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(11) § 3.4 Taylor polynomial shortcuts

It helps to know a few single-variable Taylor polynomials/series, and then some shortcuts.

e.g. PROP 3.4.2:

$$e^x = \underbrace{\underbrace{\underbrace{1}_{P^0_{e^x,0}} + x}_{P^1_{e^x,0}} + \frac{x^2}{2!}}_{P^2_{e^x,0}\ \text{etc.}} + \frac{x^3}{3!} + \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

"Generalized binomial theorem"
$$(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1}x^1 + \binom{\alpha}{2}x^2 + \dots \quad \text{where } \binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(k-1))}{k!}$$
for $\alpha \in \mathbb{R}$

proof: All straightforward using $P^k_{f,0}(x) = \sum_{m=0}^{k} \frac{f^{(m)}(0)}{m!} x^m$,

e.g. $f(x) = (1+x)^\alpha$ has
$$f^{(1)}(x) = \alpha(1+x)^{\alpha-1}$$
$$f^{(2)}(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$$
$$\vdots$$
$$f^{(m)}(x) = \alpha(\alpha-1)\cdots(\alpha-(m-1))(1+x)^{\alpha-m}$$

so $$\frac{f^{(m)}(0)}{m!} = \frac{\alpha(\alpha-1)\cdots(\alpha-(m-1))}{m!} = \binom{\alpha}{m} \blacksquare$$

Once one has these, one can derive many others using some shortcuts...

PROP 3.4.3, 3.4.4: (i) Given $f, g : \underset{\mathbb{R}^n}{\overset{\cap}{U^{open}}} \to \mathbb{R}$ with both in $C^k(U)$,

then $f+g \in C^k(U)$ with $P^k_{f+g,\bar{a}}(\bar{a}+\bar{h}) = P^k_{f,\bar{a}}(\bar{a}+\bar{h}) + P^k_{g,\bar{a}}(\bar{a}+\bar{h})$

and $P^k_{fg,\bar{a}}(\bar{a}+\bar{h})$ = terms of deg $\leq k$ in $P^k_{f,\bar{a}}(\bar{a}+\bar{h}) P^k_{g,\bar{a}}(\bar{a}+\bar{h})$.

(ii) Given $\mathbb{R}^m \supseteq U^{open} \xrightarrow{f} V^{open} \subseteq \mathbb{R}^n \xrightarrow{g} \mathbb{R}$ with $f, g \in C^k$

then $g \circ f \in C^k(U)$ and $P_{g\circ f,\bar{a}}(\bar{a}+\bar{h})$ = terms of deg $\leq k$ in $P^k_{g,f(\bar{a})}(P_{f,\bar{a}}(\bar{a}+\bar{h}))$

(12) EXAMPLES:

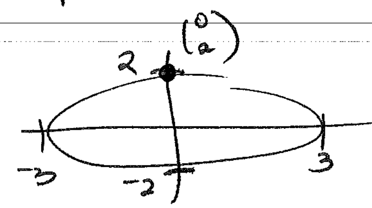
① $f(x) = xe^x = x\left(1+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right) = x+\frac{x^2}{1!}+\frac{x^3}{2!}+\frac{x^4}{3!}+\dots$

② $f\binom{x}{y} = \sin(x^2+y) = (x^2+y) - \frac{(x^2+y)^3}{3!} + \frac{(x^2+y)^5}{5!} \dots$

$= x^2+y - \frac{x^6+3x^4y+3x^2y^2+y^3}{3!} + \dots$

so, for example,

$$P^3_{f,\binom{0}{0}}\binom{x}{y} = x^2+y-\frac{y^3}{3!}$$, as we computed earlier.



③ On the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ near $\binom{0}{2}$, y is an implicitly defined function of x, namely $y=g(x)=2\sqrt{1-\frac{x^2}{9}}$, with quadratic Taylor polynomial obtained ~~from~~ expanding

$$g(x) = 2\left(1-\frac{x^2}{9}\right)^{\frac12}$$

$$= 2\left(\binom{1/2}{0} + \binom{1/2}{1}\left(-\frac{x^2}{9}\right)^1 + \underbrace{\binom{1/2}{2}\left(\frac{-x^2}{9}\right)^2+\dots}_{O(x^4)}\right)$$

$$= 2\left(1 - \frac12\cdot\frac{x^2}{9} + O(x^4)\right)$$

$$= \underbrace{2-\frac{x^2}{9}}_{P^2_{g(x),0}(x)} + O(x^4)$$

One could also approach this last example using only the implicit relation on x,y, via this rule:

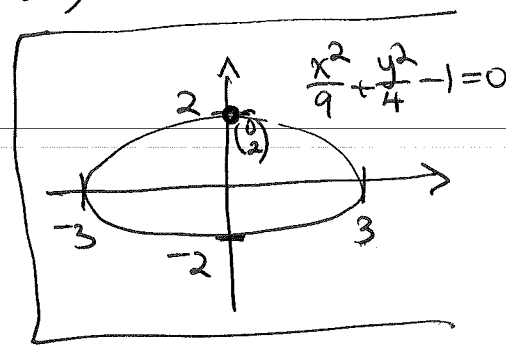
PROP: If $\bar c = \binom{\bar a}{\bar b}$ lies on $\bar F(\bar z)=\bar 0$ for some $\bar F$ which is C^k
(THM 3.4.7)
and defines $\bar y=\bar g(\bar x)$ implicitly on $\bar F\binom{\bar x}{\bar y}=\bar 0$ near $\bar x=\bar a$, (meaning $\bar F\binom{\bar x}{g(\bar x)}=\bar 0$) then one can solve for $P^k_{g,\bar a}(\bar a+\bar h)$ using the relation

$$P^k_{\bar F,\bar c}\begin{pmatrix}\bar a+\bar h\\ P^k_{g,\bar a}(\bar a+\bar h)\end{pmatrix} \in O(|\bar h|^k)$$

(and g is also in C^k)

(13) <u>EXAMPLE</u>: In ③, above $F\begin{pmatrix}x\\y\end{pmatrix} = \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0$ defines the ellipse, and near $\bar{c} = \begin{pmatrix}\bar{a}\\\bar{b}\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$, it implicitly defines $y = g(x)$, whose quadratic Taylor polynomial $P^2_{g(x),0}(x) = a_0 + a_1x + a_2x^2$ should satisfy

$$P^2_{F,\binom{0}{2}}\begin{pmatrix}0+x\\ P^2_{g,0}(0+x)\end{pmatrix} \in O(x^3)$$



$$\frac{(0+x)^2}{9} + \frac{\left(P^2_{g,0}(0+x)\right)^2}{4} - 1$$

$$= \frac{x^2}{9} + \frac{(a_0+a_1x+a_2x^2)^2}{4} - 1$$

$$= \frac{x^2}{9} + \frac{a_0^2 + a_1^2x^2 + a_2^2x^4 + 2a_0a_1x + 2a_0a_2x^2 + 2a_1a_2x^3}{4} - 1$$

$$= \left(\frac{a_0^2}{4} - 1\right)x^0 + \left(\frac{2a_0a_1}{4}\right)x^1 + \left(\frac{1}{9} + \frac{a_1^2 + 2a_0a_2}{4}\right)x^2 + O(x^3)$$

$$\Rightarrow \begin{cases} \frac{a_0^2}{4} - 1 = 0 \Rightarrow a_0 = \pm 2, \text{ but } \underline{\text{we}}\ \underline{\text{know}}\ a_0 = g(0) = 2, \text{ since we are near } \binom{0}{2} = \binom{0}{g(0)} \\ \frac{2a_0a_1}{4} = 0 \Rightarrow \frac{2\cdot 2a_1}{4} = 0 \Rightarrow a_1 = 0 \\ \frac{1}{9} + \frac{a_1^2 + 2a_0a_2}{4} = 0 \Rightarrow \frac{1}{9} + \frac{2\cdot 2\cdot a_2}{4} = 0 \Rightarrow a_2 = -\frac{1}{9} \end{cases}$$

i.e. $g(x) = a_0 + a_1x + a_2x^2 + O(x^3)$
$= 2 - \frac{1}{9}x^2 + O(x^3)$, as before

How do we <u>justify</u> these rules/shortcuts for $P^k_{f,\bar{a}}$?

We've already discussed, in the exercise at the end of our "linear algebra loose ends," why $f, g \in C^k$ implies $f+g \in C^k$, $f\cdot g \in C^k$, $\frac{f}{g} \in C^k$, $g \circ f \in C^k$ (with appropriate extra hypotheses in each case) using sum, product, quotient, chain rules and <u>induction on k</u>.

(14) We also saw how for inverse functions

$$\begin{array}{ccc} U^{open} & \underset{\bar{g}=\bar{f}^{-1}}{\overset{\bar{f}}{\rightleftarrows}} & V^{open} \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^n \end{array}$$

one can similarly use
$$[D\bar{g}(\bar{y})] = [D\bar{f}(\bar{x})]^{-1} = (D\bar{f}(\bar{g}(\bar{y})))^{-1} = \frac{1}{\det\left[\frac{\partial f_i}{\partial x_j}(\bar{g}(\bar{y}))\right]} \operatorname{adj}\left[\frac{\partial f_i}{\partial x_j}(\bar{g}(\bar{y}))\right]^T$$

to show $\bar{f} \in C^k \Rightarrow \bar{g} \in C^k$ again via induction on k.

Similarly, for implicit functions $\bar{F}\begin{pmatrix}\bar{x}\\ g(\bar{x})\end{pmatrix} = \bar{0}$

the relation $D\bar{g}(\bar{a}) = -\left[D\bar{F}(\bar{c})\Big|_{\substack{\text{pivot}\\ \text{vars}}}\right]^{-1}\left[D\bar{F}(\bar{c})\Big|_{\substack{\text{nonpivot}\\ \text{variables}}}\right]$

can be used to show $\bar{F} \in C^k \Rightarrow \bar{g} \in C^k$ via induction on k.

2/2/2017 But then, one still needs to show that if $\underset{f(\bar{a}+\bar{h})}{f(\bar{x})} = P_{f,\bar{a}}(\bar{a}+\bar{h}) + o(|\bar{h}|^k)$

and $g(\bar{x}) = P_{g,\bar{a}}(\bar{a}+\bar{h}) + o(|\bar{h}|^k)$

then $f(\bar{x})+g(\bar{x}) = P_{f,\bar{a}}(\bar{a}+\bar{h}) + P_{g,\bar{a}}(\bar{a}+\bar{h}) + o(|\bar{h}|^k)$

easy since $o(|\bar{h}|^k) + o(|\bar{h}|^k) \subset o(|\bar{h}|^k)$

$$f(\bar{x})g(\bar{x}) = (P_{f,\bar{a}} + o(|\bar{h}|^k))(P_{g,\bar{a}} + o(|\bar{h}|^k))$$
$$= P_{f,\bar{a}}P_{g,\bar{a}} + P_{f,\bar{a}}\,o(|\bar{h}|^k) + P_{g,\bar{a}}\,o(|\bar{h}|^k)$$
$$= \begin{pmatrix}\text{deg} \le k \text{ terms}\\ \text{of } P_{f,\bar{a}}P_{g,\bar{a}}\end{pmatrix} + \underbrace{o(|\bar{h}|^k) + (P_{f,\bar{a}} + P_{g,\bar{a}})\,o(|\bar{h}|^k)}_{\text{all in } o(|\bar{h}|^k)\text{; not hard, see appendix A.11}}$$

... etc for $(g\circ f)(\bar{x})$ and for $F\begin{pmatrix}\bar{x}\\ \bar{g}(\bar{x})\end{pmatrix} = 0$; see appendix A.11.