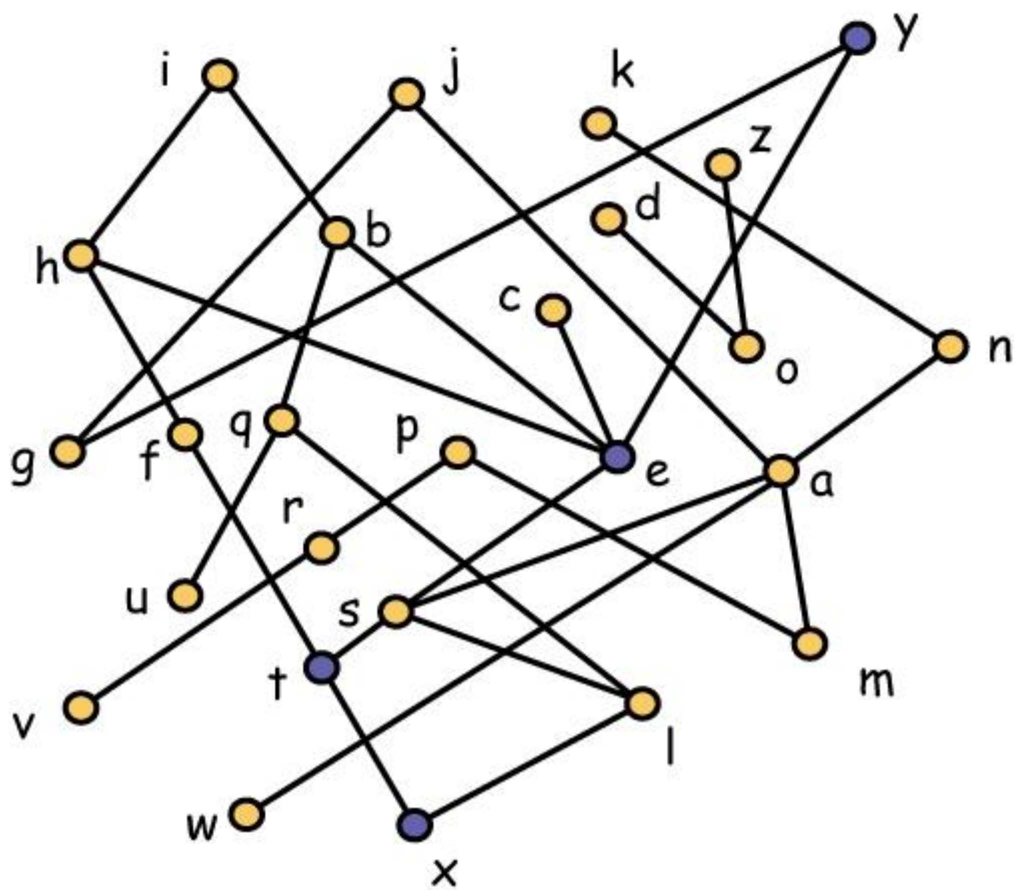
Terminology:

- b < i and s < y.
- j covers a.
- b > e and k > w.
- s and y are comparable.
- j and p are incomparable.
- c is a maximal element.
- u is a minimal element.



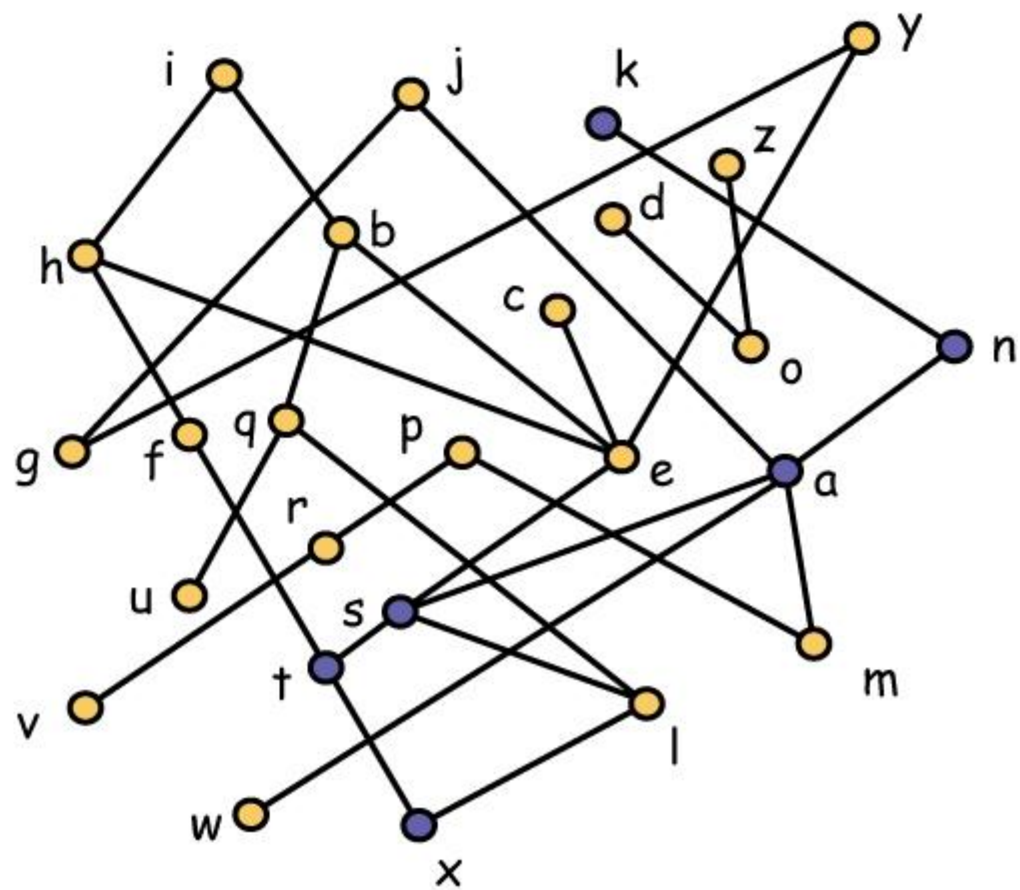
A Chain of Size 4

Definition *A* **chain** is a subset in which every pair is comparable.



A Maximal Chain of Size 6

Definition A chain is **maximal** when no superset is also a chain. Is the chain in the picture maximum?



Height of a Poset

Definition The **height** of a poset P is the maximum size of a chain in P.

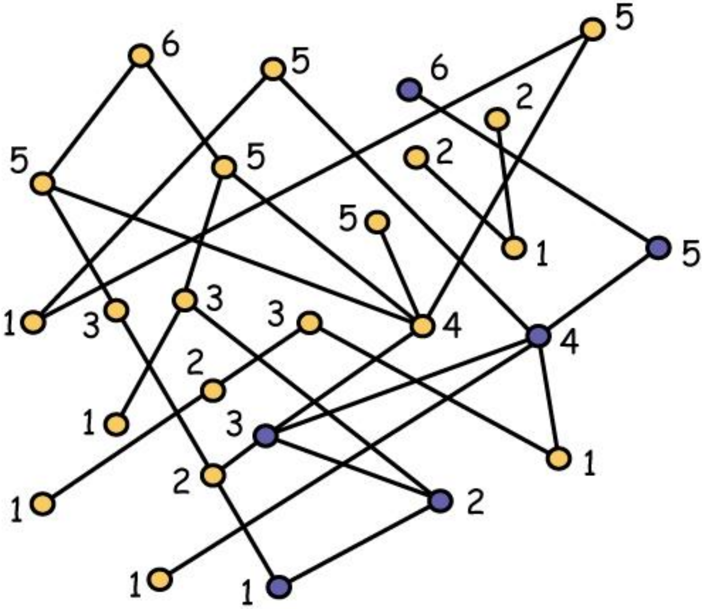
Proposition To partition a poset P of height h into antichains, at least h antichains are required.

Question How hard is it to find the height of a poset and the minimum size of a partition of the poset into antichains?

Scholarship and Attribution

Observation Getting it right is not as easy as some would believe. In fact, it is not always easy for people to agree on what "right" means.

Mirsky's Theorem

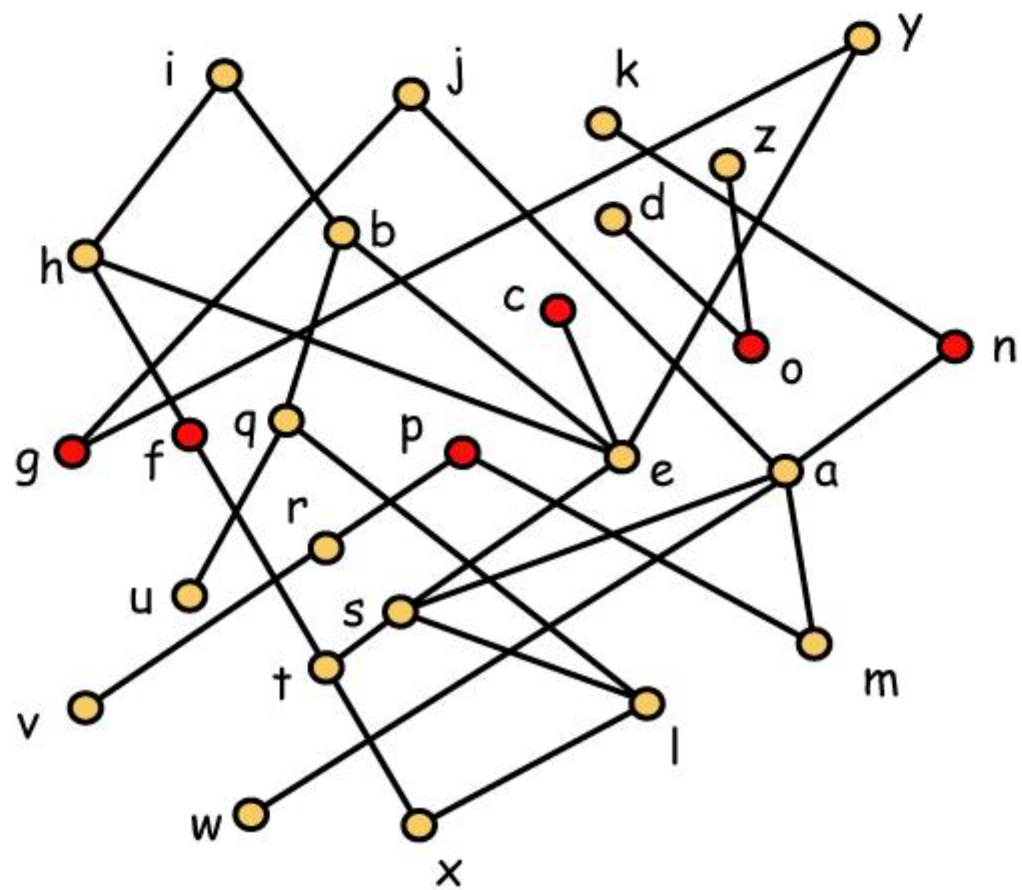


Theorem (1971) A poset of height h can be partitioned into h antichains.

Proof A_i is the set of elements at height i.

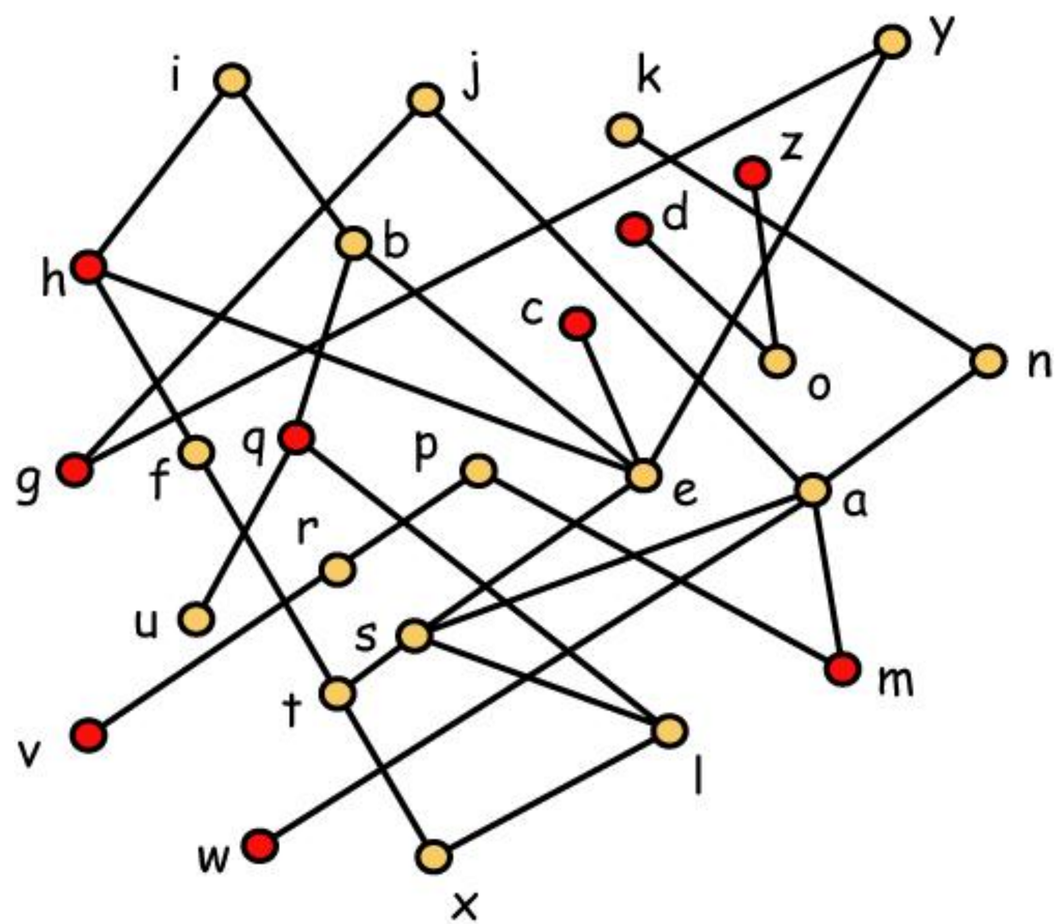
An Antichain of Size 6

Definition A subset is an **antichain** when every pair is incomparable.



A Maximal Antichain of Size 9

Definition *An* antichain is **maximal** when no superset is also an antichain. Is the antichain in the picture maximum?



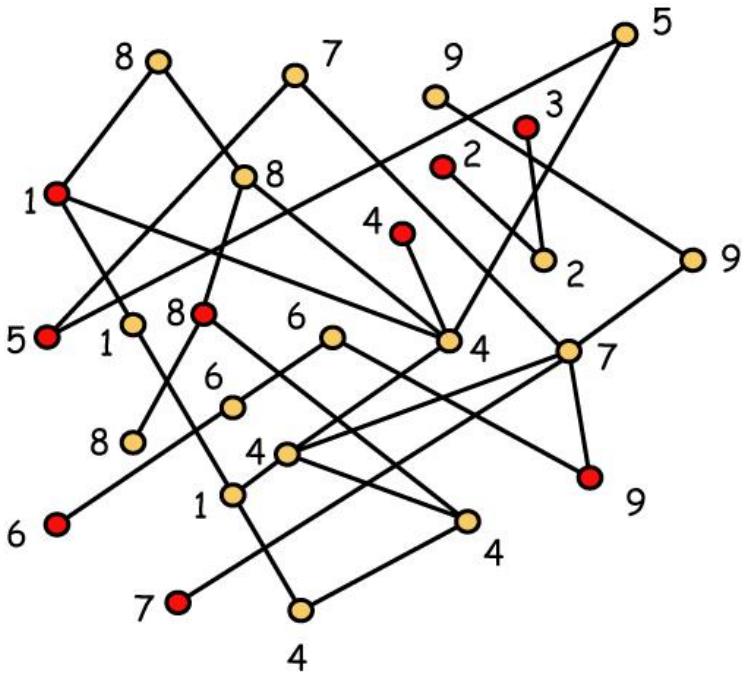
Width of a Poset

Definition The **width** of a poset P is the maximum size of an antichain in P.

Proposition To partition a poset P of width w into chains, at least w chains are required.

Question How hard is it to find the width of a poset and the minimum size of a partition of the poset into chains?

Dilworth's Theorem



Theorem (1950) A poset of width w can be partitioned into w chains.

Note The original proof is one page long!

Alternate Proofs of Dilworth's Theorem

Fulkerson (1954) Used bipartite matching algorithm (network flows) to find minimum chain partition and maximum antichain simultaneously.

Gallai/Milgram (1960) Path decompositions in oriented graphs.

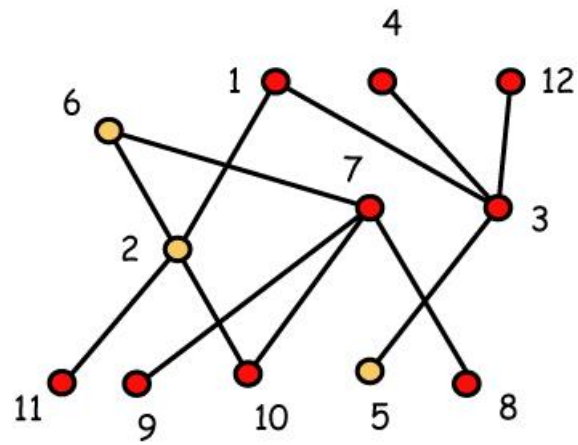
Perles (1963) Simple induction depending on whether there is a maximum antichain A with $U(A)$ and $D(A)$ non-empty. This is the proof found in most combinatorics textbooks today.

Posets and Perfect Graphs

Theorem (Lovász, 1972) A graph G is perfect if and only if its complement is perfect.

Remark Dilworth's theorem then follows then as an immediate corollary to the trivial theorem on height.

Sperner k-families

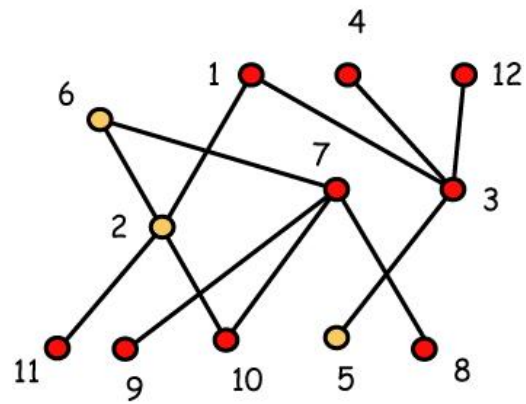


Definition When $k \geq 1$, a set S of elements of P is called a **Sperner k-family** when height$(S) \leq k$.

In the diagram, the red points form a Sperner 2-family.

A Sperner 1-family is just an antichain.

Sperner k-families and Chains



Notation When $k \geq 1$, the maximum size of a subposet S with $\text{height}(S) \leq k$ is denoted $w_k(P)$. Here, $w_2(P) = 9$.

Observation When $\text{height}(S) \leq k$ and C is a chain, then $|S \cap C| \leq \min\{k, |C|\}$.

Chain Partitions and Sperner k-families

Observation When $k \geq 1$, if height$(S) \leq k$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ is any chain partition of P, then

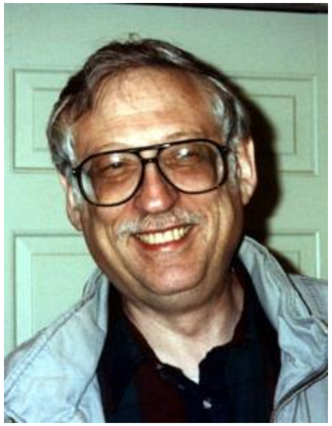
$$|S| \leq \textstyle\sum_i \min\{k, |C_i|\}.$$

Definition A chain partition $\mathcal{C}$ is **k-saturated** when

$$w_k(P) = \textstyle\sum_i \min\{k, |C_i|\}.$$

Greene/Kleitman Theorem

Theorem (1976) For every $k \geq 1$ and for every poset P, there is a chain partition C of P that is simultaneously k and $k+1$-saturated.



The G/K Theorem is Tight

Theorem (West, 1986) For every $k \geq 1$ and for every $h \geq 4$, there is a poset P of height h which does not have a chain partition that is simultaneously k and k'-saturated whenever

$$1 \leq k < k' < h \text{ and } k + 2 \leq k'.$$

Note "Minimal" examples were given in 2002 by G. Chappell.

Duality - Greene's Theorem

Notation When $k \geq 1$, the maximum size of a subset T in a poset P with $\text{width}(T) \leq k$ is denoted $h_k(P)$.

Note $h_1(P)$ is just the height of P.

Theorem (1976) For every $k \geq 1$ and for every poset P, there exists an antichain partition $\mathcal{A}$ that is simultaneously k and $k+1$-saturated.

Alternative Proofs

Note Combinatorial proofs of the Greene-Kleitman theorem have been provided by H. Perfect (1984) and M. Saks (1979). The argument by Saks results in an effective algorithm for finding for each $k \geq 1$ a chain partition which is both k-saturated and $k+1$-saturated.

Note A. Frank (1980) has given a unified approach proving both Greene-Kleitman and Greene using network flows. A. J. Hoffman and D. E. Schwarz (1977) have given such a proof using linear algebra.

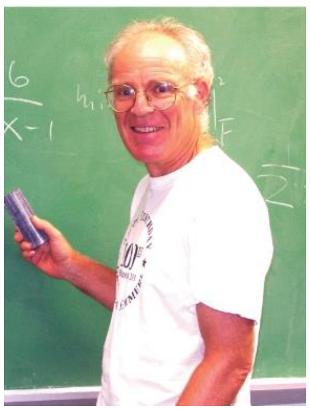
On-line Antichain Partition Problems

Builder reveals comparabilities between the new point and all preceding points.
Partitioner makes an irrevocable assignment of the new point to an antichain.

Basic Question Is there some function $f(h)$ so that if Builder is constrained to posets of height at most h, then Partitioner can construct an on-line partition into $f(h)$ antichains.

Subtlety Does it matter if Partitioner does or doesn't know h?

On-Line Antichain Partitioning



Theorem (Schmerl, Szemerédi, 1983) There is an on-line algorithm that will partition a poset of height h into

$$h(h + 1)/2$$

antichains. Furthermore, this is best possible. Also, Partitioner does not need to know h in advance.

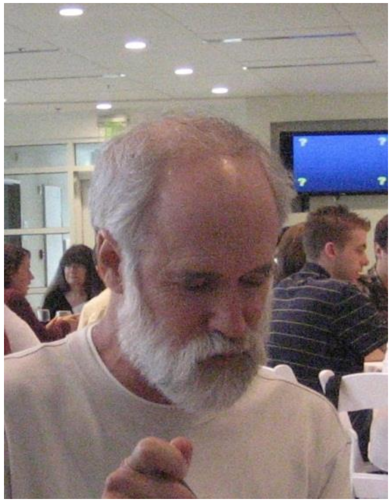
On-line Chain Partition Problems

Builder reveals comparabilities between the new point and all preceding points.
Partitioner makes an irrevocable assignment of the new point to a chain.

Basic Question Is there some function $g(w)$ so that if Builder is constrained to posets of width at most w, then Partitioner can construct an on-line partition into $g(w)$ antichains.

Subtlety Does it matter if Partitioner does or doesn't know w?

On-Line Chain Partitioning



Theorem (Kierstead, 1981) There is an on-line algorithm that will partition a poset of width w into

$$(5^w - 1)/4$$

chains.

Note From below, $w(w+1)/2$ chains are required.

On-Line Chain Partitioning

Theorem (Bosek and Krawczyk, 2010) There is an on-line algorithm that will partition a poset of width w into $w^{16 \log w}$ chains.

Remark The best lower bound to date is $(1 - o(1))\, w^2$.

Intersecting Maximal Chains

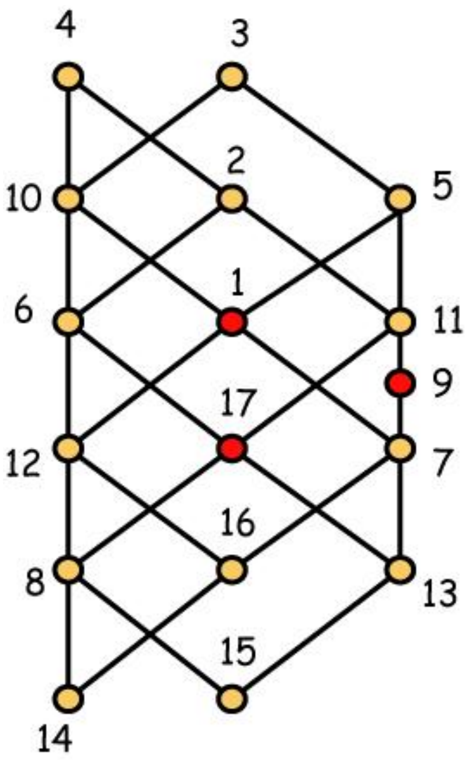
Lemma (Lonc and Rival, 1987) In a poset P on n points, there is always a set S of at most $n/2$ points that meets every non-trivial maximal chain.

Question Does the same result hold for antichains?

Intersecting Maximal Antichains

Example (Sands, 1991) There is a poset P on 17 points for which any set intersecting all non-trivial maximal antichains has at least 9 elements.

Sands' Example



Fact For i = 1, 2, …, 16, i and i + 1 form a maximal antichain.

Fact 1, 9 and 17 form a 3-element maximal antichain.

Intersecting Maximal Antichains

Theorem (Maltby, 1992) For every $\varepsilon > 0$, there is some n_0 so that if $n > n_0$, then there is a poset P on n points for which any set intersecting all non-trivial maximal antichains has at least $(8/15 - \varepsilon)n$ points.

Intersecting Maximal Antichains

Theorem (Duffus, Kierstead and Trotter, 1991) The chromatic number of the hypergraph of non-trivial maximal antichains of a poset P has chromatic number at most 3.

As a consequence, if P has n elements, there is a subset S of size at most $2n/3$ meeting every non-trivial maximal antichain.

Pairwise Disjoint Maximal Antichains

Theorem (Duffus and Sands, 2009)
Let $s \geq k \geq 3$. If

$$s \leq |C| \leq s + (s-k)/(k-2)$$

for every maximal chain C in P, then P has k pairwise disjoint maximal antichains.

The Dual Theorem

Theorem (Howard and Trotter, 2009) Let $s \geq k \geq 3$. If

$$s \leq |A| \leq s + (s-k)/(k-2)$$

for every maximal antichain A in P, then P has k pairwise disjoint maximal chains.

Transversals for Chains

Theorem (Greene and Kleitman, 1976) The minimum size of a set intersecting all maximum chains is equal to the maximum number of pairwise disjoint maximum chains.

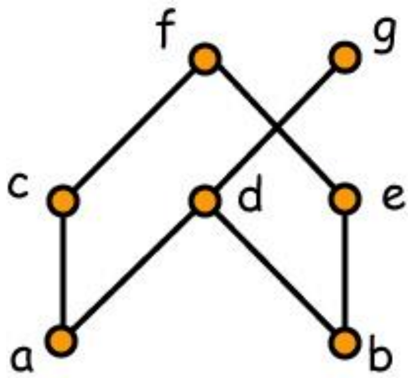
Theorem (Howard and Trotter, 2009) The minimum size of a set intersecting all maximal chains is equal to the maximum number of pairwise disjoint maximal chains.

Transversals for Antichains

Theorem (Folklore but maybe 2014 ?) The minimum size of a set intersecting all maximum antichains is equal to the maximum number of pairwise disjoint maximum antichains.

Theorem (Howard and Trotter, 2009) When there is a finite projective plane of order q, there is a poset in which the maximum number of pairwise disjoint maximal antichains is 2, yet the minimum size of a set intersecting all maximal antichains has size $2q$.

The Dimension of a Poset



$L_1 = b < e < a < d < g < c < f$

$L_2 = a < c < b < d < g < e < f$

$L_3 = a < c < b < e < f < d < g$

The **dimension** of a poset is the minimum size of a realizer. This realizer shows $\dim(P) \leq 3$. In fact,

$$\dim(P) = 3$$

Bounds on Dimension

Theorem (Dilworth, 1950) The dimension of a poset is at most its width.

Theorem (Kimble 1974, Trotter 1975) If A is a maximum antichain in a poset P, then

$$\dim(P) \leq \max \{ 2, |P - A| \}.$$

Corollary (Hiraguchi, 1951) If P is a poset on n points and $n \geq 4$, then $\dim(P) \leq n/2$.

Comparability and Incomparability Graphs

Definition With a poset P, we associate a **comparability graph** G whose vertex set is the ground set of P with xy an edge in G if and only if x and y are comparable in P.

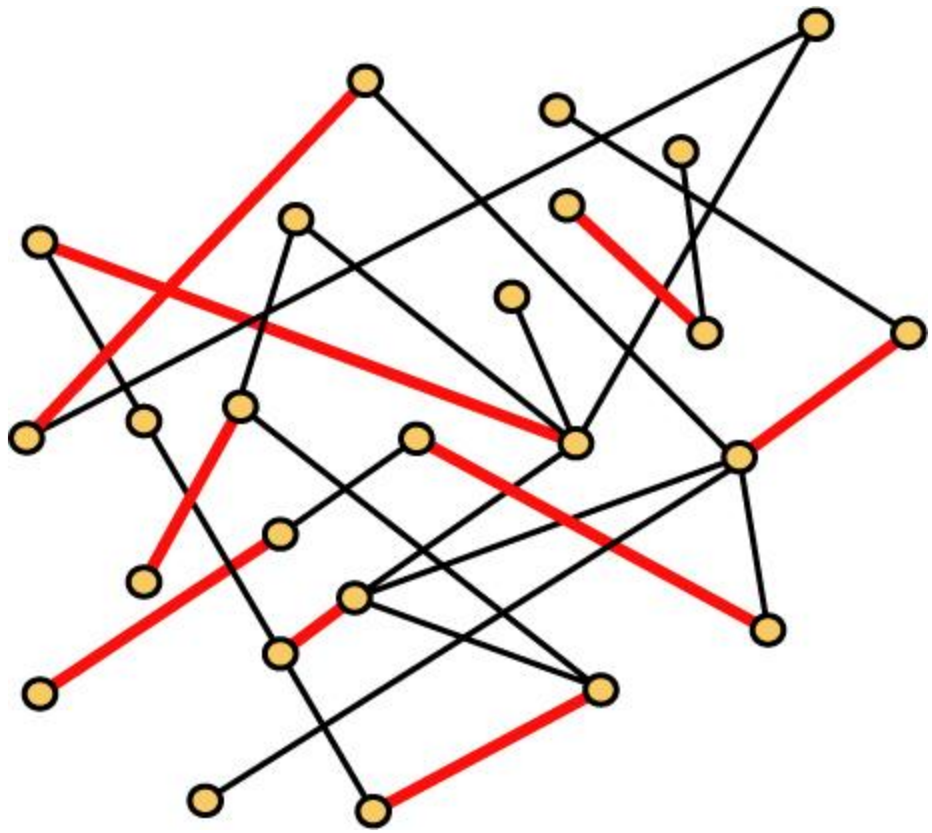
Definition The **incomparability graph** of a poset is just the complement of the comparability graph.

Theorem Height, width, dimension and number of linear extensions are comparability invariants.

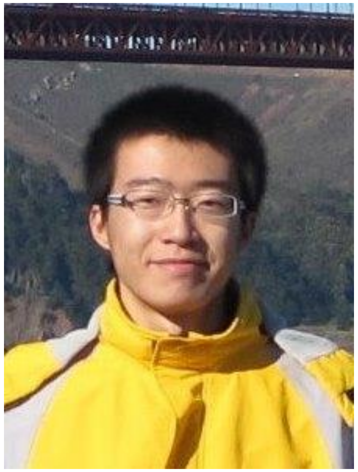
Matchings in Graphs

Definition *A* matching in a graph is a set of edges with no common end points.

Fact The matching shown has size 9 and is maximal



Dimension and Matchings



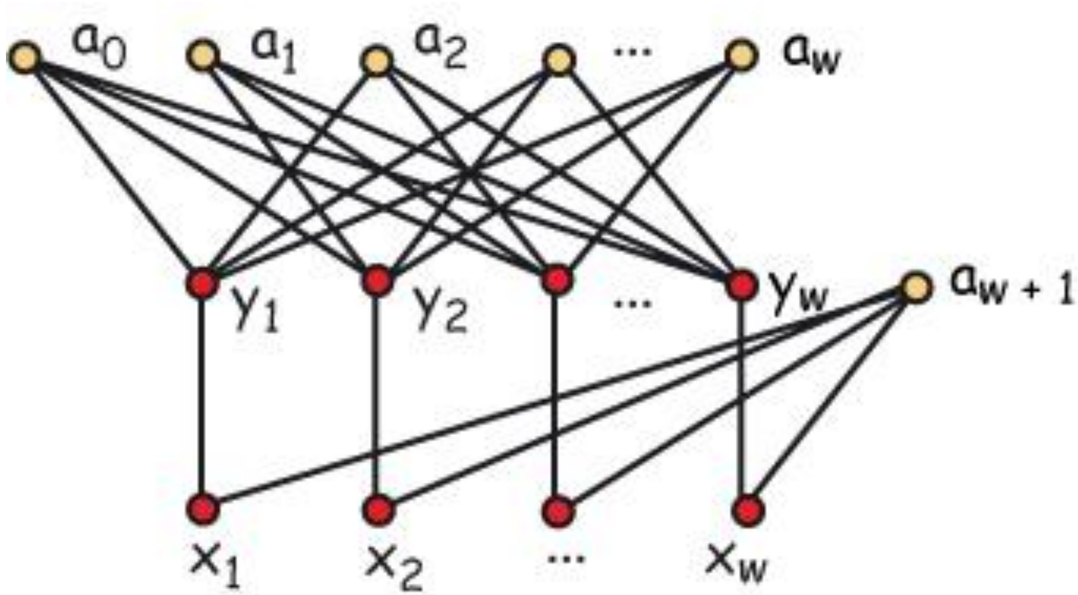
Theorem (Trotter and Wang, 2014) If $\dim(P) = d \geq 3$, there is a matching of size d in the comparability graph of P.

Theorem (Trotter and Wang, 2014) If $\dim(P) = d \geq 3$, there is a matching of size d in the incomparability graph of P.

Corollary (Hiraguchi, 1951) If P is a poset on n points and $n \geq 4$, then $\dim(P) \leq n/2$.

Some Ideas Behind The Proofs (1)

Theorem (Trotter, 1975) For every $w \geq 1$, if P is a poset and $\mathrm{width}(P - \max(P)) = w$, then $\dim(P) \leq w + 1$.



Some Ideas Behind The Proofs (2)

Theorem (Trotter and Wang, 2014+) For every $w \geq 2$, if P is a poset, $\text{width}(P - \max(P)) = w$, $P - \max(P) = C_1 \cup C_2 \cup \ldots \cup C_w$ with $|C_w| = 1$, then

$$\dim(P) \leq w.$$

Theorem (Trotter and Wang, 2014+) For every $w \geq 2$, if P is a poset, A is a subset of $\max(P)$ and $P - A$ has a complete matching of size w which is maximum in P, then

$$\dim(P) \leq w.$$

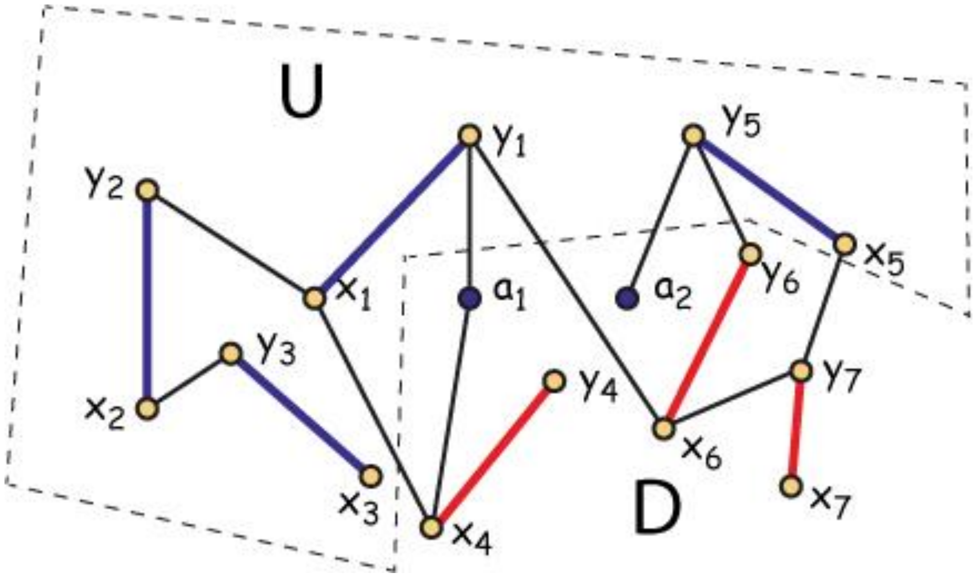
Some Ideas Behind The Proofs (3)

Theorem (Trotter and Wang, 2014+) If $P = U \cup D$ is a partition into an up set U and a down set D, then

$$\dim(P) \leq \mathrm{width}(U) + \dim(D).$$

Some Ideas Behind The Proofs (4)

Overview Look for a suitable maximum matching in P. This maximum matching splits the partitions the poset into an up set U and its complement D. The dimension of D is at most 3 and the width of U is at most 4. So $\dim(P) \leq 3 + 4 = 7$.



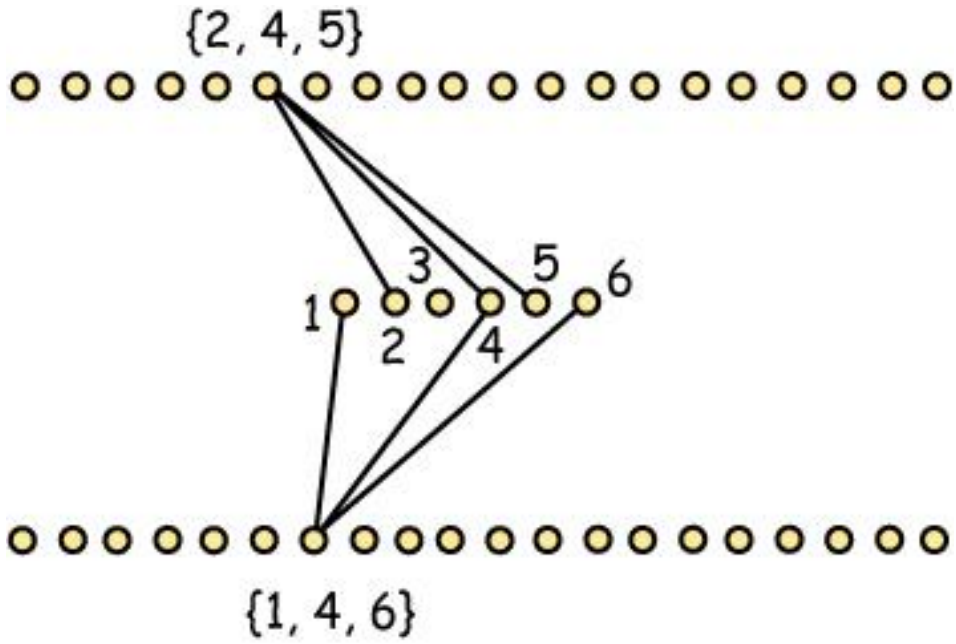
Comparability Graphs – Not Cover Graphs

Theorem (Trotter and Wang, 2014+) For every $n \geq 1$, there is a poset of dimension \binom(2n, n) for which the maximum size of a matching in the cover graph has size $2n$.

Theorem (Trotter and Wang, 2014+) If P is a poset and the maximum size of a matching in the cover graph of P has size m, then

$$\dim(P) \leq (5^m + 2m)/2.$$

Coding the Standard Example



Example Points in {1,2, …, 2n} in middle. Minimal element for every n-element subset. Same for maximal elements

Matchings in Graphs

Observation A point not in the matching has a signature from a set of size 5^m.

Fact WLOG, no two points have same signature.

Conclusion P has size at most $5^m + 2m$.