(33)

2/10/2017 →

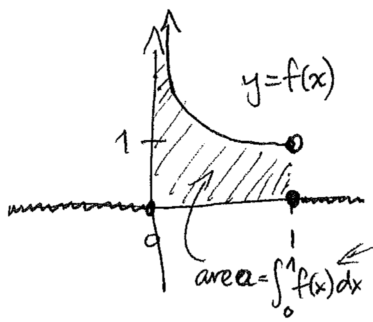
Chapter 4 Integration (of functions over all of $\mathbb{R}^n$)

(Ch. 5 = integrating functions over curves, surfaces, k-dim'l manifolds $\subset \mathbb{R}^n$, $k \le n$, e.g. arc length, surface area, etc.)

§4.1 Defining integrals

Initially, we'll start out sounding very restrictive about the functions $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ for which we'll define the (Riemann) integral $\int_{\mathbb{R}^n} f(\bar{x})\, |d^n\bar{x}|$

- We'll assume f is bounded, i.e. $\exists M$ with $|f(\bar{x})| \le M\ \forall \bar{x} \in \mathbb{R}^n$



(e.g. $\mathbb{R}^1 \xrightarrow{f} \mathbb{R}$

$x \longmapsto f(x) = \begin{cases} 0 & \text{if } x \notin (0,1) \\ \frac{1}{\sqrt{x}} & \text{if } x \in (0,1) \end{cases}$

would be disallowed initially, even though it's a convergent improper integral from 1-variable calc)

- We'll assume the support of f in $\mathbb{R}^n$ is bounded, where

$$\operatorname{supp}(f) \underset{\text{DEFN 4.1.2}}{:=} \overline{\{\bar{x} \in \mathbb{R}^n : f(\bar{x}) \neq 0\}}$$

← the closure, i.e. all the points in $\mathbb{R}^n$ arbitrarily close to points $\bar{x}$ with $f(\bar{x}) \neq 0$

supp(f) bounded means it lies in some such ball, i.e. $\exists$ such an M

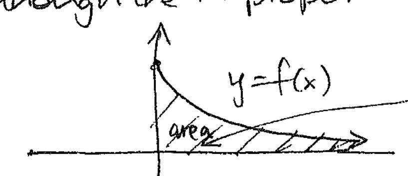
$\cap$

$\underbrace{B_M}(\bar{0})$ ball of radius M about $\bar{0}$

(e.g. $\mathbb{R}^1 \xrightarrow{f} \mathbb{R}$

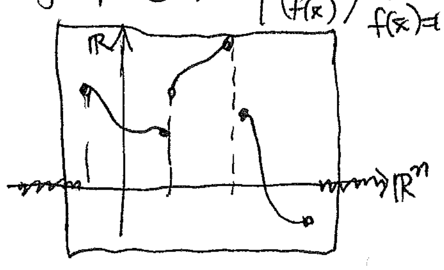
$x \longmapsto f(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-x} & \text{if } x > 0 \end{cases}$

would be disallowed initially, even though the improper integral area $= \int_0^\infty e^{-x} dx$ converges)



In other words, we're assuming the nonzero part of the graph of $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$

$$\text{nonzero graph}(f) := \left\{ \begin{pmatrix} \bar{x} \\ f(\bar{x}) \end{pmatrix} : \begin{array}{l} \bar{x} \in \mathbb{R}^n \\ f(\bar{x}) \neq 0 \end{array} \right\} \subset \mathbb{R}^{n+1} \text{ is bounded in } \mathbb{R}^{n+1}$$

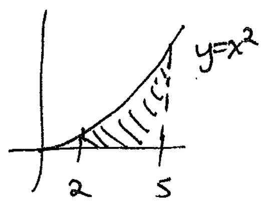


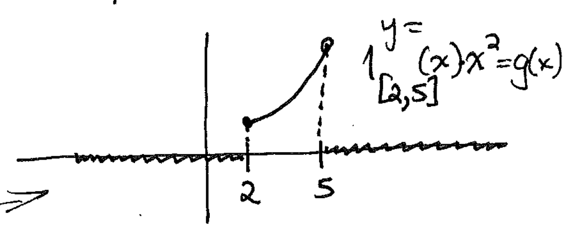
• Instead of integrating over subsets $A \subset \mathbb{R}^n$, say $\int_A f(\bar{x})|d^n\bar{x}|$ we'll always integrate $g(\bar{x}) := 1_A(x) f(\bar{x})$ over all of $\mathbb{R}^n$,

i.e. $$\int_A f(\bar x)|d^n\bar x| := \int_{\mathbb{R}^n} 1_A(\bar x) f(\bar x)\,|d^n \bar x| \quad \text{where } 1_A(\bar x) = \begin{cases}1 & \text{if } \bar x\in A\\ 0 & \text{if } \bar x\notin A\end{cases}$$

the indicator function of A (DEF'N 4.1.1)

e.g. instead of $\int_2^5 x^2\,dx$, we'll compute $\int_{\mathbb{R}^1} 1_{[2,5]}(x)\cdot x^2\,|d^1x|$





note: $g(x)$ is bounded, of bounded support!

• We'll use higher dimensional versions of limits of Riemann sums to define $\int_{\mathbb{R}^n} f(\bar x)\,d\bar x$, but the subdivisions of $\mathbb{R}^n$ will initially always come the dyadic pavings

DEF'NS 4.1.5, 4.1.7 The level N or N^{th} dyadic paving of $\mathbb{R}^n$ is the collection of (semi-open) cubes

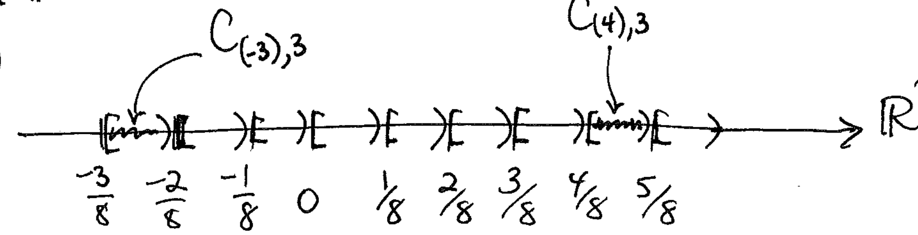
$$D_N(\mathbb{R}^n) = \Big\{ \underbrace{C_{\begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix},N}}_{\text{the semi-open cube of side length } \frac{1}{2^N} \text{ with a corner at } \frac{1}{2^N}\begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix}} : k_1, k_2, \dots, k_n \in \mathbb{Z} \Big\}$$

$$:= \Big\{ \bar x \in \mathbb{R}^n : \quad x_i \in \Big[\frac{k_i}{2^N}, \frac{k_i+1}{2^N}\Big) \Big\}$$

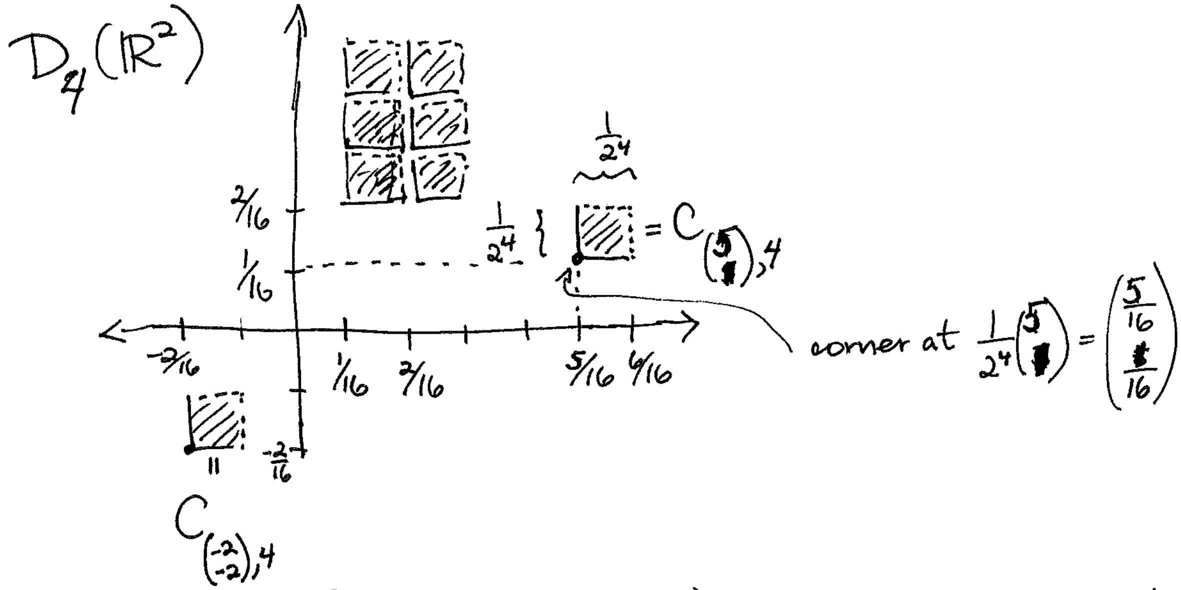
---

e.g. $N=3$ in $\mathbb{R}^1$

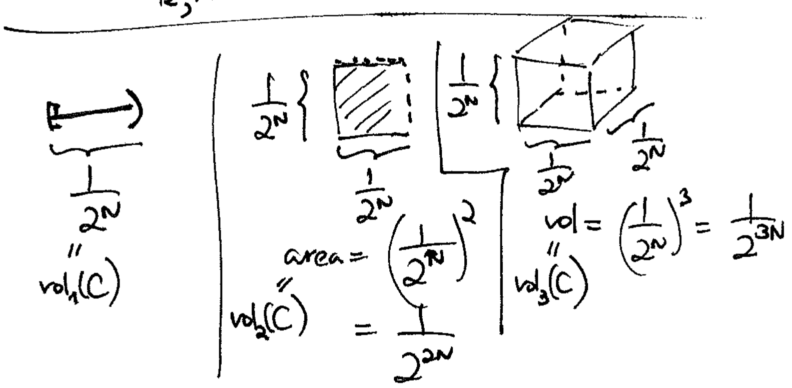
$D_3(\mathbb{R}^1)$



(35)



Note that each cube $C = C_{\bar{k},N} \in D_N(\mathbb{R}^n)$ has n-dim'l volume $\mathrm{vol}_n(C) = \left(\frac{1}{2^N}\right)^n = \frac{1}{2^{nN}}$



So we approximate $\int_{\mathbb{R}^n} f(\bar{x})\,d\bar{x}$ from below and above with....

DEF'N 4.1.8: $U_N(f) := \sum'_{\text{cubes } C \in D_N(\mathbb{R}^n)} M_C(f) \cdot \mathrm{vol}_n(C)$ — N^{th} upper sum

$L_N(f) := \sum_{C \in D_N(\mathbb{R}^n)} m_C(f) \cdot \mathrm{vol}_n(C)$

where $m_A(f) := \sup\{ f(\bar{x}) : \bar{x} \in A\} \in \mathbb{R}$ $\begin{pmatrix} \neq +\infty \\ \neq -\infty \end{pmatrix}$ since f is bounded

$M_A(f) := \inf\{ f(\bar{x}) : \bar{x} \in A\} \in \mathbb{R}$

NOTE: Really finite sums since $\mathrm{supp}(f)$ is bounded, i.e. most cubes C in $D_N(\mathbb{R}^n)$ have $f(\bar{x}) = 0$ $\forall \bar{x} \in C$ so $m_C(f) = M_C(f) = 0$

Of course, $m_C(f) \leq M_C(f)$ $\forall C$
$\Rightarrow L_N(f) \leq U_N(f)$ $\forall N$.