3/31/2017

(89) So who are all the k-forms on $\mathbb{R}^n$, for all k?

One can add them and scale them,

e.g. $5\,dx_1 \wedge dx_3 \wedge dx_4 - 7\,dx_2 \wedge dx_1 \wedge dx_5$ is a 3-form on $\mathbb{R}^5$

so they form a vector space, denoted $A_c^k(\mathbb{R}^n)$ (DEF'N 6.1.7)

A for "alternating"

c for "constant" (we'll let the coefficients be functions on $\mathbb{R}^n$ later)

THEOREM (6.1.8, 6.1.10)

Every k-form φ in $A_c^k(\mathbb{R}^n)$ has a unique expression

$$\varphi = \sum_{1 \le i_1 < \cdots < i_k \le n}' a_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

namely with $a_{i_1 \cdots i_k} = \varphi(\bar{e}_{i_1}, \ldots, \bar{e}_{i_k})$.

Hence $A_c^k(\mathbb{R}^n)$ has as a basis the elementary k-forms

$$\{ dx_{i_1} \wedge \cdots \wedge dx_{i_k} : 1 \le i_1 < \cdots < i_k \le n \}$$

and therefore has dimension $\underbrace{\binom{n}{k}}_{\text{binomial coefficient}} := \#$ k-element subsets $\{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$

$$\overset{\text{EXERCISE (or look up a proof?)}}{=} \frac{n!}{k!(n-k)!}$$

REMARK: In particular, any k-form φ on $\mathbb{R}^n$ with $k > n$ must be identically 0. This is because if $\bar{v}_1, \ldots, \bar{v}_k \in \mathbb{R}^n$ with $k > n$, then $\exists$ a linear dependence among them, say $\bar{v}_k = \sum_{i=1}^{k-1} c_i \bar{v}_i$,

so $\varphi(\bar{v}_1, \ldots, \bar{v}_{k-1}, \bar{v}_k) = \varphi(\bar{v}_1, \ldots, \bar{v}_{k-1}, \sum_{i=1}^{k} c_i \bar{v}_i)$

$$= \sum_{i=1}^{k} c_i \underbrace{\varphi(\bar{v}_1, \ldots, \bar{v}_{k-1}, \bar{v}_i)}_{\substack{=0 \text{ since}}} = 0.$$

φ is alternating, so it vanishes when two arguments are equal (swapping those 2 arguments should negate its sign) proof of THM 6.1.8:

Uniqueness: If a k-form $\varphi \in A^k_c(\mathbb{R}^n)$ has such an expression, then the coefficient $a_{i_1\cdots i_k} = \varphi(\bar{e}_{i_1},\ldots,\bar{e}_{i_k})$ due to this computation:

Given $1 \le i_1 < \ldots < i_k \le n$
and $1 \le j_1 < \ldots < j_k \le n$,

$$dx_{j_1}\wedge\ldots\wedge dx_{j_k}(\bar{e}_{i_1},\ldots,\bar{e}_{i_k}) = \text{det of rows } j_1,\ldots,j_k \text{ in } \begin{array}{cc} \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \vdots & & \vdots \\ 1 & \vdots & & \vdots \\ 0 & 0 & & \vdots \\ \vdots & 1 & \ddots & \vdots \\ \vdots & 0 & & 0 \\ \vdots & \vdots & & 1 \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow i_1 \\ \\ \leftarrow i_2 \\ \vdots \\ \leftarrow i_k \\ \\ \\ \end{array} \end{array}$$

$$= \begin{cases} \det\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = 1 & \text{if } i_1 = j_1,\ldots,i_k = j_k \\ \left(\substack{\text{det of a matrix} \\ \text{with a zero row}}\right) = 0 & \text{if some } i_\ell \neq j_\ell \end{cases}$$

Existence: Given $\varphi \in A^k_c(\mathbb{R}^n)$, want to show
that $\psi := \sum_{1 \le i_1 < \ldots < i_k \le n} \varphi(\bar{e}_{i_1},\ldots,\bar{e}_{i_k})\, dx_{i_1}\wedge\ldots\wedge dx_{i_k}$
has $\varphi(\bar{v}_1,\ldots,\bar{v}_k) \overset{(*)}{=} \psi(\bar{v}_1,\ldots,\bar{v}_k) \quad \forall\, \bar{v}_1,\ldots,\bar{v}_k \in \mathbb{R}^n$.

Expanding each $v_\ell = \sum_{m=1}^{n} a_{\ell m}\bar{e}_m$, and using multilinearity of φ, ψ, one sees that $(*)$ will hold if it's true whenever each $\bar{v}_i$ is a standard basis vector $\bar{e}_m$,

That is, whenever $\bar{v}_1,\ldots,\bar{v}_k$ are $\bar{e}_{j_1},\ldots,\bar{e}_{j_k}$ for some $1 \le j_r \le n$.

Using alternating property of φ, ψ, both vanish if $j_1,\ldots,j_k$ have any repeats, and when the j_r are distinct, the values of φ, ψ are determined by the case where $1 \le j_1 < \ldots < j_k \le n$.

Finally, in this case one has

$$\psi(\bar{e}_{j_1},\ldots,\bar{e}_{j_k}) = \sum_{1 \le i_1 < \ldots < i_k \le n} \varphi(\bar{e}_{i_1},\ldots,\bar{e}_{i_k}) \underbrace{dx_{i_1}\wedge\ldots\wedge dx_{i_k}(\bar{e}_{j_1},\ldots,\bar{e}_{j_k})}_{= \begin{cases} 1 & \text{if } i_\ell = j_\ell\ \forall \ell \\ 0 & \text{else} \end{cases}}$$

by the computation in the uniqueness proof

(97) What do the wedge symbols "$\wedge$" mean in $dx_i \wedge dx_j$?

They're consistent with a multiplication one can define called <u>wedge product</u> (DEF'N 6.1.12)

$$A_c^k(\mathbb{R}^n) \times A_c^\ell(\mathbb{R}^n) \longrightarrow A^{k+\ell}(\mathbb{R}^n)$$

$$(\varphi \quad , \quad \omega) \longmapsto \varphi \wedge \omega \text{ defined by}$$

$$\varphi\wedge\omega(\bar v_1, \dots, \bar v_{k+\ell}) :=$$

$$\sum_{\sigma \in \operatorname{Perm}(k,\ell)} \operatorname{sgn}(\sigma)\, \underbrace{\varphi(\bar v_{\sigma(1)}, \dots, \bar v_{\sigma(k)})}_{\text{increasing subscripts}} \cdot \underbrace{\omega(\bar v_{\sigma(k+1)}, \dots, \bar v_{\sigma(\ell)})}_{\text{increasing subscripts}}$$

<u>EXAMPLES</u>:

① $k=\ell=1$ $\qquad \varphi\wedge\omega(\bar v_1, \bar v_2) = +\varphi(\bar v_1)\omega(\bar v_2) - \varphi(\bar v_2)\omega(\bar v_1)$

② $k=1$, $\ell = n-1$

$$\begin{aligned}\varphi\wedge\omega(\bar v_1, \bar v_2, \dots, \bar v_n) = &+\varphi(\bar v_1)\,\omega(\bar v_2, \dots, \bar v_n)\\ &-\varphi(\bar v_2)\,\omega(\bar v_1, \bar v_3, \dots, \bar v_n)\\ &+\varphi(\bar v_3)\,\omega(\bar v_1, \bar v_2, \bar v_4, \dots, \bar v_n)\\ &- \cdots\\ &+(-1)^{n-1}\varphi(\bar v_n)\,\omega(\bar v_1, \bar v_2, \dots, \bar v_n)\end{aligned}$$

③ $k=\ell=2$

$$\begin{aligned}\varphi\wedge\omega(\bar v_1, \bar v_2, \bar v_3, \bar v_4) = &+\varphi(\bar v_1, \bar v_2)\,\omega(\bar v_3, \bar v_4)\\ &-\varphi(\bar v_1, \bar v_3)\,\omega(\bar v_2, \bar v_4)\\ &+\varphi(\bar v_1, \bar v_4)\,\omega(\bar v_2, \bar v_3)\\ &+\varphi(\bar v_2, \bar v_3)\,\omega(\bar v_1, \bar v_4)\\ &-\varphi(\bar v_2, \bar v_4)\,\omega(\bar v_1, \bar v_3)\\ &+\varphi(\bar v_3, \bar v_4)\,\omega(\bar v_1, \bar v_2)\end{aligned}$$

④ Note if $\left.\begin{array}{l}\varphi = dx_i\\ \omega = dx_j\end{array}\right\} \in A^1(\mathbb{R}^n)$, then

$$\begin{aligned}\varphi\wedge\omega\left(\underbrace{\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}}_{\bar a}, \underbrace{\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}}_{\bar b}\right) &= \varphi(\bar a)\omega(\bar b) - \varphi(\bar b)\omega(\bar a)\\ &= dx_i(\bar a)\,dx_j(\bar b) - dx_i(\bar b)\,dx_j(\bar a)\\ &= a_i b_j - b_i a_j\\ &= dx_i \wedge dx_j(\bar a, \bar b) \text{ from before}\end{aligned}$$

One can check that our old $dx_{i_1}\wedge\dots\wedge dx_{i_k} = (((dx_{i_1}\wedge dx_{i_2})\wedge dx_{i_3})\wedge\dots)\wedge dx_{i_k}$ in this new sense. (see EXAMPLES 6.1.14 for $k=3$)

In fact one doesn't need to worry about the parenthesization order:

PROP 6.1.15: $\varphi\wedge\omega$ has these properties:

Not obvious! (ignored in book) 0. $\varphi\wedge\omega\in A_c^{k+l}(\mathbb{R}^n)$

(easy) 1. $\varphi\wedge(\omega_1+\omega_2) = \varphi\wedge\omega_1 + \varphi\wedge\omega_2$

EXER. 6.1.13 (let's skip it!) (tricky) 2. $\varphi_1\wedge(\varphi_2\wedge\varphi_3) = (\varphi_1\wedge\varphi_2)\wedge\varphi_3$

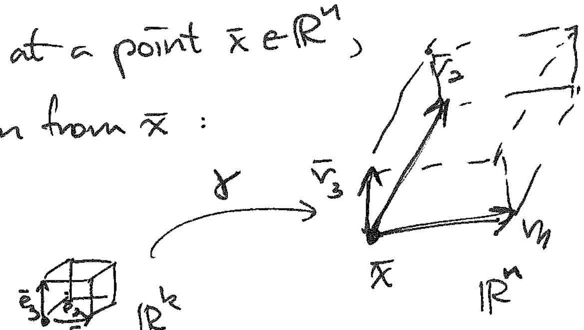
(not hard) 3. $\varphi\wedge\omega = (-1)^{kl}\,\omega\wedge\varphi$ if $\varphi\in A_c^k(\mathbb{R}^n)$, $\omega\in A_c^l(\mathbb{R}^n)$

(e.g. $dx\wedge dy = -dy\wedge dx$
but $dx\wedge(dy\wedge dz) = +(dy\wedge dz)\wedge dx$)

4/3/2017 >

Most often, the $\bar v_1,\dots,\bar v_k$ on which we evaluate a k-form are thought of as anchored at a point $\bar x\in\mathbb{R}^n$, spanning a parallelogram from $\bar x$:

$$P_{\bar x}(\bar v_1,\dots,\bar v_k)$$



(think $\underset{\mathbb{R}^k}{\overset{\cup}{U}} \xrightarrow{\gamma} \underset{\mathbb{R}^n}{\overset{\cup}{M}}$

$\bar u \longmapsto \bar x=\gamma(\bar u)$

$\bar e_1 \longmapsto D\gamma(\bar u)(\bar e_1)=\bar v_1$

$\vdots$

$\bar e_k \longmapsto D\gamma(\bar u)(\bar e_k)$)

So we'll want our k-form to have coefficients $a_{i_1\dots i_k}(\bar x)$ that are functions of $\bar x$

DEF'N 6.1.16: A k-form field on $U\subset\mathbb{R}^n$ is a map $U\xrightarrow{\varphi}A_c^k(\mathbb{R}^n)$,

so $\varphi = \sum_{1\le i_1<\dots<i_k\le n} a_{i_1\dots i_k}(\bar x)\, dx_{i_1}\wedge\dots\wedge dx_{i_k}$.

$$A^k(U) := \{\text{all } k\text{-form fields on } U\}$$

sometimes called "differential k-forms" (on U)

EXAMPLE (6.1.7) $\varphi = \cos(xz)\,dx\wedge dy \in A^2(\mathbb{R}^3)$

with $\varphi\left(P_{\begin{pmatrix}1\\2\\\pi\end{pmatrix}}\left(\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}2\\2\\3\end{bmatrix}\right)\right) = \cos(1\cdot\pi)\det\begin{bmatrix}1&2\\0&2\end{bmatrix} = -2$

$\varphi\left(P_{\begin{pmatrix}1/2\\2\\\pi\end{pmatrix}}\left(\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}2\\2\\3\end{bmatrix}\right)\right) = \cos(\tfrac12\cdot\pi)\det\begin{bmatrix}1&2\\0&2\end{bmatrix} = 0$