Math 3593H Honors Math II
Quiz 4, Thursday April 20, 2017

Instructions:
20 minutes, closed book, no electronic devices,
but an 8.5×11 page of notes is OK.
There are three problems, worth a total of 20 points.

1. Let $\bar{F}$ be the vector field on $\mathbb{R}^3$ defined by $\bar{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} x^2 \\ y^3 \\ z^4 \end{bmatrix}$

(a) (3 points) Write down the associated work 1-form $W_{\bar{F}}$ in $A^1(\mathbb{R}^3)$.

$$W_{\bar{F}} = x^2\,dx + y^3\,dy + z^4\,dz$$

(b) (3 points) Write down the associated flux 2-form $\Phi_{\bar{F}}$ in $A^2(\mathbb{R}^3)$.

$$\Phi_{\bar{F}} = +x^2\,dy\wedge dz - y^3\,dx\wedge dz + z^3\,dx\wedge dy$$

2. (7 points) Parametrize $C \subset \mathbb{R}^3$ via the map from $U = (1,2) \subset \mathbb{R}$

$$\begin{array}{ccc} U & \xrightarrow{\bar{\gamma}} & C \\ t & \longmapsto & \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}, \end{array}$$

and orient C via $\bar{\gamma}$, that is, $C = [\bar{\gamma}(U)]$. Calculate $\int_C x^2 z^2\; dy$.

$$\begin{array}{ccc} \mathbb{R}^1 & \xrightarrow{D\bar{\gamma}(t)} & \mathbb{R}^3 \\ t & \longmapsto & \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix} \end{array}$$

$$\text{so } \int_C x^2 z^2\,dy = \int_{t=1}^{t=2} (x^2 z^2\,dy)\,[D\bar{\gamma}(t)]\;|d^1t|$$

$$= \int_{t=1}^{t=2} t^2\cdot(t^3)^2\cdot 2t\;dt = 2\int_1^2 t^9\,dt = 2\left[\frac{t^{10}}{10}\right]_1^2$$

$$= \frac{1}{5}\left(2^{10}-1\right)$$

3. (7 points) Prove or disprove:
The parametrization of the strict upper-halfplane

$$M = \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : y > 0\} \subset \mathbb{R}^2$$

via the polar coordinate map from

$$U := \{\left(\begin{smallmatrix} r \\ \theta \end{smallmatrix}\right) : r > 0 \text{ and } 0 < \theta < \pi\} \subset \mathbb{R}^2$$

given by

$$\begin{array}{ccc} U & \xrightarrow{\bar{\gamma}} & M \\ \begin{pmatrix} r \\ \theta \end{pmatrix} & \longmapsto & \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \end{array}$$

is **order-preserving**, when U, M are both given their standard orientations as open subsets of $\mathbb{R}^2$.

True. The map $\mathbb{R}^2 \xrightarrow{D\bar{\gamma}\left(\begin{smallmatrix} r \\ \theta \end{smallmatrix}\right)} \mathbb{R}^2$

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \longmapsto \begin{array}{cc} \partial/\partial r & \partial/\partial\theta \\ \end{array}$$
$$\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

has $\det D\bar{\gamma}\begin{pmatrix} r \\ \theta \end{pmatrix} = \cos\theta \cdot r\cos\theta - \sin\theta(-r\sin\theta)$

$= r(\cos^2\theta + \sin^2\theta)$

$= r > 0 \quad \forall \begin{pmatrix} r \\ \theta \end{pmatrix} \in U,$

hence $\bar{\gamma}$ is order-preserving.