(44)

Since X is compact, by Bolzano-Weierstrass, can extract convergent subsequences $\bar{x}_{i_j} \longrightarrow \bar{a} \in X$

$\bar{y}_{i_j} \longrightarrow \bar{b} \in X$

But then $\lim_{i\to\infty} |\bar{x}_i - \bar{y}_i| = 0$ forces $\bar{a} = \bar{b}$

By continuity of f, $\exists J$ such that $j \geq J \Rightarrow$ $|f(\bar{x}_{i_j}) - f(\bar{a})| \leq \frac{\epsilon_0}{3}$
$|f(\bar{y}_{i_j}) - f(\bar{a})| \leq \frac{\epsilon_0}{3}$

hence $|f(\bar{x}_{i_j}) - f(\bar{y}_{i_j})| \leq |f(\bar{x}_{i_j}) - f(\bar{a})| + |f(\bar{a}) - f(\bar{y}_{i_j})| \leq \frac{\epsilon_0}{3} + \frac{\epsilon_0}{3} < \epsilon_0$, a contradiction. ∎

2/20/2017 This is the essence behind...

THM 4.3.6: $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ continuous, with bounded support, is integrable.

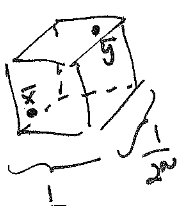
proof: Its (closed) support is closed, bounded, so compact, Hence f is uniformly continuous (by THM 4.3.7 just proven), and so given $\epsilon > 0$, we can find $\delta > 0$ with

$|f(\bar{x}) - f(\bar{y})| < \epsilon$ whenever $|\bar{x} - \bar{y}| < \delta$.

Pick N large enough that $\frac{\sqrt{n}}{2^N} \leq \delta$, so whenever $\bar{x}, \bar{y} \in C$ a cube from $\mathcal{D}_N(\mathbb{R}^n)$, one has $|\bar{x} - \bar{y}| \leq \frac{\sqrt{n}}{2^N} < \delta$

and hence $|f(\bar{x}) - f(\bar{y})| < \epsilon$, thus $\operatorname{osc}_C(f) < \epsilon$ for all cubes in $\mathcal{D}_N(\mathbb{R}^n)$ (!), i.e. $\sum_{\substack{C \in \mathcal{D}_N: \\ \operatorname{osc}(f) \geq \epsilon}} \operatorname{vol}_n C = 0 < \epsilon$ for sure. ∎

$\bar{x}, \bar{y} \in C \in \mathcal{D}_N(\mathbb{R}^n)$



$\frac{1}{2^N}$

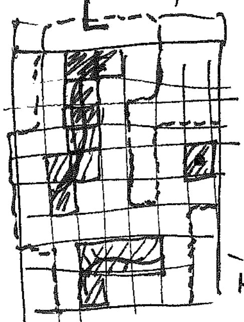
$\Rightarrow |\bar{x} - \bar{y}| \leq \sqrt{\underbrace{\frac{1}{2^N} + \frac{1}{2^N} + \dots + \frac{1}{2^N}}_{n \text{ times}}} = \frac{\sqrt{n}}{2^N}$

Better yet, f could have a few discontinuities, as promised earlier.

THM 4.3.10: $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ (bounded with bounded support) which is continuous except on a (pavable) set of zero volume will always be integrable.

(95)

<u>proof:</u> let $\Delta := \{\bar{x}_0 \in \mathbb{R}^n : f \text{ is discontinuous at } \bar{x}_0\}$



Since Δ is pavable and $\mathrm{vol}_n \Delta = 0$,

$$\underset{\lim_{N\to\infty} U_N(1_\Delta)}{\overset{\|}{}} = \lim_{N\to\infty} \sum_{\substack{C \in \mathcal{D}_N(\mathbb{R}^n):\\ C\cap\Delta\neq\emptyset}} \mathrm{vol}_n(C),$$

if we are given $\epsilon > 0$, we can pick N large enough so that

$\displaystyle\sum_{\substack{C\in\mathcal{D}_N(\mathbb{R}^n)\\ C\cap\Delta\neq\emptyset}} \mathrm{vol}_n(C) < \frac{\epsilon}{3^n}$. Then the "buffer zone" L around Δ

in which one adds all cubes <u>adjacent</u> to those with $C \cap \Delta \neq \emptyset$

has total volume of those cubes $< \dfrac{\epsilon}{3^n}\cdot \underbrace{(3^n - 1)}_{\substack{\#\text{ of neighboring}\\ \text{cubes to any}\\ C \text{ in } \mathcal{D}_N(\mathbb{R}^n)}} < \epsilon$.

Outside this buffer zone L, f is definitely continuous
and want to pick $M \geq N$ so that every other cube C has $\mathrm{osc}_C(f) \leq \epsilon$
(and then f is <u>integrable</u> by our characterization)

If no such M exists, then $\forall M \geq N$ $\exists$ a cube $C \not\subset L$
with $\bar{x}_M, \bar{y}_M \in C$ and $|f(\bar{x}_M) - f(\bar{y}_M)| \overset{(*)}{>} \epsilon$.

By Bolzano-Weierstrass (f has ~~bounded~~ support, so compact),

$\exists$ convergent subsequences $\bar{x}_{M_i} \longrightarrow \bar{a}$, $\bar{y}_{M_i} \longrightarrow \bar{b}$, and again $\bar{a} = \bar{b}$ since $\bar{x}_{M_i}, \bar{y}_{M_i}$ lie in a cube in $\mathcal{D}_{M_i}(\mathbb{R}^n)$

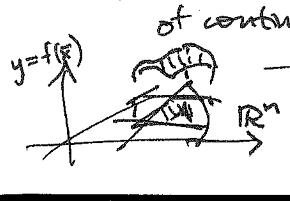
Either $\displaystyle\lim_{i\to\infty} f(\bar{x}_{M_i}) \neq f(\bar{a})$

or $\displaystyle\lim_{i\to\infty} f(\bar{y}_{M_i}) \neq f(\bar{a})$, because of $(*)$

So f is <u>not</u> <u>continuous</u> at $\bar{a}$, i.e. $\bar{a} \in \Delta$, even though $\bar{a}$ is a limit of points <u>not in</u> L. Impossible! ∎

Further reassurance comes from...

COR 4.3.12: If $A \subset \mathbb{R}^n$ is compact and bounded by a finite union of graphs of continuous functions, then any $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ which is $\left\{\begin{matrix}\text{continuous on } A\\ \text{zero outside } A\end{matrix}\right\}$ is <u>integrable</u>.



... but deducing it needs not only THM 4.3.10 just proven, but also COR 4.3.8 (& 4.3.9), which are more technical cube-bounding arguments. Read it in book!

§4.2 Probability & centers of gravity

- a glimpse of Math 5651, but it won't do it justice (and one should really do all of Chap. 4 first!)

Start with a slew of definitions...

DEF'N 4.2.3, 4.2.4, ~~4.2.9~~: A sample space S ("outcomes") is a set on which we have defined a probability measure that assigns to subsets $A \subset S$ (called events) a probability $Prob(A) \in [0,1]$ such that

$$(1)\ Prob(S) = 1$$
$$(2)\ Prob(A \cap B) = 0 \Rightarrow Prob(A \cup B) = Prob(A) + Prob(B)$$

A random variable on S is just a function $S \xrightarrow{f} \mathbb{R}$.

EXAMPLES will come from • discrete sample spaces where $S = \{x_1, x_2, ...\}$ is finite or countable and each $x_i \in S$ has a probability function $\mu(x_i) \geq 0$ normalized so that $\sum_{x_i \in S} \mu(x_i) = 1$

Defining $Prob(A) := \sum_{x_i \in A} \mu(x_i)$ then gives a prob. measure on S

• continuous sample spaces where $S = \mathbb{R}^n$ and one specifies a (prob.) density function $S = \mathbb{R}^n \xrightarrow{\mu} \mathbb{R}$ integrable with $\mu(\bar{x}) \geq 0\ \forall \bar{x} \in \mathbb{R}^n$ normalized so $\int_{\mathbb{R}^n} \mu(x) |d^n x| = 1$

Defining $Prob(A) := \int_{\mathbb{R}^n} 1_A(\bar{x}) \mu(\bar{x}) |d^n \bar{x}|$ gives a prob. meas. on S

DEF'N 4.2.10, 4.2.13: Given the random variable $S \xrightarrow{f} \mathbb{R}$,

its expectation (expected payoff of f) $E(f) := \sum_{x_i \in S} f(x_i) \mu(x_i)$ (discrete) or (cont.) $\int_{\mathbb{R}^n} f(\bar{x}) \mu(\bar{x}) |d^n \bar{x}|$

variance $var(f) := \sum_{x_i \in S} (f(x_i) - E(f))^2 \mu(x_i)$ or $\int_{\mathbb{R}^n} (f(\bar{x}) - E(f))^2 \mu(\bar{x}) |d^n \bar{x}|$

$= E((f - E(f))^2) = E(f^2) - E(f)^2$

$\boxed{\text{standard deviation of } f = \sqrt{var(f)}}$