The proof (in Appendix A.18) is rather technical, and we'll skip it.

But the idea is to try and show, given $\epsilon>0$ one can pick $N>0$ so that $U_N(f)-L_N(f)<\frac{\epsilon}{2}$

and then pick $N''>0$ so that $|U_{\mathcal{P}_{N'}}(f)-U_N(f)|\le\frac{\epsilon}{4}$

$|L_{\mathcal{P}_{N'}}(f)-L_N(f)|\le\frac{\epsilon}{4}$

by ensuring most P in $\mathcal{P}_{N''}$ lie inside a single dyadic cube $C\in\mathcal{D}_N(\mathbb{R}^n)$ and those that straddle dyadic cubes have total volume small enough, relative to $\sup\{|f(\bar{x})|:\bar{x}\in X\}$.

3/6/2017

4.8, 4.9 Determinants & volume

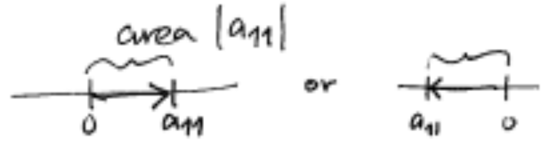
We know $\det A$ for a square matrix tells us when A is invertible ($\det A\neq 0$).

But what does it mean, as a number, when it's nonzero?

Recall for $n=1,2,3$ we interpreted already (back in §1.4 discussing cross-products) (from Chap. 1)

$|\det A|$ = volume of parallelepiped spanned by columns $\bar{v}_1,\ldots,\bar{v}_n$ of $A=\begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix}$

e.g. $n=1$ $A=[a_{11}]$



$n=2$

$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$

$A=\begin{bmatrix} | & | & | \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 \\ | & | & | \end{bmatrix}$

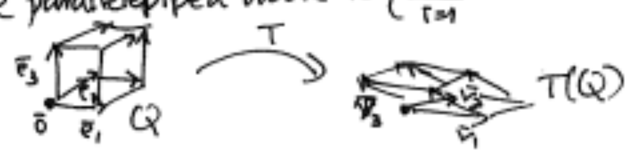
volume $=|(\bar{v}_1\times\bar{v}_2)\cdot\bar{v}_3|$

$=\left|\det\begin{bmatrix} | & | & | \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 \\ | & | & | \end{bmatrix}\right|$

Now we'll prove it in general, as it is the essence of the change-of-variable formula!

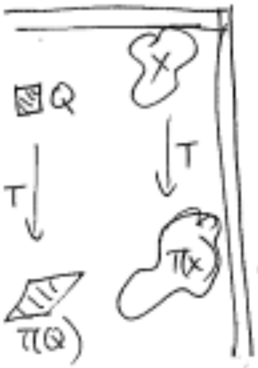
Note that if A is $n\times n$ matrix for $\mathbb{R}^n\xrightarrow{T}\mathbb{R}^n$ a linear transformation then the parallelepiped above is $\left\{\sum_{i=1}^n c_i\bar{v}_i : c_i\in[0,1]\right\}=T(Q)$ where $Q=\left\{\sum_{i=1}^n c_i\bar{e}_i : c_i\in[0,1]\right\}$

unit cube (of volume 1)



(63)

THM 4.9.1
PROP 4.9.4: Given $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ and $\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{T} & \mathbb{R}^n \\ \vec{x} & \longmapsto & A\vec{x} \end{array}$



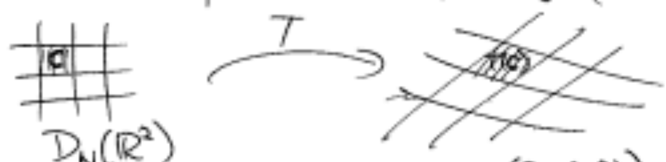
then $|\det A| \overset{(*)}{=} \mathrm{vol}_n T(Q)$

and more generally, for any pavable set $X \subset \mathbb{R}^n$, $T(X)$ is pavable,

with $|\det A|\, \mathrm{vol}_n X \overset{(**)}{=} \mathrm{vol}_n T(X)$

proof: Takes a little work; here's the plan:

(1st) Show the collections $T(\mathcal{D}_N(\mathbb{R}^n)) := \{T(C) : C \in \mathcal{D}_N(\mathbb{R}^n)\}$ give a nested partition of pavings (of $\mathbb{R}^n$):



$\mathcal{D}_N(\mathbb{R}^2)$ $\qquad$ $T(\mathcal{D}_N(\mathbb{R}^2))$

(2nd) Show every cube $C \in \mathcal{D}_N(\mathbb{R}^n)$ has $\dfrac{\mathrm{vol}_n T(C)}{\mathrm{vol}_n C} = \dfrac{\mathrm{vol}_n T(Q)}{\underbrace{\mathrm{vol}_n Q}_{1}} \left(= \mathrm{vol}_n T(Q)\right)$, i.e. $\boxed{\mathrm{vol}_n T(C) = \mathrm{vol}_n T(Q) \cdot \mathrm{vol}_n(C)}$

(3rd) Show every pavable $X \subset \mathbb{R}^n$ also has

$$\boxed{\mathrm{vol}_n T(X) = \mathrm{vol}_n T(Q) \cdot \mathrm{vol}_n(X)}$$

(4th) Show $\mathrm{vol}_n T(Q) \overset{(*)}{=} |\det A|$.

First note that W.L.O.G., T is invertible, else leftside of $(*)$, $(**)$ vanish because $\det A = 0$, and right side vanishes because $T(Q), T(X)$ lie in the proper subspace $\mathrm{img}(T) \subset \mathbb{R}^n$ and are bounded (e.g. since Q, X bounded and $|T\vec{x}| \le |T||\vec{x}|\ \forall \vec{x} \in \mathbb{R}^n$), so they have volume $\underline{0}$.

$\{T(C) : C \in \mathcal{D}_N(\mathbb{R}^n)\}$

1st step: To show the $T(\mathcal{D}_N(\mathbb{R}^n))$ are pavings of $\mathbb{R}^n$, and a nested sequence, need to check...

Checking each $T(\mathcal{D}_N(\mathbb{R}^n))$ is a paving of $\mathbb{R}^n$ (DEF'N 4.7.2)

1. $\bigcup_{P \in T(\mathcal{D}_N(\mathbb{R}^n))} P \overset{?}{=} \mathbb{R}^n$

$$\bigcup_{P \in T(\mathcal{D}_N(\mathbb{R}^n))} P = T\Big(\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n)} C\Big) = T(\mathbb{R}^n) = \mathbb{R}^n$$

$\leftarrow$ since T is invertible (so injective, surjective)

2. $\mathrm{vol}(P_1 \cap P_2) \overset{?}{=} 0$ if $P_1 \neq P_2$

$$\mathrm{vol}(T(C_1) \cap T(C_2)) = \mathrm{vol}(T(\underbrace{C_1 \cap C_2}_{=\emptyset}))$$

3. $A \subset \mathbb{R}^n$ bounded $\overset{?}{\Longrightarrow}$ A intersects only finitely many P

$\Downarrow \leftarrow$ T^{-1} can only dilate distances, lengths by at most $|T^{-1}|$: $|T^{-1}\vec{v}| \le |T^{-1}||\vec{v}|$

$T^{-1}(A)$ bounded

$\Downarrow$

$T^{-1}(A)$ intersects only finitely many $C \in \mathcal{D}_N(\mathbb{R}^n)$ $\Uparrow$ $P = T(C)$, $A = T(T^{-1}(A))$

4. $\mathrm{vol}(\partial P) \overset{?}{=} 0$

$$\mathrm{vol}(\partial P) = \mathrm{vol}(\partial T(C)) = \mathrm{vol}(T(\partial C)) = \mathrm{vol}(\underbrace{\text{union of translates of linear subspaces}}_{\text{finitely many}})$$

$\leftarrow$ because T, T^{-1} are linear, so continuous

5. Each P is pavable?

Yes, since $\mathbf{1}_P$ is discontinuous only at ∂P, which we just saw has $\mathrm{vol}_n \partial P = 0$ so measure 0, so $\mathbf{1}_P$ is integrable ✓.

Checking they form a nested family (DEF'N 4.7.3)

6. $T(\mathcal{D}_{N+1}(\mathbb{R}^n))$ refines $T(\mathcal{D}_N(\mathbb{R}^n))$?

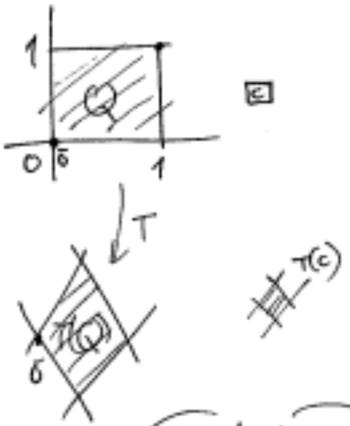
Yes, because $\mathcal{D}_{N+1}(\mathbb{R}^n)$ refines $\mathcal{D}_N(\mathbb{R}^n)$.

7. The $P \in T(\mathcal{D}_N(\mathbb{R}^n))$ shrink to points as $N \to \infty$?

Yes, because same is true for $C \in \mathcal{D}_N(\mathbb{R}^n)$, and T, T^{-1} can't dilate distances more than $|T|, |T^{-1}|$.

2nd step: Want to show each $C \in \mathcal{D}_N(\mathbb{R}^n)$ has

$$\mathrm{vol}_n(T(C)) \overset{?}{=} \mathrm{vol}_n T(Q) \cdot \mathrm{vol}_n(C)$$



$$\begin{aligned}
\mathrm{vol}_n(T(C)) &\overset{\text{if } C = C_{\vec{a},N}}{=} \mathrm{vol}_n(T(C_{\vec{a},N})) \\
&\overset{\text{translate } C \text{ so lower upper is at } \vec{0}}{=} \mathrm{vol}_n(T(C_{\vec{0},N})) \\
&= \mathrm{vol}_n\left(T\left(\tfrac{1}{2^N} Q\right)\right) \\
&\overset{T \text{ is linear}}{=} \mathrm{vol}_n\left(\tfrac{1}{2^N} T(Q)\right) \\
&= \left(\tfrac{1}{2^N}\right)^n \mathrm{vol}_n(T(Q)) = \mathrm{vol}_n T(Q) \cdot \mathrm{vol}_n(C)
\end{aligned}$$

$\mathrm{vol}_n(C) = \left(\frac{1}{2^N}\right)^n$

$\boxed{}$

3rd step: Want to show, if A is pavable then $T(A)$ is also, and $\mathrm{vol}_n T(A) = \mathrm{vol}_n T(Q) \cdot \mathrm{vol}_n(A)$.

<u>Use the fact already proved</u> that $T(A)$'s pavability (and volume) can be tested (and computed) using upper & lower sums with respect to $T(\mathcal{D}_N(\mathbb{R}^n))$, i.e. want to show

$$\underbrace{\lim_{N\to\infty} \sum_{\substack{P \in T(\mathcal{D}_N(\mathbb{R}^n)): \\ P \subset T(A)}} \mathrm{vol}_n(P)}_{L_{\mathcal{D}_N}(\mathbf{1}_{T(A)}) =} \overset{?}{=} \underbrace{\lim_{N\to\infty} \sum_{\substack{P \in T(\mathcal{D}_N(\mathbb{R}^n)): \\ P \cap T(A) \neq \emptyset}} \mathrm{vol}_n(P)}_{U_{\mathcal{D}_N}(\mathbf{1}_{T(A)}) =} \overset{?}{=} \mathrm{vol}_n T(Q) \cdot \mathrm{vol}_n(A)$$

$$\lim_{N\to\infty} \sum_{\substack{C \in \mathcal{D}_N(\mathbb{R}^n): \\ T(C) \subset T(A)}} \mathrm{vol}_n(T(C)) \qquad\qquad \lim_{N\to\infty} \sum_{\substack{C \in \mathcal{D}_N(\mathbb{R}^n): \\ T(C) \cap T(A) \neq \emptyset}} \mathrm{vol}_n(T(C))$$

$\boxed{\text{used 2nd step + } T \text{ invertible}}$

$$\lim_{N\to\infty} \sum_{\substack{C \in \mathcal{D}_N(\mathbb{R}^n): \\ C \subset A}} \mathrm{vol}_n T(Q) \cdot \mathrm{vol}_n(C) \qquad\qquad \lim_{N\to\infty} \sum_{\substack{C \in \mathcal{D}_N(\mathbb{R}^n): \\ C \cap A \neq \emptyset}} \mathrm{vol}_n T(Q)\, \mathrm{vol}_n(C)$$

$$\mathrm{vol}_n T(Q) \lim_{N\to\infty} \sum_{\substack{C \in \mathcal{D}_N(\mathbb{R}^n): \\ C \subset A}} \mathrm{vol}_n(C) = \mathrm{vol}_n T(Q) \lim_{N\to\infty} \sum_{\substack{C \in \mathcal{D}_N(\mathbb{R}^n): \\ C \cap A \neq \emptyset}} \mathrm{vol}_n(C) = \mathrm{vol}_n T(Q) \cdot \mathrm{vol}_n(A)$$

used pavability of A