(68) In general, change-of-variables looks like this:

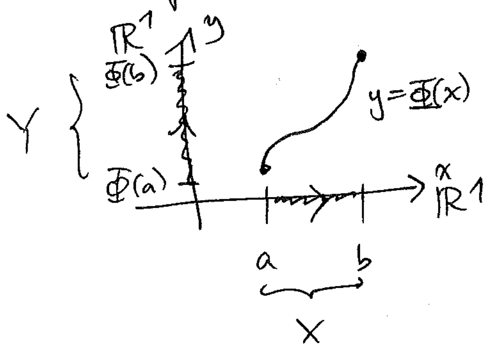
THM 4.10.2 (roughly)

Under hypotheses saying $Y \subset \mathbb{R}^n$ is, roughly speaking, nicely parametrized by $\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^n \end{array}$,

any integrable $Y \xrightarrow{f} \mathbb{R}$ has

$$\int_Y f(\vec{y})\,|d^n\vec{y}| = \int_X (f\circ\Phi)(\vec{x})\,\big|\det[D\Phi(x)]\big|\,|d^n\vec{x}|$$

Compare with 1-variable picture of substitution



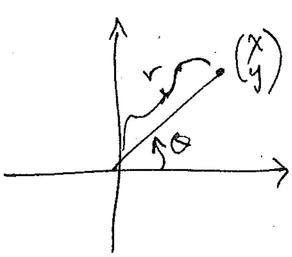
$$\int_{y=\Phi(a)}^{y=\Phi(b)} f(y)\,dy = \int_{x=a}^{x=b} f(\Phi(x))\,\underbrace{\Phi'(x)}_{\substack{\text{positive, if}\\ \Phi \text{ is monotone}\\ \text{increasing}\\ \text{on } X=[a,b]}}\,dx$$

$y=\Phi(x)$
$dy = \Phi'(x)dx$

3/10/2017 EXAMPLES:

① Polar coordinates (DEF'N 4.10.2, PROP 4.10.3) If

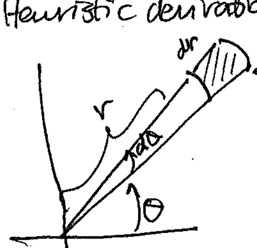
$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\Phi} & \mathbb{R}^2 \\ \binom{r}{\theta} & \longmapsto & \binom{x}{y} = \binom{r\cos\theta}{r\sin\theta} \end{array}$$



maps $\begin{array}{ccc} B & \longrightarrow & A \\ \cap & & \cap \\ [0,\infty)\times[0,2\pi) & & \mathbb{R}^2 \end{array}$ nicely, then

$$\int_A f\binom{x}{y}|dx\,dy| = \int_B f\binom{r\cos\theta}{r\sin\theta}\,r\,|dr\,d\theta|$$

Heuristic derivation of $r\,dr\,d\theta$:



$\leftarrow$ area $= \frac{d\theta}{2\pi} \times (\pi(r+dr)^2 - \pi r^2)$

$\approx \cancel{\pi r^2} + 2\pi r\,dr + \underbrace{\pi(dr)^2}_{\text{negligible}} - \cancel{\pi r^2}$

$\approx \frac{d\theta}{2\pi}\cdot 2\pi r\,dr$

$= r\,dr\,d\theta$

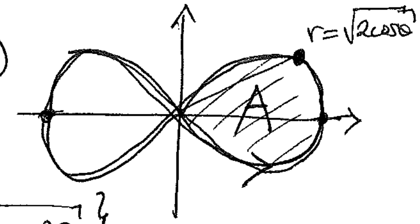
because
$$|\det[D\Phi]| = \left|\det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix}\right| = \left|\det\begin{bmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{bmatrix}\right|$$

$$= |r\cos^2\theta + r\sin^2\theta|$$

$$= |r| = r.$$

(69)

e.g. EXAMPLE 4.10.5: The lemniscate $r^2 = \cos(2\theta)$ has right lobe traced out as $\theta \in \left[\frac{-\pi}{4}, \frac{+\pi}{4}\right]$,



so can parametrize its interior A via $B = \left\{ \begin{pmatrix} r \\ \theta \end{pmatrix} : \theta \in \left[\frac{-\pi}{4}, \frac{+\pi}{4}\right], r \in [0, \sqrt{\cos 2\theta}] \right\}$,

and it has area $\displaystyle\int_A 1 \cdot dx\,dy = \int_B r\,dr\,d\theta$

$$= \int_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \int_{r=0}^{\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= \int_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{1}{2}\int_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \cos 2\theta\,d\theta = \left[\frac{\sin 2\theta}{4}\right]_{\theta=-\pi/4}^{\theta=\pi/4} = \frac{1-(-1)}{4} = \frac{1}{2}.$$

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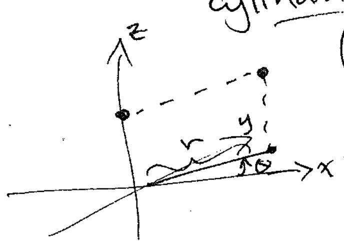
② In $\mathbb{R}^3$, it's sometimes (similarly) convenient to parametrize via

cylindrical coordinates (DEF'N 4.10.9, PROP 4.10.10)

$$\mathbb{R}^3 \xrightarrow{\Phi} \mathbb{R}^3$$

$$\begin{pmatrix} r \\ \theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$



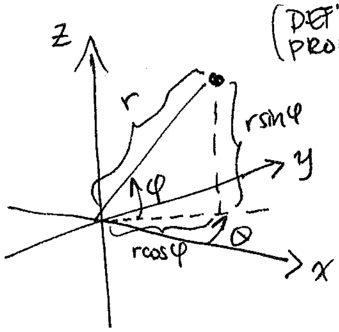
and then $\displaystyle\int_A f\begin{pmatrix} x \\ y \\ z \end{pmatrix} |dx\,dy\,dz| = \int_B f\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} r\,|dr\,d\theta\,dz|$

similar $|\det[D\Phi]|$ calculation

or sometimes spherical coordinates (DEF'N 4.10.6, PROP. 4.10.7)

$$\mathbb{R}^3 \xrightarrow{\Phi} \mathbb{R}^3$$

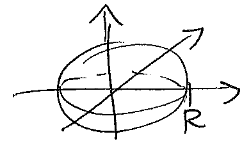
$$\begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ r\sin\varphi \end{pmatrix}$$



and then $\displaystyle\int_A f\begin{pmatrix} x \\ y \\ z \end{pmatrix} |dx\,dy\,dz| = \int_B f\begin{pmatrix} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ r\sin\varphi \end{pmatrix} \underbrace{r^2\cos\varphi}\,|dr\,d\theta\,d\varphi|$

EXERCISE: Check this is $|\det[D\Phi]|$

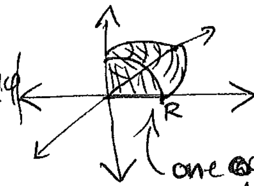
e.g. volume of sphere of radius R in $\mathbb{R}^3$



ought to be easy in spherical coordinates,

and it is:

$$8\int_A 1\cdot|dx\,dy\,dz| = 8\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{\varphi=0}^{\varphi=\frac{\pi}{2}}\int_{r=0}^{r=R} r^2\cos\varphi\,|dr\,d\theta\,d\varphi|$$



one octant A parametrized by
$\theta\in[0,\pi/2]$
$\varphi\in[0,\pi/2]$
$r\in[0,R]$

$$= 8\int_{\theta=0}^{\theta=\pi/2}\int_{\varphi=0}^{\varphi=\frac{\pi}{2}}\cos\varphi\left[\frac{r^3}{3}\right]_{r=0}^{r=R}|d\theta\,d\varphi|$$

$$= \frac{8R^3}{3}\int_{\theta=0}^{\theta=\pi/2}\left(\int_{\varphi=0}^{\varphi=\pi/2}\cos\varphi\;|d\varphi|\right)|d\theta|$$

$$= \frac{8R^3}{3}\int_{\theta=0}^{\theta=\pi/2}\underbrace{\left[\sin\varphi\right]_{\varphi=0}^{\varphi=\pi/2}}_{1}|d\theta| = \frac{8R^3}{3}\cdot\left[\pi/2 - 0\right] = \frac{4\pi}{3}R^3 \checkmark$$

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③ Read EXAMPLES 4.10.18, 4.10.19 in book for some less standard coordinate changes

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Let's return to the more precise statement of change-of-variables:

THM 4.10.12 $\underset{\mathbb{R}^n}{\underset{\cap}{X \overset{\text{compact}}{\subset} U^{\text{open}}}} \xrightarrow{\ \Phi\ } \underset{Y:=\Phi(X)}{\underset{\cup}{\mathbb{R}^n}}$ with

- $\Phi\in C^1(U)$
- $D\Phi$ Lipschitz, i.e. $\exists M$ s.t. $\underbrace{|D\Phi(\bar{x})-D\Phi(\bar{y})|}_{\text{matrix length}}\le M|\bar{x}-\bar{y}|\ \ \forall \bar{x},\bar{y}\in U$
- Φ injective on $X-\partial X$
- $D\Phi(\bar{x})$ invertible for $\bar{x}\in X-\partial X$

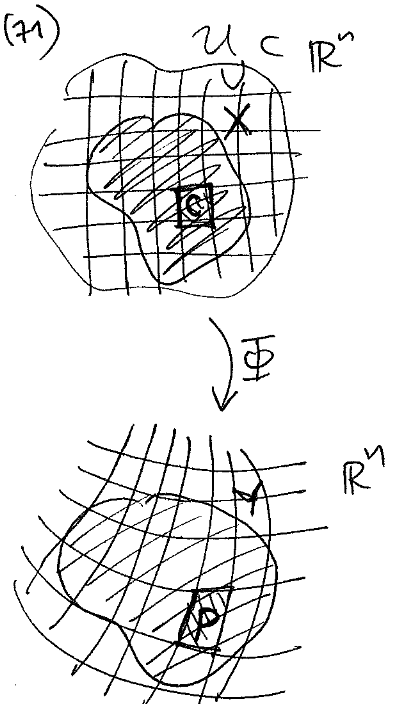
Then $Y\xrightarrow{f}\mathbb{R}$ integrable $\Rightarrow$

$X\xrightarrow{(f\circ\Phi)|\det[D\Phi]|}\mathbb{R}$ integrable,

and $\displaystyle\int_Y f(\bar{y})\,|d^n\bar{y}| = \int_X (f\circ\Phi)(\bar{x})\,|\det[D\Phi(\bar{x})]|\,|d^n\bar{x}|$

Let's try to do a heuristic derivation of this, similar to our linear change-of-variable proof (real proof in Appendix A.20)

(71)



Want to use the sequence of nested pavings $\mathcal{P}_N = \{ \underset{P}{\underset{\shortparallel}{\Phi(C)}} : \underset{C \cap X \neq \emptyset}{C \in \mathcal{D}_N(\mathbb{R}^n)} \}$

to compute $\displaystyle\int_Y f(\vec{y}) \, |d^n \vec{y}|$

(Already need $\Phi \in C^1(U)$ and $D\Phi$ Lipschitz to show some of the paving properties, like $\partial P = 0$, $\mathrm{vol}_n(P_1 \cap P_2) = 0$

and to show the $\mathrm{diam}(P) \to 0$ as $N \to \infty$;

one can bound $\underbrace{|D\Phi(\vec{x})|}_{\text{gives a bound on volume inflation } C \rightsquigarrow \Phi(C) = P \text{ and diameter inflation}}$ for $\vec{x} \in C \in \mathcal{D}_N(\mathbb{R}^n)$ having $C \cap X \neq \emptyset$)

only finitely many such C since X is compact.

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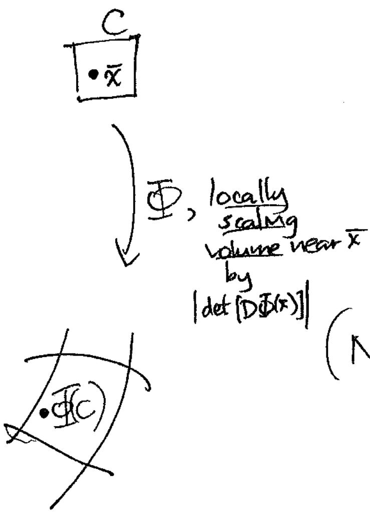
Then $$\int_Y f(\vec{y}) \, |d^n \vec{y}| = \lim_{N \to \infty} \sum_{\substack{P \in \mathcal{P}_N \\ \shortparallel \\ \Phi(C)}}{}' M_P(f) \, \mathrm{vol}_n(P)$$

$$= \lim_{N \to \infty} \sum_{C \in \mathcal{D}_N(\mathbb{R}^n)} M_C(f \circ \Phi) \, \frac{\mathrm{vol}_n \Phi(C)}{\mathrm{vol}_n(C)} \cdot \mathrm{vol}_n(C)$$



$$\text{"}\approx\text{"} \int_X (f \circ \Phi)(\vec{x}) \, \underbrace{\lim_{N \to \infty} \frac{\mathrm{vol}_n \Phi(C)}{\mathrm{vol}_n(C)}}_{\approx \, |\det[D\Phi(\vec{x})]|} \, |d^n \vec{x}|$$

(Need Lipschitz condition and $D\Phi(\vec{x})$ invertible to carefully bound $\frac{\mathrm{vol}_n \Phi(C)}{\mathrm{vol}_n(C)} \approx |\det[D\Phi(\vec{x})]|$; rather painful!)

$$= \frac{\mathrm{vol}_n \Phi(C)}{\mathrm{vol}_n D\Phi(C)} \cdot \frac{\mathrm{vol}_n D\Phi(C)}{\mathrm{vol}_n(C)}$$

approaches 1

$$\approx \int_X (f \circ \Phi)(\vec{x}) \, |\det[D\Phi(\vec{x})]| \, |d^n \vec{x}| \qquad \text{"}\blacksquare\text{"}$$