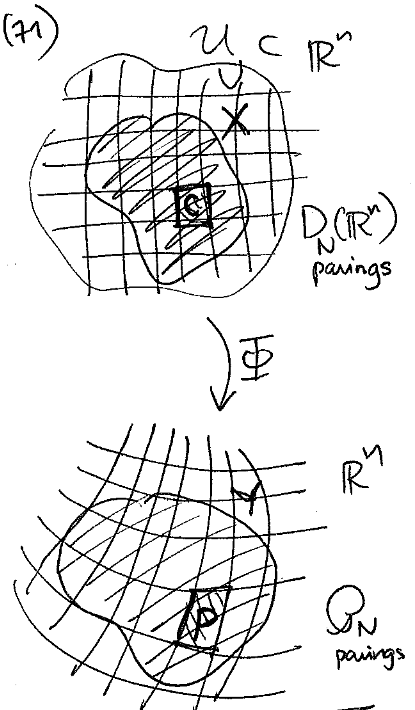


Want to use the sequence of nested pavings $\mathcal{P}_N = \{ \underset{\overset{\shortparallel}{P}}{\Phi(C)} : \underset{C\cap X \neq \emptyset}{C \in D_N(\mathbb{R}^n)} \}$

to compute $\displaystyle\int_Y f(\vec{y})\,|d^n\vec{y}|$

(Already need $\Phi \in C^1(U)$ and $D\Phi$ Lipschitz to show some of the paving properties, like $\partial P = 0$, $\mathrm{vol}_n(P_1 \cap P_2) = 0$

and to show the $\mathrm{diam}(P) \to 0$ as $N \to \infty$;

one can bound $\underbrace{|D\Phi(\vec{x})|}_{\text{gives a bound on volume inflation } C \rightsquigarrow \Phi(C) = P \text{ and diameter inflation}}$ for $\vec{x} \in C \in D_N(\mathbb{R}^n)$ having $C \cap X \neq \emptyset$)

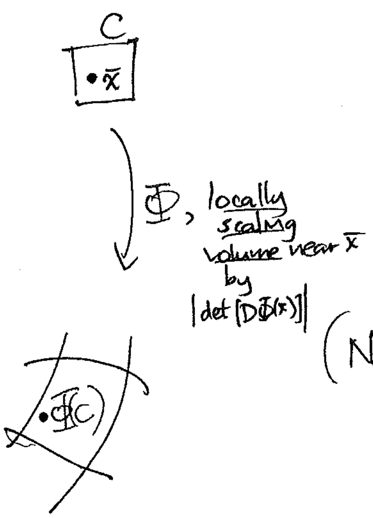
only finitely many such C since X is compact.

3/20/2017

Then $$\int_Y f(\vec{y})\,|d^n\vec{y}| = \lim_{N\to\infty} \sum_{\substack{P \in \mathcal{P}_N \\ \shortparallel \\ \Phi(C)}}{}' M_P(f)\,\mathrm{vol}_n(P)$$

$$= \lim_{N\to\infty} \sum_{C \in D_N(\mathbb{R}^n)} M_C(f\circ\Phi)\,\frac{\mathrm{vol}_n \Phi(C)}{\mathrm{vol}_n(C)} \cdot \mathrm{vol}_n(C)$$

$$\text{"}\approx\text{"} \int_X (f\circ\Phi)(\vec{x})\,\underbrace{\lim_{N\to\infty} \frac{\mathrm{vol}_n \Phi(C)}{\mathrm{vol}_n(C)}}_{\approx\, |\det[D\Phi(\vec{x})]|}\,|d^n\vec{x}|$$



(Need Lipschitz condition and $D\Phi(\vec{x})$ invertible to carefully bound $\frac{\mathrm{vol}_n \Phi(C)}{\mathrm{vol}_n(C)} \approx |\det[D\Phi(\vec{x})]|$; rather painful!)

$$\frac{\mathrm{vol}_n \Phi(C)}{\mathrm{vol}_n(C)} = \frac{\mathrm{vol}_n \Phi(C)}{\mathrm{vol}_n D\Phi(C)} \cdot \frac{\mathrm{vol}_n D\Phi(C)}{\mathrm{vol}_n(C)}$$

approaches 1

$$\approx \int_X (f\circ\Phi)(\vec{x})\,|\det[D\Phi(\vec{x})]|\,|d^n\vec{x}| \qquad \text{"}\blacksquare\text{"}$$

§4.11 Lebesgue integrals

(a real quick treatment with little proof; Math 5615 does it more fully, via a different approach)

Lebesgue integrals have the advantages of

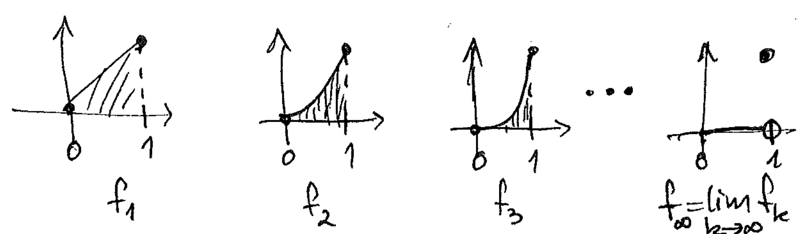
- ignoring the values of $f(x)$ on sets of measure 0,
- behaving well with limits $\int \lim_{k\to\infty} f_k = \lim_{k\to\infty} \int f_k$
  - allowing for improper integrals finally!
  - sometimes allowing $\frac{d}{dx}\int f = \int \frac{df}{dx}$

To appreciate the issues involved, note that Riemann integrals sometimes have $\int \lim_{k\to\infty} f_k = \lim_{k\to\infty}\int f_k$, sometimes not.

(In fact, sometimes $f_\infty = \lim_{k\to\infty} f_k$ is not even Riemann integrable!)

(good) EXAMPLE:

① $f_k(x) = \begin{cases} x^k & \text{for } x\in[0,1] \\ 0 & \text{else} \end{cases}$



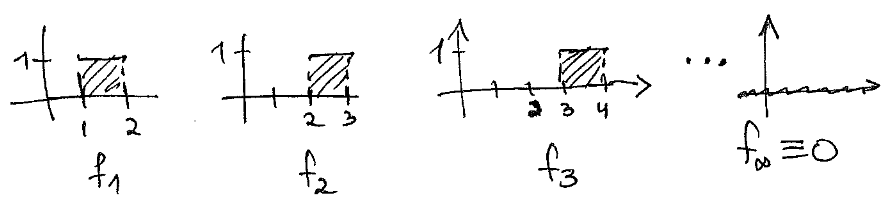
$f_\infty(x) = \begin{cases} 1 & \text{if } x=1 \\ 0 & \text{else} \end{cases}$

$\int_{\mathbb{R}} \overbrace{\lim_{k\to\infty} f_k(x)}^{f_\infty(x)} |dx| = 0$ ← equal, no problem!

$\lim_{k\to\infty}\int_{\mathbb{R}} f_k(x)\,dx = \lim_{k\to\infty}\int_0^1 x^k\,dx = \lim_{k\to\infty}\left[\frac{x^{k+1}}{k+1}\right]_0^1 = \lim_{k\to\infty}\frac{1}{k+1} = 0$

(bad)
NON-EXAMPLES:

① $f_k(x) = 1_{[k,k+1]}(x)$



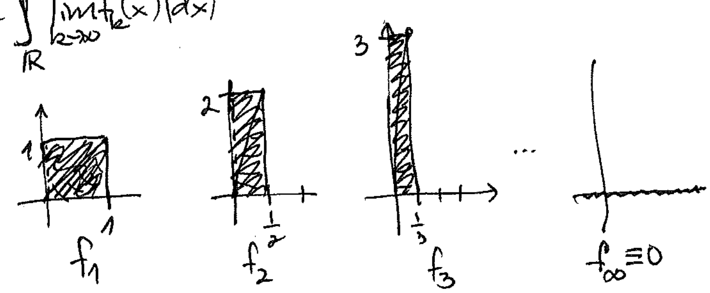
$\downarrow \lim_{k\to\infty}$

$f_\infty(x) = 0 \quad \forall x\in\mathbb{R}$

mass "escaping to ∞"

$\lim_{k\to\infty}\int_{\mathbb{R}} f_k(x)\,|dx| = \lim_{k\to\infty} 1 = 1 \neq 0 = \int_{\mathbb{R}} \lim_{k\to\infty} f_k(x)\,|dx|$

② Similarly, $f_k(x) = \begin{cases} k & \text{for } x\in[0,\frac{1}{k}] \\ 0 & \text{else} \end{cases}$



$\lim_{k\to\infty}\int_{\mathbb{R}} f_k(x)\,|dx| = \lim_{k\to\infty} 1 = 1 \neq 0 = \int_{\mathbb{R}} \lim_{k\to\infty} f_k(x)\,|dx|.$

③ Slightly different issue: if we list $\mathbb{Q}\cap[0,1]=\{\underset{a_1}{0},\underset{a_2}{1},\underset{a_3}{\tfrac12},\tfrac13,\tfrac23,\tfrac14,\tfrac24,\tfrac34,\tfrac15,\tfrac25,\tfrac35,\tfrac45,\dots\}$

and let $f_k(x)=\begin{cases}1 & \text{if } x\in\{a_1,\dots,a_k\}\\ 0 & \text{else}\end{cases}=1_{\{a_1,\dots,a_k\}}(x)$

then $\lim\limits_{k\to\infty}\int_{\mathbb{R}} f_k(x)|dx| = \lim\limits_{k\to\infty}\int_{\mathbb{R}} 1_{\{a_1,\dots,a_k\}}(x)|dx| = \lim\limits_{k\to\infty}\mathrm{vol}_1\{a_1,\dots,a_k\} = \lim\limits_{k\to\infty} 0 = 0$

but $f_\infty = 1_{\mathbb{Q}\cap[0,1]}$ is not Riemann integrable.
$(U(f_\infty)=1>0=L(f_\infty))$

There are theorems about Riemann integrals with hypotheses ~~that eliminate~~ NON-EXAMPLES ①, ②, ③, some more flexible than others, e.g.

easy, but not even flexible enough for EXAMPLE ① above

THM 4.11.2: If $\{f_k\}$ $\mathbb{R}^n\to\mathbb{R}$ are bounded, of bounded support, Riemann integrable, and they converge uniformly to f

> **DEF'N 4.11.1**
> $\forall\epsilon>0\ \exists N$ s.t. $\forall k\geq N$, $\forall \vec{x}\in\mathbb{R}^n$ $|f_k(\vec{x})-f(\vec{x})|<\epsilon$

then f is integrable, and
$$\lim_{k\to\infty}\int_{\mathbb{R}^n} f_k(\vec{x})\,|d^n\vec{x}| = \int_{\mathbb{R}^n} f(\vec{x})\,|d^n\vec{x}|$$

much harder, but at least explains EXAMPLE ① above

THM 4.11.4: If $\{f_k\}$ are bounded, of bounded support, Riemann integrable,

and $\exists\ \mathbb{R}^n\xrightarrow{f}\mathbb{R}$ integrable with $\lim\limits_{k\to\infty} f_k(\vec{x}) \underset{\text{a.e.}}{=} f(\vec{x})$

almost everywhere, that is, except on a set of measure 0

then $\lim\limits_{k\to\infty}\int_{\mathbb{R}^n} f_k(\vec{x})\,|d^n\vec{x}| = \int_{\mathbb{R}^n} f(\vec{x})\,|d^n\vec{x}|$.

proof: See App. A.22 ∎

Q: Why do NON-EXAMPLES ①, ②, ③ fail the hypotheses?

We'll define Lebesgue integrability (L-integrability) in terms of Riemann (R-integrability)

and taking certain limits (actually infinite sums)...

(74) PROP 4.11.5 / THM 4.11.7 / DEF'N 4.11.8 : Suppose $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ has $f(\bar{x}) \underset{a.e}{\overset{(*)}{=}} \sum_{k=1}^{\infty} f_k(\bar{x})$

and that $\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\bar{x})| \, |d^n\bar{x}|$ $\overset{(**)}{\text{converges}}$

(in particular, these $|f_k|$ are R-integrable $\forall k$)

Then we say f is L-integrable and define its

L-integral as $\int_{\mathbb{R}^n} f(\bar{x}) \, |d^n\bar{x}| := \underbrace{\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} \underset{\text{also R-integrable } \forall k}{f_k(\bar{x})} \, |d^n\bar{x}|}_{\text{converges absolutely, since } \left|\int f_k\right| \le \int |f_k|}$.

In particular, this will be independent of the choice of sequences $\{f_k\}$ satisfying $(*)$, $(**)$.

3/21/2017

EXAMPLES:

① R-integrable $\overset{\text{PROP 4.11.9}}{\Longrightarrow}$ L-integrable, since if f is R-integrable then one can always take $f_1 = f$, $f_2 = f_3 = f_4 = \dots \equiv 0$ to make $(*)$ hold,

and $\sum_{k=1}^{\infty} \int |f_k(\bar{x})| \, |d^n\bar{x}| \overset{(**)}{=} \int \underbrace{|f_1(\bar{x})|}_{|f(x)| \text{ is R-integrable because } f(x) \text{ was}} |d^n\bar{x}|$

by PROP 4.1.14 (iv)

$\sum_{k=1}^{\infty} \int f_k(x) \, |d^n\bar{x}| = \int \underbrace{f_1(x)}_{f(x)} \, |d^n\bar{x}|$ ✓

② NON-EXAMPLE ③ from before, where $f = f_\infty = 1_{\mathbb{Q} \cap [0,1]}$ is now no problem, since we can list $\mathbb{Q} \cap [0,1] = \{a_1, a_2, a_3, a_4, a_5, \dots\}$ as before
$$a_1 = 0,\ a_2 = 1,\ a_3 = \tfrac{1}{2},\ a_4 = \tfrac{1}{3},\ a_5 = \tfrac{2}{3}$$

and write $f = \sum_{k=1}^{\infty} f_k$ where $f_k = 1_{\{a_k\}} = |f_k|$
has $\int_{\mathbb{R}} f_k(x) \, |dx| = \int_{\mathbb{R}} |f_k(x)| \, |dx| = 0$

(In fact, if we take $f_k \equiv 0 \ \forall k$, still have $f \underset{a.e.}{=} \sum_{k=1}^{\infty} f_k$!)

$\Rightarrow f = 1_{\mathbb{Q} \cap [0,1]}$ is L-integrable, with $\int_{\mathbb{R}} f(x) \, |dx| = 0$

Similarly $f = 1_{\mathbb{Q}}$ or $f = 1_A$ for any subset $A \subset \mathbb{R}^n$ with measure 0 will be L-integrable, with $\int_{\mathbb{R}^n} f(\bar{x}) \, |d^n\bar{x}| = 0$.