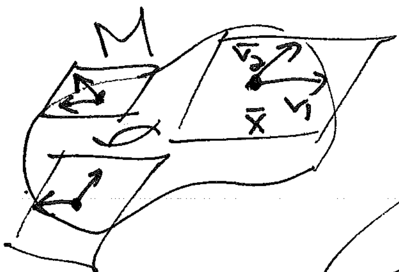
(98)

On a manifold $M \subset \mathbb{R}^n$, one simply orients each tangent space $T_{\bar{x}}M$ in a continuous fashion:

DEF'N 6.3.3: For a $k$-dim'l manifold $M \subset \mathbb{R}^n$, an orientation is a continuous map




$$\mathcal{B}(M) := \left\{ (\bar{x}, \bar{v}_1, \dots, \bar{v}_k) : \bar{x} \in M, \; \underset{\text{an ordered basis of } T_{\bar{x}}M}{(\bar{v}_1, \dots, \bar{v}_k)} \right\} \xrightarrow{\Omega} \{\pm 1\}$$

$$\cap$$

$$\mathbb{R}^n \times \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k \text{ times}} = \mathbb{R}^{n(k+1)}$$

Think of this as the space of anchored parallelopipeds $P_{\bar{x}}(\bar{v}_1, \dots, \bar{v}_k)$ on $M$

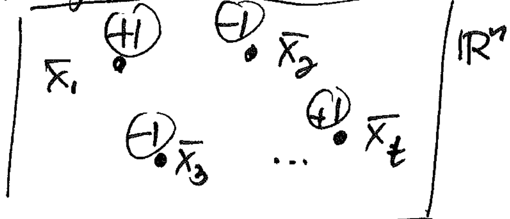
such that $\forall \bar{x} \in M$, the restriction of $\Omega$ to $\mathcal{B}_{\bar{x}}(M) = \left\{ \underset{\text{an ordered basis for } T_{\bar{x}}M}{(\bar{v}_1, \dots, \bar{v}_k)} \right\} \xrightarrow{\Omega} \{\pm 1\}$ is an orientation of $T_{\bar{x}}M$.

4/7/2017 >

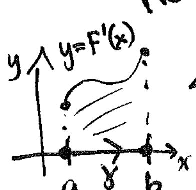
EXAMPLES (some from PROP 6.3.4)

(0) $k=0$: A 0-dim'l manifold $M$ is a finite set of points
$$M = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_t\}$$
and each $T_{\bar{x}_i}M$ is a 0-dim'l vector space, with only one basis, the empty list () (!)

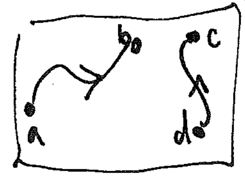
so you have to assign a +1 or −1 to each point $\bar{x}_i$ arbitrarily:



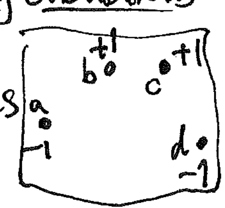

REMARK: This really will come up when we talk about boundary orientations (§6.6) for manifolds-with-boundary:



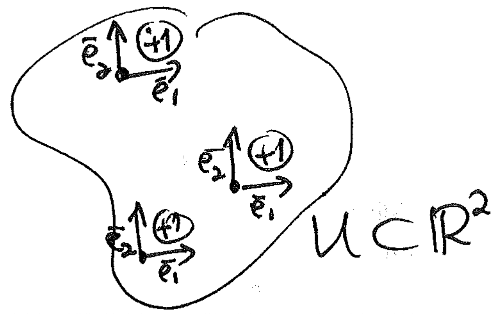

FTC: $\int_\gamma F' = +F(b) - F(a)$




induces

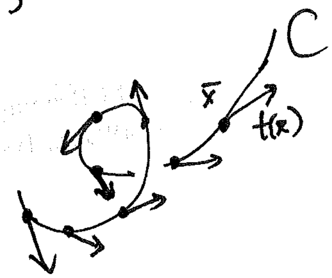


(99) ① For $\mathcal{U} \overset{open}{\subset} \mathbb{R}^n$, there is a standard orientation that makes every $\bar{x} \in \mathcal{U}$ have $(\bar{e}_1, \ldots, \bar{e}_n)$ a direct ($\Omega = +1$) basis for $T_{\bar{x}}(\mathcal{U})$




② For $k=1$, so the 1-dim'l manifold is a curve $C$, one way to get an orientation is from a choice of a non-vanishing vector field on $C$, i.e. a choice of $\bar{t}(\bar{x}) \in T_x C - \{\bar{0}\}$ for every $\bar{x} \in C$, that varies continuously, i.e. so that ~~this is a~~ continuous map:

$$C \longrightarrow \mathbb{R}^n - \{\bar{0}\}$$
$$\bar{x} \longmapsto \bar{t}(\bar{x})$$




Then one can define

$$\mathcal{B}(C) = \{(\bar{x}, \bar{v}) : \bar{x} \in C, \bar{v} \in T_{\bar{x}}(C) - \{\bar{0}\}\} \xrightarrow{\Omega} \{\pm 1\}$$
$$(\bar{x}, \bar{v}) \longmapsto \operatorname{sgn}(\bar{t}(\bar{x}) \cdot \bar{v})$$

This $\Omega$ is continuous because it's a composite of continuous maps

$$\begin{array}{ccccccc} (\mathbb{R}^n)^2 & \longrightarrow & (\mathbb{R}^n)^2 & \longrightarrow & \mathbb{R} & \longrightarrow & \{+1, -1\} \\ (\bar{x}, \bar{v}) & & (\bar{t}(\bar{x}), \bar{v}) & & \bar{t}(\bar{x}) \cdot \bar{v} & & \operatorname{sgn}(\bar{t}(\bar{x}) \cdot \bar{v}) \end{array}$$

and on each $\mathcal{B}_{\bar{x}}(C)$ its restriction is an orientation, since if $\bar{v}, \bar{w} \in T_x(C) - \{\bar{0}\}$
have $\bar{v} = c \cdot \bar{w}$
then $\operatorname{sgn}(\bar{t}(\bar{x}) \cdot \bar{v}) = \operatorname{sgn}(c) \operatorname{sgn}(\bar{t}(\bar{x}) \cdot \bar{w})$

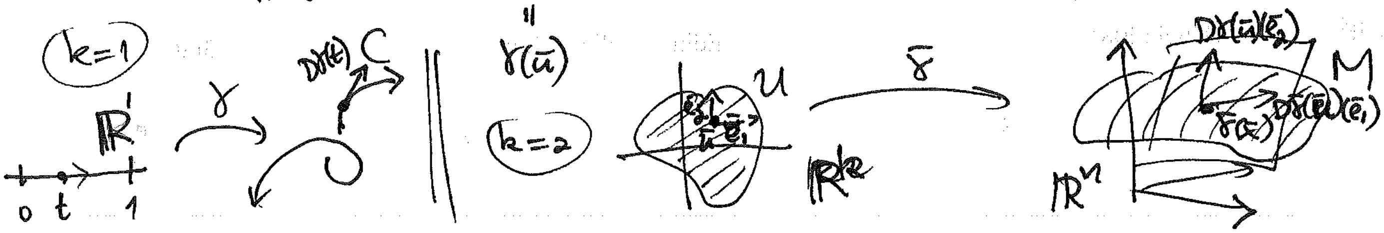
One way to get such a non-vanishing vector field is to parametrize $C$ bijectively ...

(100) ③ A bijective parametrization $\underset{\mathbb{R}^k}{\overset{\cap}{U}} \xrightarrow{\ \bar\gamma\ } \underset{\mathbb{R}^n}{\overset{\cap}{M}}$ of a k-dim'l manifold

$\boxed{\text{rare!}}$

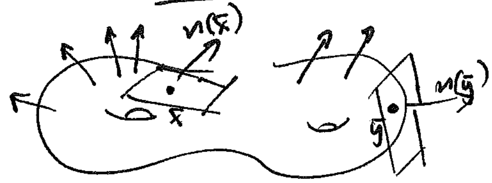
that has $\underset{\mathbb{R}^k \to \mathbb{R}^n}{D\bar\gamma(\bar u)}$ injective $\forall \bar u \in U$ will induce an orientation

in which at $\underset{\bar\gamma(\bar u)}{\overset{\Vert}{\bar x \in M}}$ one has $\Omega(D\bar\gamma(\bar u)(\bar e_1), \ldots, D\bar\gamma(\bar u)(\bar e_k)) = +1$




④ For surfaces $S \subset \mathbb{R}^3$ (2-dim'l manifolds), one can sometimes get an orientation

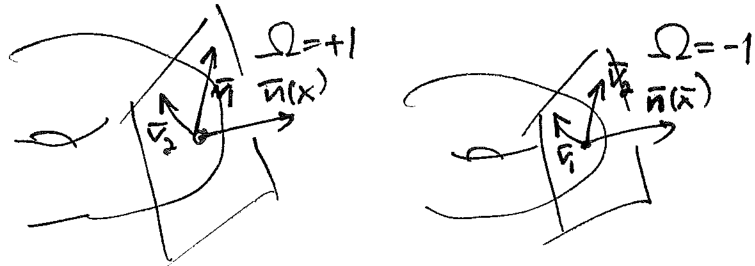
from a transverse vector field $S \xrightarrow{\ \bar n\ } \mathbb{R}^3$, continuous
$\bar x \longmapsto \bar n(\bar x) \notin T_{\bar x}(S)$




by decreeing for $(\bar x, \underbrace{\bar v_1, \bar v_2}_{\in T_{\bar x}(S)}) \in \mathcal{B}(S)$ that $\Omega(\bar x, \bar v_1, \bar v_2) :=$

$$\operatorname{sgn}\det\begin{bmatrix} | & | & | \\ \bar n(\bar x) & \bar v_1 & \bar v_2 \\ | & | & | \end{bmatrix}$$

$$= \operatorname{sgn}(\bar n(\bar x) \bullet (\bar v_1 \times \bar v_2))$$



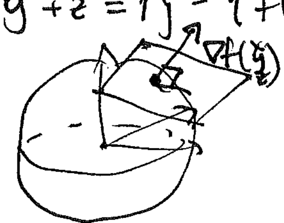

Sometimes one can get $S \xrightarrow{\ \bar n\ } \mathbb{R}^3$ when $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$

by setting $\bar n(\bar x) = \nabla f(\bar x) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = Df(\bar x)$, since we know $T_{\bar x}(S) = \ker Df(\bar x)$

i.e. $\nabla f(\bar x) \perp T_{\bar x}(S)$

e.g. $S = \{\text{unit sphere } x^2+y^2+z^2 = 1\} = \left\{ f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$ where $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2+y^2+z^2-1$

$$\nabla f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$



(101)

(5) PROP 6.3.8 generalizes this:

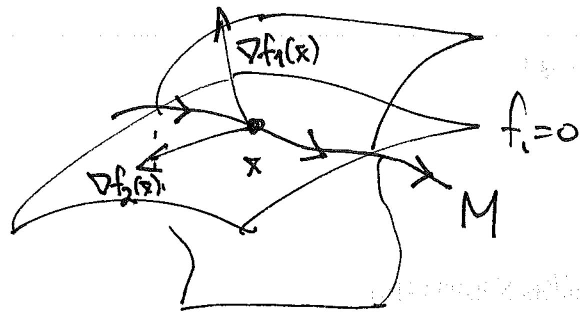
If $M$ is a $k$-dim'l manifold in $\mathbb{R}^n$ defined as $\bar{f}(\bar{x}) = \bar{0}$ for a $C^1$ map $\begin{matrix} U & \xrightarrow{\bar{f}} & \mathbb{R}^{n-k} \\ \cap & & \\ \mathbb{R}^n & & \end{matrix}$ with $D\bar{f}(\bar{x})$ surjective $\forall \bar{x} \in M$

then decreeing for $(\bar{x}, \underbrace{\bar{v}_1, \ldots, \bar{v}_k}_{\in T_{\bar{x}}M}) \in \mathcal{B}(M)$ that

$$\Omega(\bar{x}, \bar{v}_1, \ldots, \bar{v}_k) = \operatorname{sgn} \det(\nabla f_1(\bar{x}), \ldots, \nabla f_{n-k}(\bar{x}), \bar{v}_1, \ldots, \bar{v}_k)$$

gives an orientation on $M$

e.g. $n=3$
$k=1$

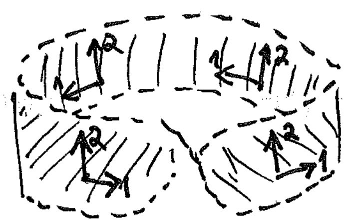



Discuss this first!

NON-EXAMPLES

(6) - Not all manifolds are orientable, e.g.

Möbius band $S$



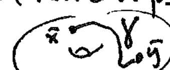

It doesn't make sense to do flux integrals over $S$, although surface area of $S$ makes sense!

(7) EXAMPLE 6.4.9: $X := \{ \underset{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}}{\overset{\shortparallel}{A}} \in \mathrm{Mat}(2,3) : \operatorname{rank}(A) = 1 \}$ will turn out to be a 4-dim'l manifold inside $\mathrm{Mat}(2,3) \cong \mathbb{R}^6$, but non-orientable (later).

In showing this, it's helpful to note this fact:

PROP 6.3.10: A (path-) connected manifold $M$ having an orientation $\Omega$ will have only two orientations: $\Omega$ and $-\Omega$.

every $\bar{x}, \bar{y} \in M$ have a path $[0,1] \xrightarrow[\text{continuous}]{\gamma} M$




$\gamma(0) = \bar{x}$
$\gamma(1) = \bar{y}$

proof: Enough to show that if 2 orientations $\Omega, \Omega'$ agree at some $(\bar{x}, \bar{v}_1, \ldots, \bar{v}_k) \in \mathcal{B}(M)$, they agree on all of $\mathcal{B}(M)$.