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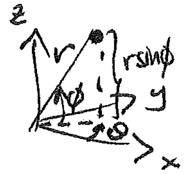
(106) We sometimes need to be careful about whether $\bar{\gamma}$ is o.p.

EXAMPLE: (6.4.5) If we use spherical coords



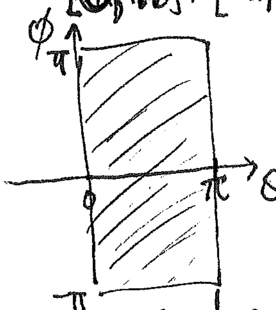
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\phi \cos\theta \\ r\cos\phi \sin\theta \\ r\sin\phi \end{pmatrix}$$

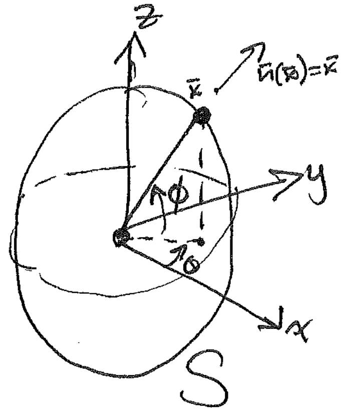
to parametrize the unit sphere S via

$$\overset{\theta}{[0,\pi]} \times \overset{\phi}{[-\pi,+\pi]} = U \xrightarrow{\bar{\gamma}} \mathbb{R}^3$$

$$U \subset \mathbb{R}^2$$

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} \xmapsto{\bar{\gamma}} \begin{pmatrix} \cos\phi \cos\theta \\ \cos\phi \sin\theta \\ \sin\phi \end{pmatrix}$$





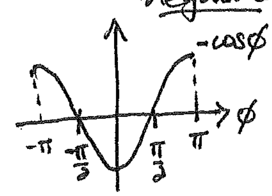
and orient S via the outward normal $\bar{n}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$,

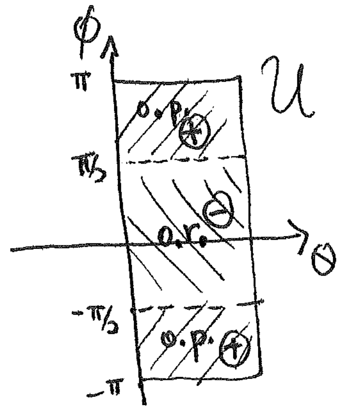
then to have $\bar{\gamma}$ be o.p. we would need

$$\det\left[\bar{n}\begin{pmatrix} \theta \\ \phi \end{pmatrix}, \underbrace{D\bar{\gamma}\begin{pmatrix} \theta \\ \phi \end{pmatrix}(\bar{e}_1)}_{\text{i.e. } \partial/\partial\theta}, \underbrace{D\bar{\gamma}\begin{pmatrix} \theta \\ \phi \end{pmatrix}(\bar{e}_2)}_{\text{i.e. } \partial/\partial\phi}\right] \overset{?}{>} 0 \quad \forall \begin{pmatrix} \theta \\ \phi \end{pmatrix} \in U \quad (\text{or } U - X \text{ for some } X)$$

$$= \det\begin{bmatrix} \cos\phi\cos\theta & -\cos\phi\sin\theta & -\sin\phi\cos\theta \\ \cos\phi\sin\phi & \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \underset{\text{(algebra omitted)}}{= \cdots =} -\cos\phi,$$

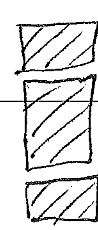
which is both positive and negative on U:





When $\phi = \pm\frac{\pi}{2}$, the orientation can reverse.

Note $\bar{\gamma}$ only has $D\bar{\gamma}(\bar{u})$ injective on $\underbrace{U - X}_{\text{a disconnected set}}$, where X contains $\phi = \pm\frac{\pi}{2}$ lines



EXERCISE: Check that if we instead use

$$\overset{\theta}{[0,2\pi]} \times \overset{\phi}{\left[-\frac{\pi}{2},\frac{\pi}{2}\right]} = U \xrightarrow{\bar{\gamma}} \mathbb{R}^3$$

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} \xmapsto{\bar{\gamma}} \underbrace{\begin{pmatrix} \cos\phi \cos\theta \\ \cos\phi \sin\theta \\ \sin\phi \end{pmatrix}}_{\text{same!}}$$

then $\bar{\gamma}$ is o.p.

§6.6, 6.7 Boundaries & exterior derivatives

Now we're headed toward a big goal:

Stokes's Theorem:

There is a notion of k-dim'l manifold-with-boundary $X \subset \mathbb{R}^n$ (§6.6)

such that its boundary ∂X is a $(k-1)$-dim'l manifold,

and an orientation Ω on X induces one on ∂X,

along with an exterior derivative operation on k-forms (§6.7)

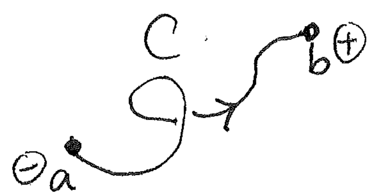
$$A^{k-1}(\mathbb{R}^n) \xrightarrow{d} A^k(\mathbb{R}^n)$$
$$\varphi \longmapsto d\varphi$$

such that
$$\boxed{\int_{\partial X} \underbrace{\varphi}_{(k-1)\text{-form}} = \int_X \underbrace{d\varphi}_{k\text{-form}}}$$

(rough idea)

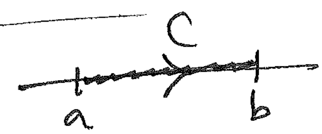
EXAMPLES:

$k=1$: $X = C$ a curve with boundary endpoints a, b



$$\underbrace{\int_C \overbrace{d\varphi}^{1\text{-form}}}_{\text{a line integral}} = \int_{\partial C} \overbrace{\varphi}^{0\text{-form, i.e. function } \mathbb{R}^n \xrightarrow{\varphi} \mathbb{R}} = +f(b) - f(a)$$

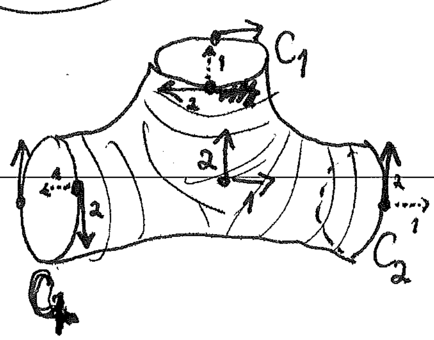
special case $n=1$ $\rightsquigarrow$



$$\int_a^b \varphi'(x)\,dx = \int_C d\varphi = \int_{\partial C} \varphi = \varphi(b) - \varphi(a)$$

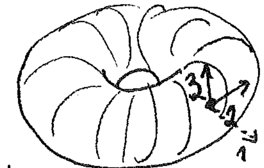
Fund'l Thm. Calc.

$k=2$: $X = S$ a surface with boundary curves $C_1, C_2, \ldots, C_r$



$$\underbrace{\int_S \overbrace{d\varphi}^{2\text{-form}}}_{\text{a flux integral}} = \underbrace{\int_{\partial S} \overbrace{\varphi}^{1\text{-form}}}_{\text{a sum of line integrals}} = \sum_{i=1}^{r} \int_{C_i} \varphi$$

$k=3$ $\quad X$ = a 3-dim'l manifold with boundary surfaces $S_1, S_2, \dots, S_r$



X = solid torus (filled donut)

∂X = torus (skin/crust of donut)

$$\underbrace{\int_X \overbrace{d\varphi}^{\text{3-form}}}_{\text{a volume integral}} = \underbrace{\int_{\partial X} \overbrace{\varphi}^{\text{2-form}}}_{\text{a sum of flux integrals}} = \sum_{i=1}^{r} \int_{S_i} \varphi$$

Turns out we'll need to integrate more generally over "pieces-with-boundary" $X \subset M$ when M is a manifold ...

Our old notion of boundary $\partial X = \underbrace{\bar{X}}_{\text{closure of } X} - \underbrace{\mathring{X}}_{\text{interior of } X}$ is not what we want, inside of M....

DEF'N 6.6.1: For $X \subset M$ a k-dim'l manifold

- the boundary of X in M

$$\partial_M X := \{\bar{x} \in M : \text{every open set } U \ni \bar{x} \text{ has both } U \cap X \neq \emptyset, \; U \cap (M - X) \neq \emptyset\}$$

- the smooth points in $\partial_M X$

$\partial_M^s X := \{ \bar{x} \in M :$ not only is there some $\underset{\mathbb{R}^n}{\underset{\cap}{U}}^{\text{open}} \xrightarrow{\bar{f}} \mathbb{R}^{n-k}$ in $C^1(U)$, U containing $\bar{x}$, such that $U \cap M = \bar{f}^{-1}(\bar{0})$, $D\bar{f}(\bar{x})$ surjective

but one can extend it to $U^{\text{open}} \xrightarrow{\binom{\bar{f}}{g}} \mathbb{R}^{n-k+1}$ in $C^1(U)$ such that $g(\bar{x})=0$, and $U \cap X = \bar{f}^{-1}(\bar{0}) \cap \{g \geq 0\}$, $D\binom{\bar{f}}{g}(\bar{x})$ surjective, $\}$

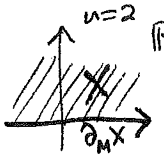
$U \cap M = \{\bar{f} = 0\}$

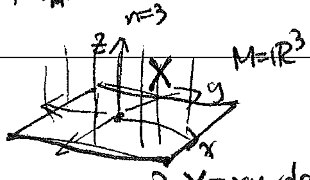
4/17/2017 $\rightarrow$ EXAMPLES:

(1) Inside $M = \mathbb{R}^n$ itself,

$$\underset{\cup}{M}$$
$$X = \left\{ \bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_n \geq 0 \right\}$$
$$\cup$$
$$\partial_M X = \left\{ \bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_n = 0 \right\} \stackrel{s}{=} \partial_M^s X$$

i.e. the whole boundary $\partial_M X$ in M is smooth





$\partial_M X$ = xy-plane