(1)

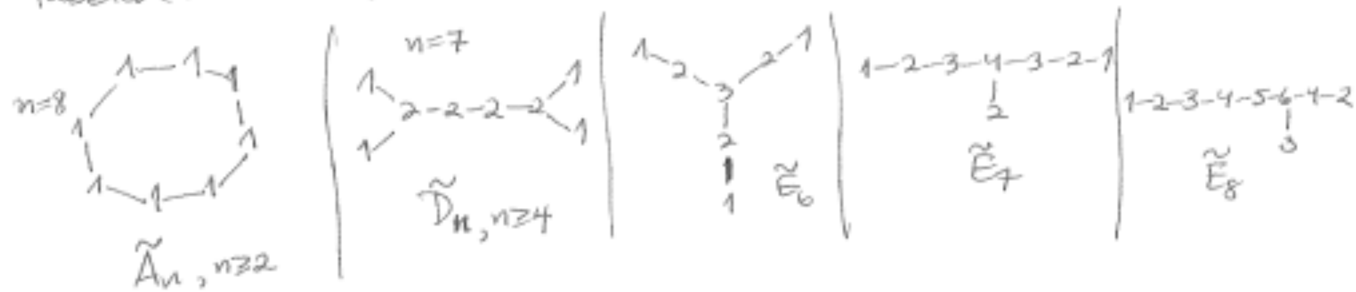
McKay Correspondence   Feb. 1, 2016

(Refs: Yi Sun, Sternberg "Fin. subgps of SU_2, ...")

Let's classify finite subgroups $G \hookrightarrow SL_2(\mathbb{C}) = SL(V)$, $V = \mathbb{C}^2$ by proving two things...

(Old) THM: Γ a connected finite graph has a vertex-labeling $f: V \to \{1, 2, \dots\}$ which is $\boxed{\text{additive } 2f(v) = \sum_{\substack{w \in V: \\ w - v}} f(w)}$

$\iff \Gamma$ is an $\boxed{\widetilde{ADE} \text{ (simply-laced affine) Dynkin diagram}}$ labeled as follows, up to scaling:



(McKay ~1980) THM: For a finite subgroup $G \hookrightarrow SL_2(\mathbb{C})$, its McKay quiver Γ having vertex set $\mathrm{Irr}(G) = \{\chi_0, \chi_1, \dots, \chi_\ell\}$ (irred. $\mathbb{C}$ G-characters, $\chi_0 = 1_G$), and m_{ij} arcs $\chi_i \xrightarrow{m_{ij}} \chi_j$ if $\boxed{\chi_i \otimes \chi_V = \sum_{j=0}^{\ell} m_{ij} \chi_j}$

is always connected, loopless, and $m_{ij} = m_{ji} \in \{0, 1\}$ (except $m_{12}=2$ in affine A_1) (so it corresponds to a graph Γ)

$\chi_i \rightleftarrows \chi_j$

and $f(\chi_i) = \deg(\chi_i) := \chi_i(e)$ is additive, so Γ is $\widetilde{ADE}$ as above.

EXAMPLE: $G = \mathbb{Z}/n\mathbb{Z} \longrightarrow SL_2(\mathbb{C}) = SL(V)$ has $\mathrm{Irr}(G) = \{\chi_0, \chi_1, \chi_2, \dots, \chi_{n-1}\}$ (all degree 1)

$\{e, g, g^2, \dots, g^{n-1}\} \quad g \longmapsto \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}$, $\zeta := e^{2\pi i/n}$

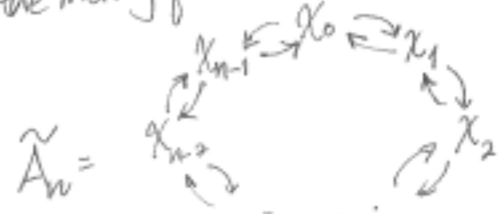
$\chi_0 = 1_G$ where $\chi_1(g) = \zeta^1$

$\chi_i(g) = \zeta^i$

$\chi_V = \chi_1 + \chi_{n-1}$

$\chi_i \otimes \chi_V = \chi_{i-1} + \chi_{i+1}$ (subscripts mod n)

and the McKay quiver for V is



(2)

proof of THM 1:

- If Γ has a cycle, it is that cycle:

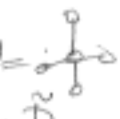
$2f(v_i) = v_{i-1} + v_{i+1} + \dots$ mod k

$\geq v_{i-1} + v_{i+1}$

$\Rightarrow 2\sum_{i=0}^{k-1} f(v_i) \geq \sum_{i=0}^{k-1} f(v_i) + \sum_{i=0}^{k-1} f(v_i)$

$\Rightarrow$ equality everywhere, so no neighbors outside the cycle.

So WLOG, Γ is a tree.

- There are no vertices of degree ≥ 5, and if one has degree 4, then $\Gamma =$  $\tilde{D}_4$:

$\Rightarrow 2f(v) = f(v_1) + \dots + f(v_5) + \dots$

$\geq \sum_{i=1}^{d} f(v_i)$

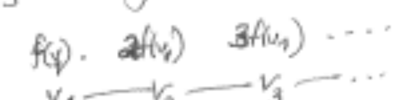
and $2\sum_{i=1}^{d} f(v_i) = 5f(v) + \dots \geq 5f(v)$

$\rightarrow \frac{d}{4} f(v) \leq \frac{1}{2}\sum_{i=1}^{d} f(v_i) \leq f(v)$

$\Rightarrow$ contradiction unless $d \leq 4$, and if $d=4$ it forces equality everywhere, so no other vertices.

So WLOG, Γ has max degree ≤ 3.

- There exist vertices of degree 3, else Γ is a path, forcing this labeling:



Contradiction to finiteness!

- If there are 2 vertices of degree 3, connected by a path, then there is nothing else, i.e. it is $\tilde{D}_n$:

by this calculation:

(3)

$$\begin{array}{c} a \\ b \end{array} \!\!>\! x_1 - x_2 - \cdots - x_{k-1} - x_k \!<\! \begin{array}{c} c \\ d \end{array}$$

$$\left.\begin{array}{l} 2a \geq x_1 \\ 2b \geq x_1 \\ 2c \geq x_k \\ 2d \geq x_k \end{array}\right\} \Rightarrow 2(a+b+c+d) \geq 2(x_1+x_k)$$

$$a+b+c+d \underset{(*)}{\geq} x_1 + x_k$$

Also

$$\begin{aligned} 2x_1 &= a+b+x_2 \\ 2x_2 &= x_1+x_3 \\ 2x_3 &= x_2+x_4 \\ 2x_4 &= x_3+x_5 \\ &\vdots \\ 2x_{k-1} &= x_{k-2}+x_k \\ 2x_k &= c+d+x_{k-1} \end{aligned}$$

$$2\sum_{i=1}^{k} x_i = a+b+c+d+2\sum_{i=1}^{k} x_i - (x_1+x_k)$$

$$\Rightarrow x_1+x_k = a+b+c+d \underset{(*)}{\geq} x_1+x_k$$

forces equality, so no other vertices

So WLOG, Γ has only one deg 3 vertex.

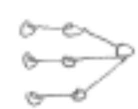
It looks like this:

$$\begin{array}{l} a - 2a - 3a - \cdots\cdots - (i-1)a \\ \qquad b - 2b - \cdots\cdots - (j-1)b \\ \qquad\quad c - 2c - \cdots\cdots - (k-1)c \end{array} \!\!>\! x = ia = jb = kc$$

with $a \leq b \leq c$
and $i \geq j \geq k$

Then $2x = 3x - (a+b+c)$

i.e. $\underset{kc}{\overset{}{x}} = a+b+c \leq 3c \Rightarrow k \leq 3$

- If $k=3$, then $x = 3c = a+b+c$ forces $a=b=c$ and $\Gamma = \tilde{E}_6$ 

- If $k=2$, then $c = a+b$ and $\Gamma =$

$$\begin{array}{l} a - 2a - 3a - \cdots - (i-1)a \\ \qquad b - 2a - \cdots - (j-1)b \end{array} \!\!>\! \underset{\substack{\| \\ ia \\ \| \\ jb}}{2a+2b} - a+b$$

and $jb = 2a+2b \leq 4b$

$\Rightarrow j \leq 4$. If $j=4$ then $a=b$ and $\Gamma = \tilde{E}_7$

If $j=3$, then $3b = jb = 2a+2b$

so $b = 2a$

$c = 3a$

and $\Gamma = \tilde{E}_8$

$$\begin{array}{l} a - 2a - 3a - 4a - 5a \\ \qquad\qquad\qquad\qquad 2a - 4a \end{array} \!\!>\! \underset{2a+2b}{\overset{}{6a}} - \underset{a+b}{3a}$$

(4)

Before proving THM 2,
let's note some general things about the McKay matrix $(m_{ij}) =: M$
for a rep'n $G \longrightarrow GL_n(\mathbb{C}) = GL(V)$ i.e. $\chi_i \otimes \chi_V = \sum_{j=0}^{\ell} m_{ij}\chi_j$

PROP: If $\chi_{i^*} = \bar{\chi}_i$ for $i = 0, 1, \ldots, \ell$ then $m_{ji} = m_{i^* j^*}$
(i.e. $M^T = PMP$ for a certain involutive perm. matrix $P = P^T = P^{-1}$ enacting $i \mapsto i^*$)

proof: $m_{ji} = \langle \chi_j \otimes \chi_V, \chi_i \rangle_G = \frac{1}{|G|}\sum_{g \in G} \chi_j(g)\chi_V(g)\chi_i(g^{-1})$
$= \frac{1}{|G|}\sum_{g \in G} \chi_{j^*}(g^{-1})\,\chi_V(g)\,\chi_{i^*}(g) = \underbrace{\langle \chi_{i^*} \otimes \chi_V, \chi_{j^*} \rangle}_{= m_{i^* j^*}}$ ∎

PROP: Each column $\begin{bmatrix} \chi_0(g) \\ \chi_1(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}$ of the character table for G
is an M-eigenvector, with eigenvalue $\chi_V(g)$

proof: $\underbrace{\chi_i(g)\,\chi_V(g)}_{i^{th} \text{ entry of } \chi_V(g)\begin{bmatrix} \chi_0(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}} = \underbrace{\sum_{j=0}^{\ell} m_{ij}\chi_j(g)}_{i^{th} \text{ entry of } M\begin{bmatrix} \chi_0(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}}$ for $i = 0, 1, \ldots, \ell$ ∎

COR: The vector $\delta = \begin{bmatrix} \chi_0(e) \\ \vdots \\ \chi_\ell(e) \end{bmatrix}$ of irreducible degrees satisfies $n\delta = M\delta$

Also, this n-eigenspace $\ker(M - n\cdot I) = \mathbb{C}\delta$ is <u>simple</u> if it is a <u>faithful</u> rep'n $G \hookrightarrow GL_n(\mathbb{C})$

proof: Since the columns $\begin{bmatrix} \chi_0(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}$ give a basis of eigenvectors,
one only needs to check $\chi_V(g) = n$ implies $g = e$:
$n = \chi_V(g) = \sum_{i=1}^{n} \lambda_i$ if g has eigenvalues $\lambda_1, \ldots, \lambda_n$ on V
$\Rightarrow n = \left|\sum_{i=1}^{n}\lambda_i\right| \underset{\text{Cauchy-Schwarz}}{\leq} \sum_{i=1}^{n}|\lambda_i| = \underbrace{1+1+\ldots+1}_{n \text{ times}} = n \Rightarrow \lambda_i = 1\ \forall i \Rightarrow g$ acts as 1_V
$\overset{\text{faithful}}{\Longrightarrow} g = e$ ∎

(5)

<u>PROP</u> (Burnside) If $\underset{\text{finite}}{G} \hookrightarrow GL_n(\mathbb{C}) = GL(V)$ is faithful, then every G-irreducible χ_i appears in some tensor power $T^m(V) = \underbrace{V \otimes \cdots \otimes V}_{m \text{ times}}$,
(so $\exists$ a path $\underbrace{\chi_0 \to \cdots\cdots \to \chi_i}_{m \text{ steps}}$ in the McKay quiver Γ)

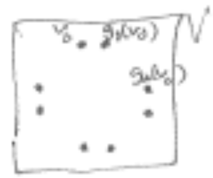
<u>proof</u>: 1st find a vector $v_0 \in V$ whose G-orbit is <u>free</u>, i.e. $g(v_0) \neq v_0$ unless $g = e$:

$$\text{pick any } v_0 \in V \setminus \underbrace{\bigcup_{g \in G \setminus \{e\}} \underbrace{\ker(g - 1_V)}_{\text{a proper subspace } \subsetneq V}}_{\text{a finite union of proper subspaces } \subsetneq V}$$

Then consider

$$\begin{array}{ccccccc}
T^\bullet(V^*) & \twoheadrightarrow & S^\bullet(V^*) \cong \mathbb{C}[x_1, \ldots, x_n] & \twoheadrightarrow & \left\{ \begin{matrix} \text{functions} \\ \varphi : G \to \mathbb{C} \end{matrix} \right\} \cong \underset{\substack{\text{regular} \\ \text{rep'n}}}{\mathbb{C}G} & \supseteq & \chi_i^* \\
\| & & \uparrow & & & & \\
\bigoplus_{m \geq 0} T^m(V^*) & & V^* = \mathbb{C}x_1 + \cdots + \mathbb{C}x_n \quad f(x) & \longmapsto & (\varphi(g) = f(g(v_0))) & &
\end{array}$$

surjectivity is an exercise in polynomial interpolation



<u>RMK</u>: If $\underset{\text{finite}}{G} \hookrightarrow GL(V)$ has only t <u>different</u> character values $\{\chi_V(g)\}_{g \in G}$
then Brauer showed every χ_i appears in some $T^m(V)$ with $m \leq t-1$ (?).

Now specialize to $\underset{\text{finite}}{G} \hookrightarrow SL_2(\mathbb{C}) = SL(V)$
and note that • $\delta = \begin{bmatrix} \chi_0(e) \\ \chi_1(e) \\ \vdots \\ \chi_k(e) \end{bmatrix}$ is <u>additive</u> on the vertices of Γ by the eigenvalue equation $2\delta = M\delta$

$$2\delta_i = \sum_{j=0}^{k} m_{ij} \delta_j$$

• $m_{ij} = m_{ji}$ since $V \cong V^*$ as every g diagonalizes to $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ with $\lambda^{-1} = \bar{\lambda} \in \mathbb{C}^\times$
so $\chi_V = \lambda + \lambda^{-1} = \chi_{V^*}$

Hence it only remains for THM 2
to show $\begin{cases} m_{ii} = 0 \\ \text{and} \\ m_{ij} \in \{0, 1\} \end{cases}$

PROP: $m_{ij} \in \{0,1\}$ for $\underset{\text{finite}}{G} \to GL_2(\mathbb{C})$ any ~~nontrivial~~ rep'n with nonscalar matrices in its image.

proof:

$$m_{ij} = |m_{ij}| = \left| \frac{1}{|G|} \sum_{g\in G} \chi_i(g)\chi_V(g)\overline{\chi_j(g)} \right|$$

$$\leq \frac{1}{|G|} \sqrt{\sum_{g\in G} |\chi_i(g)|^2 \underbrace{|\chi_V(g)|^2}_{\leq 2^2}} \cdot \sqrt{\sum_{g\in G} |\chi_j(g)|^2}$$

since $\chi_V(g) = \lambda + \lambda^{-1}$, $|\lambda| = 1$

since its a ~~nontrivial~~ rep'n with nonscalar matrices in its image.

$$< \frac{2}{|G|} \sqrt{\sum_{g\in G} |\chi_i(g)|^2} \sqrt{\sum_{g\in G} |\chi_j(g)|^2} = \frac{2}{|G|} \sqrt{|G|}\cdot\sqrt{|G|} = 2 \qquad \blacksquare$$

REMARK: For subgroups G of SL_2(C), containing only scalar matrices forces G={ I_2, -I_2 } which is type affine A_1, and in this case, indeed the McKay quiver has two nodes 1,2 and m_{12}=2.

PROP: $m_{ii} = 0$ for $\underset{\text{finite}}{G} \hookrightarrow SL_2(\mathbb{C}) = SL(V)$

proof: If V is reducible, then $g \longmapsto \left[\begin{array}{c|c} \lambda(g) & 0 \\ \hline 0 & \lambda(g)^{-1} \end{array}\right]$ forcing G cyclic, and of type $\tilde{A}_n$ as we saw

If V is irreducible, then $2 = \dim_{\mathbb{C}}(V)$ divides $|G|$ by a famous (nontrivial) result about irred. characters

so G contains an element of order 2 by Cauchy's Thm., and in $SL_2(\mathbb{C})$ this has to be $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

Since $-I \in Z(G)$, it acts as a scalar ϵ in any irreducible χ_i (ϵ =+1 or -1)

so $\chi_i(-g) = \epsilon\chi_i(g)$ $\forall g \in G$, and hence

$$m_{ii} = \frac{1}{|G|} \sum_{g\in G} \chi_i(g)\chi_V(g)\overline{\chi_i(g)} = \frac{1}{|G|} \sum_{g\in G} |\chi_i(g)|^2 \chi_V(g)$$

$$2m_{ii} = \frac{1}{|G|} \sum_{g\in G} \left(\chi_V(g)|\chi_i(g)|^2 + \chi_V(-g)|\chi_i(-g)|^2 \right)$$

$$= \frac{1}{|G|} \sum_{g\in G} \chi_V(g) \underbrace{\left(|\chi_i(g)|^2 - \underbrace{|\epsilon\chi_i(g)|^2}_{|\chi_i(g)|^2} \right)}_{=0} \qquad \blacksquare$$

(7)

<u>DEF'N:</u> Given a faithful rep'n $G \overset{\gamma}{\hookrightarrow} Gl_n(\mathbb{C})$

define its <u>McKay-Cartan</u> <u>matrix</u> $\tilde{C} := nI_{\ell+1} - M$

and Cartan matrix $C := \tilde{C} - \left\{ \begin{smallmatrix} \chi_0 \text{ row,} \\ \text{column} \\ \text{removed} \end{smallmatrix} \right\}$

and <u>critical</u> <u>group</u> $K(\gamma) := \mathrm{coker}\left(\mathbb{Z}^\ell \overset{C}{\hookrightarrow} \mathbb{Z}^\ell\right)$

$= \mathbb{Z}^\ell / \mathrm{im}\, C$

$$\tilde{C} = \begin{array}{c|cccc} & \chi_0 & \chi_1 & \cdots & \chi_\ell \\ \hline \chi_0 & * & * & \cdots & * \\ \chi_1 & * & & & \\ \vdots & \vdots & & C & \\ \chi_\ell & * & & & \end{array}$$

<u>PROP:</u> Equivalently,

(see PROP 2.19 in paper with Benkart & Klivans)

$$\mathbb{Z} \oplus \underbrace{K(\gamma)}_{\substack{\text{the} \\ \text{torsion} \\ \text{part of} \\ \text{the cokernel}}} \cong \mathrm{coker}\left(\mathbb{Z}^{\ell+1} \xrightarrow{\tilde{C}} \mathbb{Z}^{\ell+1}\right)$$

$$\text{and} \quad K(\gamma) \cong \left(\delta^{(e)}\right)^{\perp} / \mathrm{im}(\tilde{C})$$

<u>EXAMPLE:</u> $G = \mathfrak{A}_4 =$ alternating group in $\mathfrak{S}_4 \xrightarrow{\gamma = \text{rotational symmetries of tetrahedron}} SO_3(\mathbb{R}) \; (\subset Gl_3(\mathbb{C}))$

<u>Character</u> <u>table</u>:

$\omega := e^{\frac{2\pi i}{3}}$

	e	(123)	(132)	$(12)(34)$
$1_G = \chi_0$	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
$\left(\underset{\text{for } i=0,1,2}{\chi_i \chi_3 =}\right) \chi_\gamma = \chi_3$	3	0	0	-1
$2\chi_3 + \chi_0 + \chi_1 + \chi_2 = \chi_3^2$	9	0	0	1

$$M = (m_{ij}) = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

$$\tilde{C} = 3I_4 - M = \begin{array}{c} \\ \chi_0 \\ \\ \\ \\ \end{array} \begin{array}{c} \chi_0 \quad\quad\quad\quad \\ \begin{bmatrix} 3 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \end{array}, \quad C = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$\det \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} = 3$

$$K(\gamma) = \mathrm{coker}\left(\mathbb{Z}^3 \xrightarrow{C = \left[\begin{smallmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{smallmatrix}\right]} \mathbb{Z}^3\right)$$

$$= \mathrm{coker}\left(\mathbb{Z}^3 \xrightarrow{\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right]} \mathbb{Z}^3\right)$$

$$\cong \mathbb{Z}/3\mathbb{Z}$$

(8)

Some properties of $K(\gamma)$

- If $G \overset{\gamma}{\hookrightarrow} SL_n(\mathbb{C})$, not just $GL_n(\mathbb{C})$

  then $\exists$ a surjection $K(\gamma) \overset{\pi}{\twoheadrightarrow} \hat{G} := \mathrm{Hom}(G, \mathbb{C}^\times) \cong \underbrace{G^{ab} = G/[G,G]}_{\text{abelianization of } G}$

EXAMPLE: For $Cl_4 \overset{\gamma}{\hookrightarrow} SO_3(\mathbb{R}) \subset SL_3(\mathbb{C})$ above

$$K(\gamma) \cong \mathbb{Z}/3\mathbb{Z}$$
$$\pi \downarrow \wr$$
and $$Cl_4^{ab} \cong \mathbb{Z}/3\mathbb{Z}$$

EXAMPLE: For McKay's original setting, where $\underset{\text{finite}}{G} \overset{\gamma}{\hookrightarrow} SL_2(\mathbb{C})$

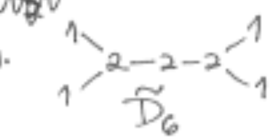
one always has an isomorphism

$$K(\gamma) \cong G^{ab}$$

but also $\quad \|$

$\underset{\substack{\text{usual Cartan matrix} \\ \text{expressing the simple roots } \{\alpha_1,\dots,\alpha_n\} \\ \text{in the basis of fund'l weights } \{\lambda_1,\dots,\lambda_n\}}}{\mathrm{coker}(C)} \cong$ "fundamental group" of Φ: $\underset{\text{weight lattice}}{P(\Phi)} \Big/ \underset{\text{root lattice}}{Q(\Phi)} \cong \pi_1$ (compact adjoint form of the semisimple Lie group)

e.g.



$\tilde{D}_6$

$$\tilde{C} = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

$$G^{ab} \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } G \leftrightarrow \tilde{D}_n \text{ } n \text{ even} \\ \mathbb{Z}/4\mathbb{Z} & \text{if } G \leftrightarrow \tilde{D}_n \text{ } n \text{ odd} \end{cases}$$

- $\mathbb{Z} \oplus K(\gamma) = \mathrm{coker}(\tilde{C})$ is a naturally a ring

  and $K(\gamma)$ ~~itself an~~ ideal in this ring:

PROP: Considering the representation ring $R(G) := \mathbb{Z}^{\mathrm{Irr}(G)}$ having $\underset{\text{as } \mathbb{Z}\text{-basis}}{\{\overset{1}{e_0}, e_1, \dots, e_\ell\}} \overset{\epsilon}{\to} \mathbb{Z}$, $\chi \mapsto \underset{\deg \chi}{\chi(e)}$

and $e_i \cdot e_j = \sum_{k=0}^{\ell} c_{ij}^k e_k$ if $\chi_i \otimes \chi_j = \sum_{k=0}^{\ell} c_{ij}^k e_k$,

then $G \overset{\gamma}{\hookrightarrow} GL_n(\mathbb{C})$ has $\mathrm{coker}(\tilde{C}) \cong R(\gamma) := R(G) \Big/ (n - e_\gamma)$ where $e_\gamma = \sum_{k=0}^{\ell} c_k e_k$ if $\chi_\gamma = \sum_{k=0}^{\ell} c_k \chi_k$

proof: $n - e_\gamma$ acts on the $\mathbb{Z}$-basis $\{e_0, e_1, \dots, e_\ell\}$ for $R(G)$ via the matrix $\tilde{C}$ ∎

(9)

EXAMPLE: $R(\underset{\mathfrak{A}_4}{G}) \cong \mathbb{Z}\{\overset{1}{\chi_0}, \overset{x}{\chi_1}, \overset{x^2}{\underset{\chi_1\otimes\chi_1}{\chi_2}}, \overset{y}{\chi_3}\} \cong \mathbb{Z}^4$

$$\cong \mathbb{Z}[x,y]/(\underline{x^3}-1,\ \underline{xy}-y,\ \underline{y^2}-(2y+1+x+x^2))$$

So $\mathfrak{A}_4 \xrightarrow{\gamma} SO_3(\mathbb{R})$

has $R(\gamma) = \mathbb{Z}[x,y]/(x^3-1,\ xy-y,\ y^2-(2y+1+x+x^2),\ 3-y)$

$$\cong \mathbb{Z}[x]/(x^3-1,\ 3(x-1),\ \underbrace{9-(6+1+x+x^2)}_{-(x^2+x-2)})$$

$$\cong \mathbb{Z}[x]/(3(x-1),\ (x-1)^2)$$

$$\cong \mathbb{Z}[u]/(3u,u^2) = \mathbb{Z}\cdot 1 \oplus (\mathbb{Z}/3\mathbb{Z})u$$

$$= \mathbb{Z}\oplus\underbrace{\mathbb{Z}/3\mathbb{Z}}_{K(\gamma)} = \mathrm{coker}(\tilde{C})$$

• If G is abelian, so $G \cong \hat{G} = \mathrm{Irr}(G)$ $\qquad \tilde{C}$ = usual digraph Laplacian
then any rep'n $G \xrightarrow[\sum a_k\chi_k]{\gamma} GL_n(\mathbb{C})$ has $K(\gamma) \cong$ usual digraph critical group
for the Cayley digraph of $(\hat{G}, \{a_k \text{ copies } \chi_k \text{ for } k=0,1,\dots,\ell\})$

EXAMPLE: $G = (\mathbb{Z}/2\mathbb{Z})^n \xrightarrow{\gamma} GL_n(\mathbb{C})$
$= \langle g_1,\dots,g_n : g_i^2 = e\rangle$

$$g_i \longmapsto \begin{bmatrix} 1 & & & & & & 0 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & \boxed{-1} & & \\ & & & & & 1 & \\ 0 & & & & & & \ddots\ 1 \end{bmatrix}$$

has $K(\gamma) = K(\overset{Q_n}{n\text{-cube}})$

so $\mathbb{Z}\oplus K(\gamma) \cong \mathbb{Z}[x_1,\dots,x_n]/(x_1^2-1,\dots,x_n^2-1,\ n-(x_1+\dots+x_n))$

Q: Does this help to understand the 2-primary structure of $K(Q_n)$?
(Hua Bai computed the p-primary structure for odd p, which is much easier.)

Q: What does $K(\chi^\lambda)$ for $\mathfrak{S}_n \xrightarrow{\chi^\lambda} GL_{f^\lambda}(\mathbb{C})$ look like?
$\lambda \neq (4), (1,1,1,1), (2,2)$, $n=4$

(1)

T. Douvropoulos Geometric McKay Correspondence 2/22/2016

Kepler: $\underbrace{\text{Saturn Jupiter}}_{\text{cube}}$ $\underbrace{\text{Jupiter Mars}}_{\text{tetrahedron}}$ $\underbrace{\text{Mars Earth}}_{\text{dodecahedron}}$ $\underbrace{\text{Earth Venus}}_{\text{icosahedron}}$ $\underbrace{\text{Venus Mercury}}_{\text{octahedron}}$

Algebraic McKay correspondence

$$\underset{\text{finite}}{\Gamma} \subset SL(2,\mathbb{C}) \longrightarrow \text{extended Dynkin diagram}$$
$$\text{irreps} \longrightarrow \text{vertices}$$

Geometric McKay correspondence

$$\underset{\text{finite}}{\Gamma} \subset SL(2,\mathbb{C}) \longrightarrow \underset{\substack{\text{unique}\\ \text{singularity}\\ \text{embedded in } \mathbb{C}^3}}{\mathbb{C}^2/\Gamma} \longrightarrow \underset{\text{Dynkin diagram}}{(\text{nonextended})}$$

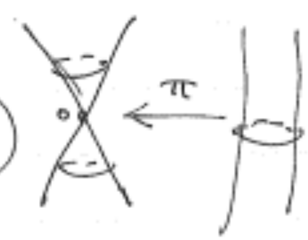
$$\text{irred. components of the exceptional divisor} \longrightarrow \text{vertices}$$

<u>Two pictures:</u>

A_1: $C_2 = \{\pm 1\} \subset \cancel{}\, SL(2,\mathbb{C})$

defining eqn. of $\mathbb{C}^2/\Gamma$ $x^2 - yz = 0$

(equiv. to $x^2 - y^2 - z^2 = 0$)

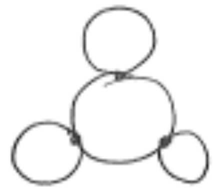


$\pi^{-1}(0) =$ 

$A_1 = \circ$ Dynkin diagram

D_4: defining eqn. of $\mathbb{C}^2/\Gamma$ $x(y^2 - x^2) + z^2 = 0$

(2)

$E = \pi^{-1}(0) =$  $\xrightarrow[\text{diagram}]{\text{intersection}}$  D_4 Dynkin diagram

§1 Invariant theory

Finite group G acts via lin. transformation on $V = \mathbb{C}^n$,

also acts on $\mathbb{C}[V] := \mathbb{C}[x_1, \ldots, x_n]$

via $g.f = f(g^{-1}x)$

Consider invariant subalgebra $\mathbb{C}[V]^G = \{ f \in \mathbb{C}[V] : f(x) = f(g^{-1}x) \ \forall g \in G \}$

FACT (Noether-Hilbert) $\mathbb{C}[V]^G$ is generated by finitely many polynomials

FACT (Klein-DuVal)

$\underset{\text{finite}}{\Gamma} \subset SL(2,\mathbb{C}) \Rightarrow \mathbb{C}[V]^G$ is gen'd by exactly 3 polynomials

EXAMPLE 1: $\Gamma = C_N$ gen'd by $\begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}$, $\zeta = e^{2\pi i/N}$

$\begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}(x, y) = (\zeta^{-1}x, \zeta y)$

Invariants: $\underset{f_2}{x^N}, \underset{f_3}{y^N}, \underset{f_1}{xy}$ $\qquad f_1^N - f_2 f_3 = 0$

G.I.T. (geom. invariant theory) say $\mathbb{C}[V]^\Gamma$ and V/Γ are deeply related

Indeed, $\mathbb{C}[f_1, f_2, f_3] \hookrightarrow \mathbb{C}[x,y]$ induces a map

$$\mathbb{C}^3 \longleftarrow \mathbb{C}^2$$

$$(f_1(x,y), f_2(x,y), f_3(x,y)) \longleftarrow\!\shortmid (x,y)$$

$$(x^N, y^N, xy)$$

(3)

This map $\mathbb{C}^2 \to \mathbb{C}^3$ realizes $\mathbb{C}^2/\Gamma$ as a topological quotient

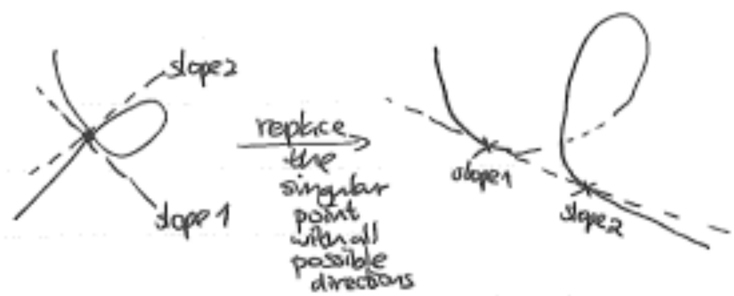
EXAMPLE: $\mathbb{C}^2 \longrightarrow \mathbb{C}^3$ $\quad \Gamma = C_N$

$(x,y) \longmapsto (xy, x^N, y^N)$

Image in $\mathbb{C}^3$ with coords (x,y,z) is cut out by $x^N - yz$

§2 | Resolution of singularity

Recall



Resolving the singularity means finding $\underset{\text{smooth}}{\tilde{S}}$ and $\tilde{S} \xrightarrow{\pi} S$ such that $\tilde{S} \setminus \pi^{-1}(0) \cong S \setminus \{0\}$

Consider $B(0,\mathbb{C}^n) = \{(v,L) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid v \in L\}$

$\underline{n=3}$: $\{((x,y,z), [s:t:u]) : xt = sy,\ xu = sz,\ tz = uy\}$

$$\begin{tikzcd} & B(0,\mathbb{C}^n) \arrow[dl, "pr_1 = \beta"'] \arrow[dr, "pr_2"] & \\ \mathbb{C}^n & & \mathbb{P}^{n-1} \end{tikzcd}$$

$\leftarrow$ tells us $B(0,\mathbb{C}^n)$ is a bundle, so smooth, for example

$\beta^{-1}(\mathbb{C}^n \setminus \{0\}) \cong \mathbb{C}^n \setminus \{0\}$

"blow-up at 0"

Given $S \subset \mathbb{C}^n$ singular at 0,
then the blow-up of $0 \in S$ is $B(0,S) := \overline{\beta^{-1}(S \setminus \{0\})}$ $\leftarrow$ Zariski closure in $\beta(0,\mathbb{C}^n)$

Exceptional divisor $E := B(0,S) \cap \beta^{-1}(0)$

(4)

Affine charts for $n=3$:

$$\begin{array}{c|c} & (x,t,u) \longmapsto ((x,xt,xu),[1:t:u]) \\ & \tilde{\mathcal{U}}_1 \hookrightarrow B(0,\mathbb{C}^3) \\ & \downarrow\beta \\ & \mathbb{C}^3 \\ & (x,t,u) \mapsto (x,xt,xu) \\ \hline \tilde{\mathcal{U}}_2 \hookrightarrow B(0,\mathbb{C}^3) & \tilde{\mathcal{U}}_3 \hookrightarrow B(0,\mathbb{C}^3) \\ \downarrow & \downarrow \\ \mathbb{C}^3 & \mathbb{C}^3 \\ (s,y,u) \mapsto (ys,y,yu) & (s,t,z) \mapsto (sz,tz,z) \end{array}$$

Resolution of type A_N, $\Gamma = C_{N+1}$

singularity has equation $x^{N+1} - yz = 0$ in $\mathbb{C}^3$,

equivalent to $x^{N+1} - y^2 + z^2 = 0$

In $\tilde{\mathcal{U}}_1$, $x^{N+1} + (xt)^2 + (xu)^2 = 0$

$\Rightarrow x^2(x^{N-1} + t^2 + u^2) = 0$

If $x=0$, get all of $((0,0,0),[1:t:u])$, which corresponds to $\beta^{-1}(0) \cap \tilde{\mathcal{U}}_1$

$x^{N-1} + t^2 + u^2 = 0$ should give me $\overline{\beta^{-1}(S \setminus \{0\}) \cap \tilde{\mathcal{U}}_1}$

$N = 1 \Rightarrow 1 + t^2 + s^2 = 0$

E is given in $\tilde{\mathcal{U}}_1$ by $[1:t:s]$ such that $1+t^2+s^2=0$

(in $\tilde{\mathcal{U}}_2$ would get $s^2(ys)^{N-1} + 1 + u^2 = 0$,

and $N=1 \Rightarrow s^2+1+u^2=0$)

$\Rightarrow E_1$ is just $s^2+t^2+u^2=0$, a copy of $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$

$N > 1 \Rightarrow E \cap \tilde{\mathcal{U}}_1 = \{ t^2+u^2=0 \} = \{ [1:a:\pm ia] : a \in \mathbb{C} \}$

$E \cap \tilde{\mathcal{U}}_2 = \{ s^{N+1} y^{N-1} + u^2 = 0 \text{ at } y=0 \} = \{ [s:1:u] : 1+u^2=0 \}$

$= \{ [b:1:\pm i] \}$

So $E = \{[1:a:\pm ia]\}$ and $\{[b:1:\pm i]\}$

If $b = \frac{1}{a}$, $[1:a:\pm ia] = [b:1:\pm i]$

E has 2 lines, 

but we have not yet resolved the singularity if $n \geq 2$

since the equation $x^{N-1} + t^2 + u^2 = 0$ is still singular.

You proceed inductively, and keep blowing up to get

 $\longrightarrow$ type A_N Dynkin diagram

A schematic picture: (Reference: Givental "Reflection groups in Singularity Theory" Trans. Amer. Math. Soc. 153, 1992 – hard to access!)