CONVERGENCE OF SOLITARY-WAVE SOLUTIONS IN A PERTURBED BI-HAMILTONIAN DYNAMICAL SYSTEM. I. COMPACTONS AND PEAKONS.

Y. A. Li[1] and P. J. Olver[1,2]

Abstract. We investigate how the non-analytic solitary wave solutions — peakons and compactons — of an integrable biHamiltonian system arising in fluid mechanics, can be recovered as limits of classical solitary wave solutions forming analytic homoclinic orbits for the reduced dynamical system. This phenomenon is examined to understand the important effect of linear dispersion terms on the analyticity of such homoclinic orbits.

1. Introduction. Classically, the solitary wave solutions of nonlinear evolution equations are determined by analytic formulae (typically a sech^2 function or variants thereof) and serve as prototypical solutions that model physical localized waves. In the case of integrable systems, the solitary waves interact cleanly, and are known as solitons. For many examples, localized initial data ultimately breaks up into a finite collection of solitary wave solutions; this fact has been proved analytically for certain integrable equations such as the Korteweg-deVries equation, [2], and is observed numerically in many others. More recently, the appearance of non-analytic solitary wave solutions to new classes of nonlinear wave equations, including peakons, [6], [13], which have a corner at their crest, cuspons, [24], having a cusped crest, and, compactons, [17], [18], [20], which have compact support, has vastly increased the menagerie of solutions appearing in model equations, both integrable and non-integrable. The distinguishing feature of the systems admitting non-analytic solitary wave solutions is that, in contrast to the classical nonlinear wave equations, they all include a nonlinear dispersion term, meaning that the highest order derivatives (characterizing the dispersion relation) do not occur linearly in the system, but are typically multiplied by a function of the dependent variable.

[1] School of Mathematics, University of Minnesota, Minneapolis, MN 55455.
[2] Research supported in part by NSF Grant DMS 95–00931 and BSF Grant 94–00283.
AMS subject classifications: 34A20, 34C35, 35B65, 58F05, 76B25

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

The first and most important of the nonlinearly dispersive, integrable equations is the equation

$$u_t + \nu u_{xxt} = \alpha u_x + \beta u_{xxx} + 3\gamma u u_x + \gamma\nu(u u_{xxx} + 2 u_x u_{xx}). \tag{1.1}$$

Here α, β, γ and ν are real constants and $u(x,t)$ is the unknown function depending on the temporal variable t and the spatial variable x. This equation contains both linear dispersion terms νu_{xxt}, βu_{xxx}, and the nonlinear dispersion terms $u u_{xxx}$. Equation (1.1) can, in certain parameter regimes, be regarded an an integrable perturbation of the well-known BBM (or regularized long wave) equation

$$u_t + u_x + u u_x - u_{xxt} = 0, \tag{1.2}$$

which was originally proposed, [4], as an alternative to the celebrated Korteweg-deVries (KdV) equation

$$u_t + u_x + u u_x + u_{xxx} = 0, \tag{1.3}$$

in the modeling of dispersive nonlinear wave phenomena. Although the integrability of the KdV equation makes it a more mathematically significant equation (see [2]), the BBM equation has better analytical properties, including the more desirable linear dispersion relation for fluid modeling, [4].

Equation (1.1) first appears (albeit with a slight error in the coefficients) in the work of Fuchssteiner (ref. [8], Equation (5.3)). Camassa and Holm, [6], derived (1.1), for certain values of the coefficients including $\nu < 0$, as a model for water waves, and established an associated linear scattering problem. They began a systematic study of the solutions of (1.1), discovering that its soliton solutions are only piecewise analytic, having a corner at their crest, and hence named them peakons. Although not classical solutions, peakons do form weak solutions of (1.1). Like classical solitons, there exist multi-peakon solutions of (1.1) with cleanly interacting peakons; see [3] for a detailed analysis of their behavior. At the same time, Rosenau, [18], discovered a wide variety of nonlinear wave equations with nonlinear dispersion that admit compactly supported solitary wave solutions — epitomized by the (presumably) nonintegrable family of equations

$$u_t \pm (u^m)_x + (u^n)_{xxx} = 0, \tag{1.4}$$

depending on positive integers m, n. If $n \geq 2$, (1.4) admits a one-parameter family of compactons, the parameter being the wave speed (which also governs its amplitude). Inspired by this discovery, Rosenau proposed the alternative form of Equation (1.1), with $\nu > 0$, and claimed that it provided an example of an integrable system supporting compacton solutions. However, very few of Rosenau's proposed compactons are actually solutions of (1.1) in the weak sense — indeed we shall prove that there is at most a single wave speed for which (1.1) with $\nu > 0$ admits a compacton solution vanishing at infinity. An additional difficulty with this equation is that, for $\nu > 0$, the dispersion relation for (1.1) has singularities which raises questions about it being a well-posed initial value problem.

The nonlinear wave equation (1.1) is just one of a wide variety of examples of "dual Hamiltonian systems" which can be constructed from classical soliton equations such as the KdV equation by either a Lagrange transformation, [19], or, more generally, a rearrangement of the operators appearing in their bi- (or, rather, multi-) Hamiltonian structure; the basic method appears in [9], and has been extensively developed in [7], [16], to which we refer the reader for many additional interesting examples. We remark that certain versions of the Harry Dym equation derived by this dualization procedure do admit parametrized families of compactons and are integrable (at least in the sense that they appear in a Hamiltonian hierarchy).

Although one can characterize the compactons and peakons as weak solutions, we remark that they are solutions in a considerably stronger (although not quite classical) sense than the more traditionally studied weak solutions such as shock waves. As a matter of fact, they are piecewise analytic, satisfying the equation in a classical sense away from singularities, and, moreover, each term (or certain combinations of terms) in the equation have well defined limits at the singularities. These facts will be demonstrated in Sections 6 and 7 in Part II of this paper. Indeed, one does not require any entropy condition to prescribe the type of singularity. Thus, we propose an apparently new and potentially useful definition of such "pseudo-classical" solutions that will handle a wide variety of such non-analytic solitary wave solutions, and distinguish them from shock waves.

For classical, linearly dispersive systems, the characterization of solitary waves and more complicated solutions by their analyticity properties has been the focus of a significant amount of study. Kruskal, [12], proposed analyzing the interactive properties of solitary wave solutions by the behavior of their poles in the complex plane. The Painlevé test for integrability of nonlinear systems, [1], [22], [25], is based on the analyticity of their solutions. The convergence of the general Painlevé series expansions has been studied in [11]. Recently Bona and Li, [5], showed that for various types of linearly dispersive systems, including equations of both KdV and BBM type, all weak solitary wave solutions which are essentially bounded and decay to zero at infinity are necessarily classical solutions, and can be analytically extended to a horizontal strip in the complex plane containing the real axis. In contrast, we will demonstrate that the system (1.1) not only has solitary wave solutions that are restrictions of analytic functions defined on a horizontal strip, but also admits both compactons, whose second order derivatives are discontinuous, and peakons, whose first order derivatives have a discontinuity, as weak (or, rather, pseudo-classical) solutions. The existence of such types of non-analytic solutions requires that the linear dispersion term vanishes for certain function values, and these are precisely the values at which the discontinuities of the solution appear. This fact indicates the important role played by the linear dispersion terms in the formation of analytic travelling wave solutions, and the significant influence of nonlinear terms on the behavior of these solutions.

The appearance of non-analytic solutions thus draws our attention to a more detailed understanding of the effects of both linear and nonlinear dispersion terms on travelling wave solutions, especially, on solitary wave solutions. In this paper, we shall discuss how such non-analytic solitary wave solutions can appear as the limits of classical, analytic solitary wave solutions. (This observation is not as paradoxical as it might initially seem — a classical instance of such loss of analyticity occurs in the convergence of Fourier series.) Such limits can be effected in two different, but essentially equivalent ways. First one can add to the equation a small linear dispersion term, having the effect of forcing analyticity of the perturbed solitary wave solution, and then allowing the coefficient of this additional term to vanish. In (1.1), the coefficient ν has this effect, provided we compensate by setting $\gamma = \tilde{\gamma}/\nu$ to leave the nonlinearly dispersive term uu_{xxx} intact. This approach is, in part, motivated by the convergence properties of the KdV equation as the dispersion term (meaning the coefficient of u_{xxx}) goes to zero; the convergence of classical solutions to the KdV equation to non-analytic shock wave solutions of the resulting dispersionless Burgers' equation $u_t + uu_x = 0$ was analyzed in great detail by Lax, Levermore and Venakides, [14], [15], [23]. Of course, our situation is analytically simpler since the limiting equation does not have shocks, and, besides, we are only attempting to analyze solitary wave solutions. Alternatively, one can replace the vanishing condition at infinity by the condition $u \to a$ as $|x| \to \pm\infty$, meaning that the wave appears as a disturbance on a fluid of uniform depth a. For most values of a, this has the effect of eliminating the effect of the nonlinearly dispersive terms, and again one can investigate how the associated analytic solitary wave solutions lose their analyticity as the undisturbed depth $a \to 0$. Of course, these two approaches are closely related — one can replace u by $\hat{u} = u - a$ to eliminate the non-zero asymptotic depth; the resulting equation will then include an additional linearly dispersive term depending on the small parameter a (as well as additional nonlinear terms). In this paper we investigate both strategies for (1.1) — here the transformation $\hat{u} = u - a$

merely redefines the parameters $\nu, \alpha, \beta, \gamma$ appropriately. Our main result is that, in all cases, analytic solitons converge to non-analytic peakons and compactons *provided* the latter are weak (i.e. pseudo-classical) solutions to the system. Thus the convergence of analytic solitary wave solutions under vanishing linear dispersion is to pick out those non-analytic solitary wave solutions which are "genuine" in the sense that they are weak, or pseudo-classical, solutions. We anticipate that this will form a rather general convergence phenomena, applicable to both integrable and nonintegrable systems, including (1.4), alike.

Our analysis breaks into three parts. First, methods from the theory of dynamical systems — in particular center manifold theory — will be employed to produce a preliminary analysis of the ordinary differential equations describing travelling wave solutions to Equation (1.1). This allows us to determine the precise parameter regimes for which (1.1) admits solitary wave solutions, which can always be characterized as the limit of periodic travelling wave solutions, as well as peakons and compactons, which are manifested by particular types of singularities in the phase plane associated with the (integrated form of) the dynamical system. To proceed further, we shall need to determine the analytic continuation of the resulting solutions in the complex plane. In contrast to the KdV equation, whose sech^2 solitons have a unique extension to a single-valued meromorphic function, the solitary wave solutions of (1.1) extend to multiply-valued analytic functions, with quite complicated branching behavior. The second part of this paper is devoted to a detailed analysis of these complex analytic extensions. To determine the convergence properties, we must restrict our attention to a region supporting a single-valued extension; branching implies that the extension is not unique, but depends on how the branch cuts are arranged in the complex plane. Interestingly, the choice of branch cuts, and hence single-valued extension, affects the convergence of the solution in the complex plane, leading to different non-analytic solitary wave solutions in the limit, which, nevertheless, restrict to the same peakon or compacton on the real axis. In the final part of the paper, we shall study the properties of branch points of these solutions, and their behavior as the corresponding solutions are converging to compactons, peakons or solitary wave solutions in order to understand how singularities influence properties of these solutions during the process of convergence.

Acknowledgements. We would like to thank Jerry Bona and Philip Rosenau for useful discussions and comments.

2. Notation. We let $C^k = C^k(\mathbb{R})$ denote the space of k times continuously differentiable functions defined on the real axis. The space of all infinitely differentiable functions with compact support in $\mathbb{R}$ is denoted by $C_c^\infty = C_c^\infty(\mathbb{R})$. The space $L^p = L^p(\mathbb{R})$ with $1 \le p \le \infty$ consists of all pth-power Lebesgue-integrable functions defined on the real line $\mathbb{R}$ with the usual modification if $p = \infty$. The standard norm of a function $f \in L^p$ will be denoted by $\| f \|_p$. The inner product of two functions f and g in L^2 is the integral

$$\langle f, g \rangle = \int_\infty^\infty f(x)\overline{g(x)}dx,$$

where the overbar denotes complex conjugation. For any integer $k \ge 0$ and constant $p \ge 1$, the Sobolev space $W^{k,p} = W^{k,p}(\mathbb{R})$ consists of all tempered distributions f such that $f^{(m)} \in L^p$ for all $0 \le m \le k$. The space $W^{k,2}$ is usually denoted by H^k.

A (classical) *travelling wave solution*[1] of Equation (1.1) of wave speed c is a solution of class C^3, having the particular form $u = \phi(x - ct)$. A travelling wave solution is called a *solitary wave* if ϕ has a well-defined limit $\lim\limits_{|x|\to\infty} \phi(x)$, which is the same at both $\pm\infty$; the limiting value represents the undisturbed depth of the fluid. A solitary wave solution, or its corresponding homoclinic orbit, is said to be analytic if the solution is a real analytic function defined on the real axis. Analytic solitary wave solutions of integrable evolution equations, such as the KdV equation, are known as *solitons*, which indicates that they emerge from collisions unchanged in form, save for a phase shift; see [2], [10].

By a *fixed point* of a dynamical system $x' = f(t, x)$, where $x \in \mathbb{R}^n$, we mean a point x_0 such that $f(t, x_0) = 0$ for all $t \in \mathbb{R}$. The fixed point is called *quasi-hyperbolic* of *degree one* if the linearized mapping derived from the system near this point has eigenvalues with their real parts different from zero except one eigenvalue with zero real part. By a *singularity* of the dynamical system $x' = f(t, x)$, we indicate a point x_0 such that $f(t, x)$ is not analytic at (t, x_0) for some t in $\mathbb{R}$.

3. Dynamical systems for solitary waves. The soliton solutions to the KdV equation can be viewed as the limits of the periodic cnoidal wave solutions; see [2], [10]. Let us review this well-known fact from a dynamical systems point of view. Substituting the travelling wave solution $u(x, t) = \phi(x - ct)$, for constant wave speed c, into the KdV equation (1.3), one obtains the ordinary differential equation

$$-(c-1)\phi' + \phi\phi' + \phi''' = 0. \tag{3.1}$$

[1] Here we give a standard definition of classical solutions of Equation (1.1) versus the definition of pseudo-classical solutions we have proposed in Section 1, which is needed to include a new class of travelling wave solutions — compactons and peakons of (1.1).

The transformation $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \phi \\ \phi' \\ \phi'' \end{pmatrix}$ reduces (3.1) to the dynamical system

$$\vec{y}\,' = \begin{pmatrix} y_2 \\ y_3 \\ (c-1)y_2 - y_1 y_2 \end{pmatrix}. \tag{3.2}$$

The fixed points of system (3.2) consists of all points of the y_1-axis. To observe properties of travelling wave solutions near each fixed point $(a, 0, 0)$ on the y_1-axis, we let $\vec{y} = \vec{\xi} + \vec{a} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$ and substitute the transformation into (3.2), leading to the system

$$\vec{\xi}\,' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & c-1-a & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\xi_1 \xi_2 \end{pmatrix}.$$

If $a < c - 1$, then there are a one-dimensional center manifold, a one-dimensional stable manifold and a one-dimensional unstable manifold near $(a, 0, 0)$ with a unique homoclinic orbit represented by the function

$$\phi(x) = a + 3(c-a-1)\,\mathrm{sech}^2 \frac{\sqrt{c-a-1}\, x}{2} \tag{3.3}$$

which is the limit of periodic cnoidal solutions of Equation (3.1). On the other hand, if $a \geq c - 1$, then there is a three-dimensional center manifold near the fixed point $(a, 0, 0)$ and $(c - 1, 0, 0)$ is a bifurcation point of the system.

Another property of Equation (3.1) worth mentioning is that it induces a homeomorphism of $(-\infty, c-1)$, the set of fixed points having homoclinic orbits, onto the interval $(c-1, \infty)$, the set of fixed points where there are periodic orbits. This fact can be verified as follows. For each $a < c - 1$, we substitute $\phi = \psi + a$ into (3.1), integrate the resulting equation once and take the integration constant to be zero, leading to the equation

$$-(c-a-1)\psi + \frac{\psi^2}{2} + \psi'' = 0. \tag{3.4}$$

The dynamical system (3.4) has two fixed points. One is the origin which supports a one-dimensional stable manifold and a one-dimensional unstable manifold with a unique homoclinic orbit representing the solitary wave solution expressed by (3.3). At the other fixed point $(2(c - a - 1), 0)$, there is a two-dimensional center manifold where there exist periodic orbits converging to the homoclinic orbit at the origin as sketched in Figure 1.

Substituting $\psi = \varphi + 2(c-a-1)$ into (3.4) and comparing the resulting equation

$$-\big(c-1-(2c-a-2)\big)\varphi + \frac{\varphi^2}{2} + \varphi'' = 0$$

with (3.1), then one may realize that periodic orbits near the point $(2(c-a-1), 0)$ of system (3.4) can be regarded as those near the fixed point $(2c-a-2, 0, 0)$. In consequence, the homeomorphism $\Phi\colon (-\infty, c-1) \longrightarrow (c-1, \infty)$ is naturally defined by

$$\Phi(a) = 2c - 2 - a. \tag{3.5}$$

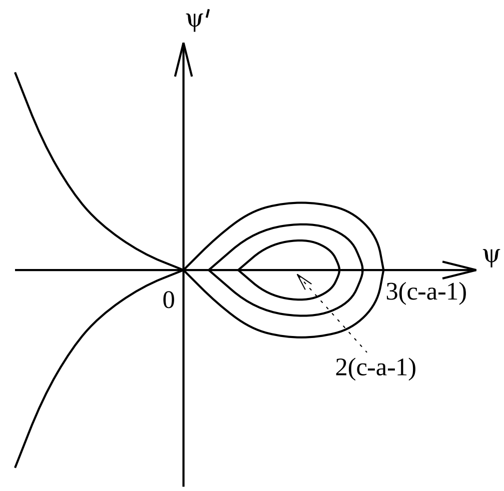

Fig. 1. The phase plane of system (3.4) with $a < c-1$

This not only shows that the quasi-hyperbolic points of system (3.1) are in one-to-one correspondence with its three-dimensional center manifolds, but also indicates that for each quasi-hyperbolic point, the corresponding three-dimensional center manifold contains a sequence of periodic orbits converging to the homoclinic orbit at the quasi-hyperbolic point. We shall see that there is a similar mapping Ψ for the dynamical system obtained by reduction from Equation (1.1) which may be used to illustrate its more complicated, but more interesting properties.

Now let us consider the nonlinearly dispersive Equation (1.1), and assume that $\gamma\nu \neq 0$. Replacing u by $u/(\gamma\nu)$, we reduce (1.1) to the simpler equation

$$u_t + \nu u_{xxt} = \alpha u_x + \beta u_{xxx} + \frac{3}{\nu} u u_x + u u_{xxx} + 2 u_x u_{xx}. \tag{3.6}$$

The resulting ordinary differential equation for travelling wave solutions $u(x,t) = \phi(x-ct)$ of speed c is

$$(\alpha + c)\phi' + (\beta + c\nu + \phi)\phi''' + \frac{3}{\nu}\phi\phi' + 2\phi'\phi'' = 0. \tag{3.7}$$

Using the same transformation $\vec{y} = \vec{\xi} + \vec{a}$ as before yields the system of equations

$$\vec{\xi}\,' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\dfrac{\alpha + c + \frac{3a}{\nu}}{\beta + c\nu + a} & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \\ + \begin{pmatrix} 0 \\ 0 \\ -\dfrac{\frac{3}{\nu}\xi_1\xi_2 + 2\xi_2\xi_3}{\beta + c\nu + a} + \dfrac{\xi_1[(\alpha + c + \frac{3a}{\nu})\xi_2 + \frac{3}{\nu}\xi_1\xi_2 + 2\xi_2\xi_3]}{(\beta + c\nu + a)(\beta + c\nu + a + \xi_1)} \end{pmatrix}. \tag{3.8}$$

Clearly, the set of fixed points and singularities of Equation (3.7) also consists of all points of the y_1-axis. Next, we discuss properties of each fixed point or singularity $(a, 0, 0)$ of system (3.7) in different cases.

Case I. When $\nu > 0$ and $\beta + c\nu > \frac{\nu(\alpha+c)}{3}$.

The constants $-(\beta + c\nu)$ and $-\frac{\nu}{3}(\alpha + c)$ divide the y_1-axis into three intervals

$$(-\infty, -(\beta + c\nu)), \quad (-(\beta + c\nu), -\frac{\nu}{3}(\alpha + c)) \text{ and } (-\frac{\nu}{3}(\alpha + c), \infty).$$

For any $a \in (-\infty, -(\beta + c\nu)) \cup [-\frac{\nu}{3}(\alpha + c), \infty)$, the system (3.8) shows that (3.7) has a three-dimensional center manifold near the point $(a, 0, 0)$ and $(-\frac{\nu}{3}(\alpha + c), 0, 0)$ is a bifurcation point. On the other hand, if $a \in (-(\beta + c\nu), -\frac{\nu}{3}(\alpha + c))$, then there is a one-dimensional center manifold, a one-dimensional stable manifold, and a one-dimensional unstable manifold at the point $(a, 0, 0)$, near which there is a unique, analytic, homoclinic orbit represented by the solution

$$\phi(x - ct) = a + \xi(x - ct).$$

Such a solution is obtained as the limit, as $\delta \to 0$, of periodic solutions

$$\phi_\delta\left(x - \left(c + \frac{\delta}{2\nu}\right)t\right) = a - \frac{\delta}{2} + \xi_\delta\left(x - \left(c + \frac{\delta}{2\nu}\right)t\right).$$

The period of $\phi_\delta(x)$ is

$$T = 2\int_\delta^{\nu B} \frac{\sqrt{\nu(A + \zeta)}\, d\zeta}{\sqrt{\zeta(\nu B - \zeta)(\zeta - \delta)}},$$

where $A = \beta + c\nu + a$, $B = -(\alpha + c + \frac{3a}{\nu})$, and $0 < \delta < \nu B$ is a constant. Note that the functions $\xi(x)$ and $\xi_\delta(x)$ satisfy the respective differential equations

$$(\xi')^2 = \frac{\xi^2(\nu B - \xi)}{\nu(\xi + A)}, \qquad \text{and} \qquad (\xi_\delta')^2 = \frac{\xi_\delta(\nu B - \xi_\delta)(\xi_\delta - \delta)}{\nu(\xi_\delta + A)}.$$

The point $(-(\beta+c\nu),0,0)$ forms a singular point of system (3.7), providing the compacton solution

$$\phi_0(x) = \begin{cases} -(\beta+c\nu) + \big(3(\beta+c\nu) - \nu(\alpha+c)\big)\cos^2\dfrac{x}{2\sqrt{\nu}}, & \text{if } |x| \le \sqrt{\nu}\,\pi \\ -(\beta+c\nu), & \text{otherwise} \end{cases} \tag{3.9}$$

occurs as a weak solution of (3.7) in the following sense.

Definition 3.1. *A solitary wave $\phi(x)$ with undisturbed depth $a = \lim_{|x|\to\infty}\phi(x)$ is a weak solution of the ordinary differential equation (3.7) if and only if $\xi = \phi - a \in H^1$, and*

$$\Big\langle (\alpha+c)\phi + \frac{3\phi^2}{2\nu} - \frac{(\phi')^2}{2}, g'\Big\rangle + \Big\langle \big(\beta+c\nu+\frac{\phi}{2}\big)\phi, g'''\Big\rangle = 0, \tag{3.10}$$

for any $g \in C_c^\infty(\mathbb{R})$.

We are interested in studying the behavior of the solitary wave solution $\phi(x) = a + \xi(x)$ as its asymptotic amplitude a approaches the singular value $-(\beta+c\nu)$.

Equation (3.7) also implicitly suggests a one-to-one mapping of its quasi-hyperbolic fixed points to its three-dimensional center manifolds. However, unlike the KdV equation (3.1), the resulting mapping is not surjective. To find the required mapping, one may use a procedure similar to that of deriving (3.5). For each $a \in (-(\beta+c\nu), -\frac{\nu}{3}(\alpha+c))$, we substitute $\phi = \psi + a$ into (3.7); integrating the resulting equation once and setting the integral constant to zero, we obtain

$$\big(\alpha+c+\frac{3a}{\nu}\big)\psi + (\beta+c\nu+a+\psi)\psi'' + \frac{3\psi^2}{2\nu} + \frac{(\psi')^2}{2} = 0. \tag{3.11}$$

The system (3.11) has two fixed points — the origin and $\big(-2(\frac{\nu}{3}(\alpha+c)+a),0\big)$. The origin is a saddle point whose unique homoclinic orbit represents an analytic solitary wave solution. Near the point $\big(-2(\frac{\nu}{3}(\alpha+c)+a),0\big)$, there exists a two-dimensional center manifold having periodic orbits converging to the homoclinic orbit at the origin as sketched in Figure 2. Substituting $\psi = \xi - 2\big(\frac{\nu}{3}(\alpha+c)+a\big)$ into (3.11) and comparing the resulting equation

$$\big(\alpha+c-2(\alpha+c)-\frac{3a}{\nu}\big)\xi + \big(\beta+c\nu-\frac{2\nu}{3}(\alpha+c)-a+\xi\big)\xi'' + \frac{3\xi^2}{2\nu} + \frac{(\xi')^2}{2} = 0$$

with (3.7), one may recognize that periodic orbits near the fixed point $\big(-2(\frac{\nu}{3}(\alpha+c)+a),0\big)$ of (3.11) come from the center manifold of the fixed point $\big(-\frac{2\nu}{3}(\alpha+c)-a,0,0\big)$ in system (3.7). Therefore the homeomorphism

$$\Psi(a) = -\frac{2\nu}{3}(\alpha+c) - a \tag{3.12}$$

of $\big(-(\beta+c\nu), -\frac{\nu}{3}(\alpha+c)\big)$ onto $\big(-\frac{\nu}{3}(\alpha+c), \beta+c\nu-\frac{2\nu}{3}(\alpha+c)\big)$ determines a one-to-one mapping from the set $\{(a,0,0); a\in\big(-(\beta+c\nu), -\frac{\nu}{3}(\alpha+c)\big)\}$ of quasi-hyperbolic points to the set of points $\{(\Psi(a),0,0); a\in\big(-(\beta+c\nu), -\frac{\nu}{3}(\alpha+c)\big)\}$ whose center manifolds contain periodic orbits converging to homoclinic orbits at the corresponding quasi-hyperbolic fixed points. One may also notice that the mapping Ψ is defined in such a way that the points $(0,0)$ and $(\Psi(a)-a,0)$ always appear as a pair of fixed points in (3.11).

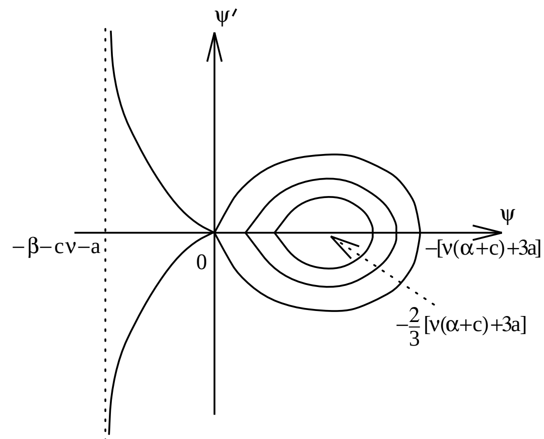

Fig. 2. The phase plane of system (3.11) with $-\beta-c\nu<a<-\frac{\nu}{3}(\alpha+c)$ in *Case I*

When $a<-\beta-c\nu$, both $(0,0)$ and $(\Psi(a)-a,0)$ have a two-dimensional center manifold with periodic orbits at each of the points, but they are separated by the singular point $(-\beta-c\nu-a,0)$ as sketched in Figure 3. On the other hand, if $a\in\big(-(\beta+c\nu), -\frac{\nu}{3}(\alpha+c)\big)$, the two points $(a,0)$ and $(\Psi(a)-a,0)$ always stay on the right-hand side of the singular point $(-\beta-c\nu-a,0)$ as shown in Figure 2. The case $a=-\beta-c\nu$ is the most unusual since the singular point $(-\beta-c\nu-a,0)$ is at the origin where a periodic orbit passes through, from which the compacton is defined as a weak solution of Equation (3.11); while the fixed point $(\Psi(a)-a,0)$ still has a two-dimensional center manifold containing periodic orbits.

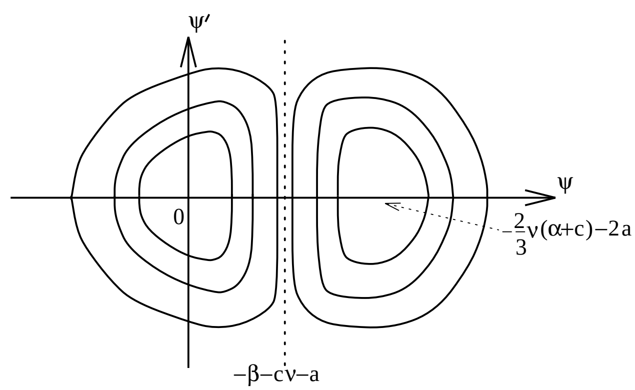

Fig. 3. The phase plane of Equation (3.11) when $a<-\beta-c\nu$ in *Case I*

Case II. When $\nu < 0$ and $\beta + c\nu > \frac{\nu(\alpha+c)}{3}$.

In this case, the interval $(-(\beta + c\nu), -\frac{\nu}{3}(\alpha + c)]$ on the y_1-axis consists of fixed points supporting three-dimensional center manifolds. The other two intervals, $(-\infty, -(\beta + c\nu))$ and $(-\frac{\nu}{3}(\alpha+c), \infty)$, consist of quasi-hyperbolic points, and $(-\frac{\nu}{3}(\alpha+c), 0, 0)$ is a bifurcation point. Unlike *Case I*, at the singular point $(-(\beta+c\nu), 0, 0)$, there exists only the stationary solution $\phi(x) \equiv -(\beta + c\nu)$ and compactons do not occur.

The mapping Ψ defined in (3.12) also offers a convenient way to describe this case as follows. When $a \in (-\infty, -\beta - c\nu)$, $\Psi(a) \in \big(\beta + c\nu - \frac{2\nu}{3}(\alpha + c), \infty\big)$. Both $(a, 0, 0)$ and $(\Psi(a), 0, 0)$ are quasi-hyperbolic points without homoclinic orbits. However, there exist cuspon solutions in which $(0, 0)$ and $(\Psi(a) - a, 0)$ appear as a pair of fixed points of (3.11), corresponding to $(a, 0, 0)$ and $(\Psi(a), 0, 0)$ of the system (3.7), respectively, and having the singular point $(-\beta - c\nu - a, 0)$ between them. This is illustrated in Figure 4.

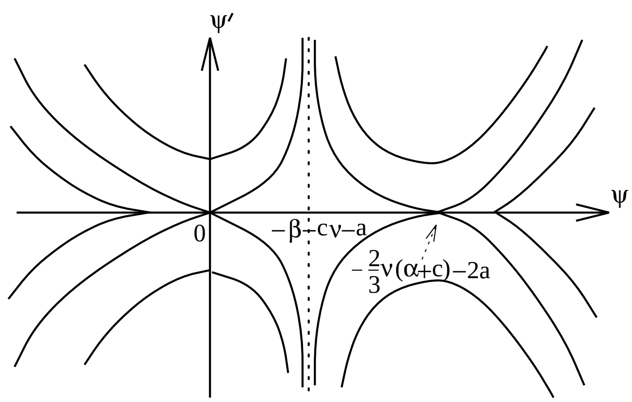

Fig. 4. The phase plane of Equation (3.11) when $a < -\beta - c\nu$ in *Case II*

If $a = -\beta - c\nu$, then the singular point $(-\beta - c\nu - a, 0)$ and the origin merge together, and as a consequence, the cuspon ceases to exist, although there is still a cuspon associated with the point $(\Psi(a) - a, 0)$.

For each $a \in \big(-\beta - c\nu, -\frac{1}{2}(\beta + c\nu) - \frac{\nu}{6}(\alpha + c)\big]$, the value of $\Psi(a)$ lies in the interval $\big[\frac{1}{2}(\beta - \nu\alpha), \beta + c\nu - \frac{2\nu}{3}(\alpha + c)\big)$ and the origin of system (3.11) changes its property to possess a two-dimensional center manifold with periodic orbits. Even though the singular point $(-\beta - c\nu - a, 0)$ is on the left-hand side of both the origin and the saddle point $(\Psi(a) - a, 0)$, there is still no homoclinic orbit, but a cuspon at the point $(\Psi(a) - a, 0)$ as illustrated in Figure 5.

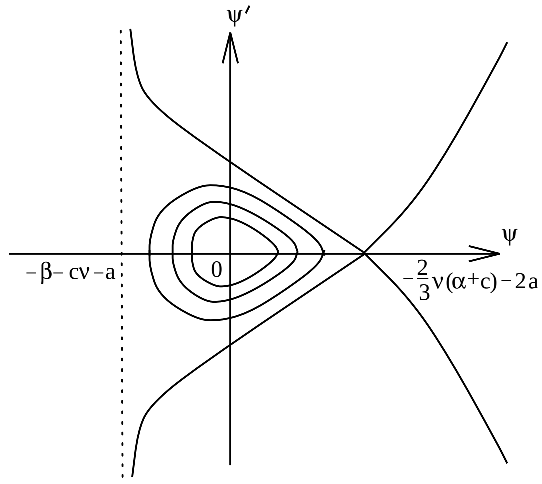

Fig. 5. The phase plane of Equation (3.11) in *Case II*
with $-\beta - c\nu < a < -\frac{1}{2}(\beta + c\nu) - \frac{\nu}{6}(\alpha + c)$

It is worth mentioning that at $a = \frac{1}{2}(\beta - \nu\alpha)$, the peakon

$$\phi_{\boldsymbol{p}}(x) = \frac{\beta - \nu\alpha}{2} - \big[\frac{3}{2}(\beta + c\nu) - \frac{\nu}{2}(\alpha + c)\big]e^{-(-\nu)^{-1/2}|x|},$$

forms a weak solution, meaning that it satisfies (3.10). Later, we shall prove that the homoclinic orbits at fixed points $(a, 0, 0)$ of system (3.7) converge to the peakon $\phi_{\boldsymbol{p}}$ as $a \in (-\frac{\nu}{3}(\alpha + c), \frac{\beta-\nu\alpha}{2})$ approaches the endpoint $\frac{\beta-\nu\alpha}{2}$.

If $a \in \big(-\frac{1}{2}(\beta + c\nu) - \frac{\nu}{6}(\alpha + c), -\frac{\nu}{3}(\alpha + c)\big)$, then $\Psi(a) \in \big(-\frac{\nu}{3}(\alpha + c), \frac{1}{2}(\beta - \nu\alpha)\big)$ and there is a homoclinic orbit at the saddle point $(\Psi(a) - a, 0)$ which is the limit of periodic orbits contained in the center manifold at the origin, as displayed in Figure 6.

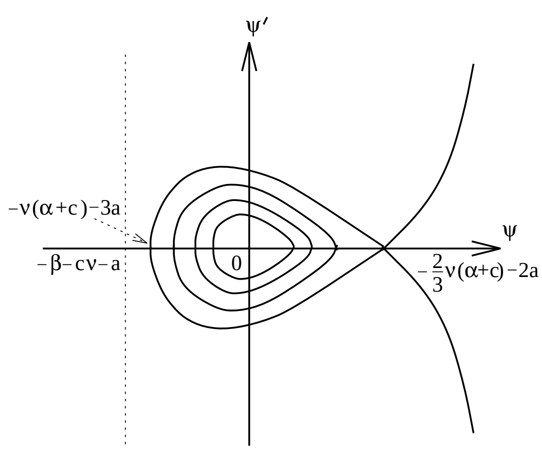

Fig. 6. The phase plane of Equation (3.11) in *Case II*
with $-\frac{1}{2}(\beta + c\nu) - \frac{\nu}{6}(\alpha + c) < a < -\frac{\nu}{3}(\alpha + c)$

Remark. A naïve explanation for the existence of so many cuspons in this case is that the family of quasi-hyperbolic points of system (3.7) outnumbers the fixed points having three-dimensional center manifolds, so that the mapping Ψ associates a great number of quasi-hyperbolic points to those of the same kind. The cuspons are present there because of the strong effect of the singular point $(-\beta - c\nu, 0, 0)$. Furthermore, we can use the equation

$$(\psi')^2 = -\frac{\psi^2(\psi + \nu(\alpha + c) + 3a) + d}{\nu(\psi + \beta + c\nu + a)} \tag{3.13}$$

derived from (3.11) by integration, where d is the integration constant, to sketch the phase plane of (3.11) for different values of a. Based on this, one may show that a necessary condition for a homoclinic orbit to exist at the point $(a, 0, 0)$ is that the mapping Ψ associates the quasi-hyperbolic point $(a, 0, 0)$ to a three-dimensional center manifold, *i.e.* if $(a, 0, 0)$ is a quasi-hyperbolic point and $(\Psi(a), 0, 0)$ does not have a three-dimensional center manifold, then homoclinic orbits can not exist at $(a, 0, 0)$. In contrast, there is a surplus of three-dimensional center manifolds in *Case I*, so that the mapping Ψ is able to associate every quasi-hyperbolic point to a three-dimensional center manifold. In addition, a homoclinic orbit is formed at each quasi-hyperbolic point because of the smaller effect of the singular point $(-\beta - c\nu, 0, 0)$ in this case than *Case II*.

Compared with system (3.7), the KdV system seems to be perfect, because the number of its quasi-hyperbolic points is balanced with the number of three-dimensional center manifolds, *i.e.* the mapping Φ defined in (3.5) is a one-to-one and onto mapping, and there are no singularities. Therefore, studying properties of one quasi-hyperbolic point of the KdV system is sufficient to understand properties of other fixed points in the system, whereas for system (3.7), we need to consider different cases in which it is also necessary to investigate fixed points in different intervals on the y_1-axis.

We summarize the remaining two cases to conclude this section.

Case III. When $\nu > 0$ and $\beta + c\nu = \frac{\nu(\alpha+c)}{3}$.

For any a with $a \neq -\frac{\nu}{3}(\alpha + c)$, there is a three-dimensional center manifold at $(a, 0, 0)$ with periodic orbits going around this point. Moreover, $(-\frac{\nu}{3}(\alpha + c), 0, 0)$ is a singular point of system (3.7) without periodic orbits.

Case IV. When $\nu < 0$ and $\beta + c\nu = \frac{\nu(\alpha+c)}{3}$.

Each $a \neq -\frac{\nu}{3}(\alpha + c)$, supports a one-dimensional center manifold, a one-dimensional stable manifold and a one-dimensional unstable manifold at $(a, 0, 0)$ without homoclinic orbits. This is not surprising since homoclinic orbits are usually accompanied by periodic orbits converging to them, but there is neither a center manifold nor a periodic orbit in this case. However we shall show that there is a cuspon at the point $(a, 0, 0)$. On the other hand, if $a = -\frac{\nu}{3}(\alpha + c)$, then $(a, 0, 0)$ is a singular point, without cuspons.

As we have seen in the above discussion, Equation (1.1) has analytic solitary wave solutions in *Cases I* and *II*, which are illustrated as homoclinic orbits in Figures 2 and 6, respectively. A question arises naturally as how these homoclinic orbits behave when the singularity $(-\beta - c\nu - a, 0)$ is close to them. The answer is that in the first case, the solitary wave solutions at points $(a, 0, 0)$ converge to the compacton ϕ_0 given by (3.9) when $a \to -\beta - c\nu$ and $-\beta - c\nu < a < -\frac{\nu}{3}(\alpha + c)$; in the second case, the solitary wave solutions at $(a, 0, 0)$ converge to the peakon at the fixed point $\big(\frac{1}{2}(\beta - \nu\alpha), 0, 0\big)$ as $a \to \frac{1}{2}\big(\beta - \nu\alpha\big)$ with $-\frac{\nu}{3}(\alpha + c) < a < \frac{1}{2}\big(\beta - \nu\alpha\big)$.

Summarizing, we let the constant c be the speed of propagation of the indicated travelling wave solutions of Equation (1.1), and a the undisturbed depth. Any such solitary wave solution takes the form $a + \phi_a(x - ct)$, where the function $\psi = \phi_a$ satisfies the ordinary differential Equation (3.11) with asymptotic boundary conditions $\lim\limits_{|x|\to\infty} \phi_a(x) = 0$.

Theorem 3.1. *If the coefficients of Equation* (1.1) *satisfy the inequalities*

$$\gamma \neq 0, \qquad \nu > 0, \qquad \textit{and} \qquad \beta + c\nu > \frac{\nu}{3}(\alpha + c),$$

then there exists an orbitally unique and analytic solitary wave solution $a + \phi_a(x - ct)$ for each $a \in \big(-\beta - c\nu, -\frac{\nu}{3}(\alpha + c)\big)$. Moreover, as a approaches $-\beta - c\nu$, the sequence of the solitary wave solutions $\{a + \phi_a(x)\}$ converges to the compacton solution given in (3.9).

Proof. Let $\varepsilon = a + \beta + c\nu$, $B = -(\alpha + c + \frac{3a}{\nu})$ and $B_0 = \frac{3}{\nu}(\beta + c\nu) - (\alpha + c)$. Then Equation (3.11) for the solitary wave solution reduces to

$$\nu(\phi_a + \varepsilon)(\phi_a')^2 = \phi_a^2(\nu B - \phi_a). \tag{3.14}$$

Using the inequality $0 \le \phi_a(x) \le \nu B$ valid for all $x \in \mathbb{R}$, one may show that sequences of functions $\{\phi_a'\}$ and $\{\phi_a''\}$ are uniformly bounded on the real axis. Therefore, the Ascoli-Arzelà Theorem shows that, as $a \to -\beta - c\nu$, there exist subsequences of the families $\{\phi_a\}$ and $\{\phi_a'\}$, without loss of generality still denoted by $\{\phi_a\}$ and $\{\phi_a'\}$, which are uniformly convergent to a function ϕ and its derivative ϕ', respectively, on any compact set of $\mathbb{R}$. Here we are relying on the fact that each ϕ_a is an even function, since ϕ_a is symmetric with respect to its elevation and translation invariant. Taking the limit on both sides of (3.14) as $a \to -\beta - c\nu$, or as $\varepsilon \to 0$ leads to the equation

$$\nu\phi{\phi'}^2 = \phi^2(\nu B_0 - \phi) \tag{3.15}$$

satisfied by the function ϕ. Since $\lim_{\varepsilon\to 0}\max_{x\in\mathbb{R}}\phi_a(x) = \lim_{\varepsilon\to 0}\nu B = \nu B_0 > 0$ and each ϕ_a is even, monotone on each side of the origin and exponentially decaying to zero at infinity, the limiting function ϕ is a nontrivial solution of of (3.15). Thus, ϕ satisfies the equation $\nu {\phi'}^2 = \phi(\nu B_0 - \phi)$. Therefore, as an even and monotone decreasing function on the positive real axis, $\phi = \phi_0 + \beta + c\nu$, that is to say, $\phi_0 = \phi - \beta - c\nu$ is the compacton solution (3.9). □

The corresponding result for peakons follows.

Theorem 3.2. *If the coefficients of Equation* (1.1)*satisfy the inequalities*

$$\gamma \neq 0, \qquad \nu < 0, \qquad \textit{and} \qquad \beta + c\nu > \frac{\nu}{3}(\alpha + c),$$

then for each $a \in \left(-\frac{\nu}{3}(\alpha + c), \frac{1}{2}(\beta - \nu\alpha)\right)$, *there exists an orbitally unique and analytic solitary wave solution* $a + \phi_a(x - ct)$. *Moreover, as* $a \to \frac{1}{2}(\beta - \nu\alpha)$, *the sequence of the solitary wave solutions* $\{\phi_a(x)\}$ *is convergent to the peakon solution*

$$\phi(x) = -\left[\frac{3}{2}(\beta + c\nu) - \frac{\nu}{2}(\alpha + c)\right] e^{-(-\nu)^{-1/2}|x|}. \tag{3.16}$$

Proof. One can straightforwardly show that the first order derivatives $\{\phi_a'\}$ of the solitary wave solutions $\{a + \phi_a\}$ at points $(a, 0, 0)$ are uniformly bounded for all $a \in \left(-\frac{\nu}{3}(\alpha + c), \frac{1}{2}(\beta - \nu\alpha)\right)$. Therefore, there exists a sequence of even functions monotonically decreasing on the positive axis, still denoted by $\{\phi_a\}$, satisfying the equation

$$\nu(\phi_a + \beta + c\nu + a)(\phi_a')^2 = -\phi_a^2(\phi_a + 3a + \nu(\alpha + c)) \tag{3.17}$$

and converging to the function ϕ as $a \to \frac{1}{2}(\beta - \nu\alpha)$. This may be derived by solving (3.17) to obtain an implicit expression of the function ϕ_a and then taking the limit as $a \to \frac{1}{2}(\beta - \nu\alpha)$; see [13] for details. □

Remark. As we pointed out in the previous discussion, solitary wave solutions do not exist if $\beta + c\nu = \frac{\nu}{3}(\alpha + c)$. In case $\beta + c\nu < \frac{\nu}{3}(\alpha + c)$, we can replace u by $-u$ in Equation (1.1), which has the effect of changing the sign of the coefficients α and β, and the wave speed c. Note that this transformation will change waves of elevation moving to the right ($c > 0$) into waves of depression, moving to the left. Otherwise, the conclusions in Theorems 3.1 and 3.2 also apply to the above equation. Therefore, if $\nu > 0$, and $a \in \left(-\frac{\nu}{3}(\alpha + c), -(\beta + c\nu)\right)$, Equation (3.7) admits a solitary wave solution in the form

$a + \phi_a(x)$, such that the sequence of solitary wave solutions $\{a + \phi_a(x)\}$ converges to a compacton as a weak solution of (3.7) when $a \to -(\beta + c\nu)$. On the other hand, if $\nu < 0$, then for each $a \in \left(\frac{1}{2}(\beta - \nu\alpha), -\frac{\nu}{3}(\alpha + c)\right)$, there is a solitary wave of elevation $a + \phi_a(x)$, such that the sequence $\{a + \phi_a(x)\}$ converges to a peakon, also as a weak solution of (3.7), as $a \to \frac{1}{2}(\beta - \nu\alpha)$. In either case, ϕ_a satisfies (3.11). In the remaining part of this paper, we shall only consider the case $\beta + c\nu > \frac{\nu}{3}(\alpha + c)$, since any result in this case can be directly applied to the case $\beta + c\nu < \frac{\nu}{3}(\alpha + c)$.

To understand how analytic solitary wave solutions converge to functions, such as compactons and peakons, having singularities on the real axis $\mathbb{R}$, we shall extend solitary wave solutions mentioned in the last two theorems to functions defined in the complex plane to study singularity distribution of these functions. This method not only provides another way to prove the last two theorems, but also makes it clear that singularities of solitary wave solutions are approaching the real axis in the process of convergence, or roughly speaking, singularities of compactons or peakons come from those of analytic solitary wave solutions, which are close to the real axis in the complex plane. This will form the subject of Part II.

REFERENCES

[1] M.J. Ablowitz, A. Ramani, and H. Segur, *A connection between nonlinear evolution equations and ordinary differential equations of P-type. I, II*, J. Math. Phys. **21** (1980), 715–721, 1006–1015.

[2] M.J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.

[3] M.S. Alber, R. Camassa, D.D. Holm, and J.E. Marsden, *The geometry of peaked solitons and billiard solutions of a class of integrable PDE's*, Lett. Math. Phys. **32** (1994), 137–151.

[4] T.B. Benjamin, J.L. Bona, and J.J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Phil. Trans. Roy. Soc. London Ser. A **272** (1972), 47–78.

[5] J.L. Bona and Y.A. Li, *Decay and analyticity of solitary waves*, to appear in J. Math. Pures Appl.

[6] R. Camassa and D.D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71** (1993), 1661–1664.

[7] A.S Fokas, P.J. Olver, and P. Rosenau, *A plethora of integrable bi-Hamiltonian equations*, to appear in Algebraic Aspects of Integrable Systems (A.S. Fokas and I.M. Gel'fand, eds.), Birkhuser, Cambridge, Mass.

[8] B. Fuchssteiner, *The Lie algebra structure of nonlinear evolution equations admitting infinite dimensional abelian symmetry groups*, Prog. Theoret. Phys. **65** (1981), 861–876.

[9] B. Fuchssteiner and A.S. Fokas, *Symplectic structure, their Bäcklund transformations and hereditary symmetries*, Physica D **4** (1981), 47–66.

[10] A. Jeffrey and T. Kakutani, *Weak nonlinear dispersive waves: a discussion centered around the Korteweg-deVries equation*, SIAM Rev. **14** (1972), 582–643.

[11] S. Kichenassamy and G.K. Srinivasan, *The structure of WTC expansions and applications*, J. Phys. A **28** (1995), 1977–2004.

[12] M.D. Kruskal, *The Korteweg-deVries equation and related evolution equations*, Nonlinear Wave Motion (A.C. Newell, ed.), Amer. Math. Soc., Providence, 1974, pp. 61–83.

[13] I.A. Kunin, Elastic Media with Microstructure I, Springer-Verlag, New York, 1982.

[14] P.D. Lax and C.D. Levermore, *The small dispersion limit of the Korteweg-deVries equation. I, II, III*, Comm. Pure Appl. Math. **36** (1983), 253–290, 571–593, 809–829.

[15] P.D. Lax, C.D. Levermore, and S. Venakides, *The generation and propagation of oscillations in dispersive initial value problems and their limiting behavior*, Important Developments in Soliton Theory (A.S. Fokas and V.E. Zakharov, eds.), Springer-Verlag, New York, 1993, pp. 205–241.

[16] P.J. Olver and P. Rosenau, *Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support*, Phys. Rev. E **53** (1996), 1900–1906.

[17] P. Rosenau, *On non-analytic solitary waves formed by a nonlinear dispersion*, preprint.

[18] P. Rosenau, *Nonlinear dispersion and compact structures*, Phys. Rev. Lett. **73** (1994), 1737–1741.

[19] P. Rosenau, *On solitons, compactons and Lagrange maps*, Phys. Lett. A **211** (1996), 265–275.

[20] P. Rosenau and J.M. Hyman, *Compactons: Solitons with finite wavelength*, Phys. Rev. Lett. **70** (1993), 564–567.

[21] W. Rudin, Real and Complex Analysis, McGraw-Hill Book Company: New York, 1987.

[22] W.-H. Steeb and N. Euler, Nonlinear Evolution Equations and Painlevé Test, World Scientific Publishing Co., Singapore, 1988.

[23] S. Venakides, *The Korteweg-deVries equation with small dispersion: higher order Lax-Levermore theory*, Comm. Pure Appl. Math. **43** (1990), 335–361.

[24] M. Wadati, Y.H. Ichikawa, and T. Shimizu, *Cusp soliton of a new integrable nonlinear evolution equation*, Prog. Theoret. Phys. **64** (1980), 1959–1967.

[25] J. Weiss, M. Tabor, and G. Carnevale, *The Painlevé property for partial differential equations*, J. Math. Phys. **24** (1983), 522–526.

E-mail address: yili@math.umn.edu; olver@ima.umn.edu