ON THE BLOW-UP OF SOLUTIONS TO THE INTEGRABLE MODIFIED CAMASSA–HOLM EQUATION

YUE LIU, PETER J. OLVER, CHANGZHENG QU, AND SHUANGHU ZHANG

ABSTRACT. We derive conditions on the initial data, including cases where the initial momentum density is not of one sign, that produce blow-up of the induced solution to the modified integrable Camassa-Holm equation with cubic nonlinearity. The blow-up conditions are formulated in terms of the initial momentum density and the average initial energy.

Keywords: Modified Camassa-Holm equation, blow up, integrable equation, peakon

AMS Subject Classification (2000): 35B44, 35G25

1. INTRODUCTION

In this paper, we are concerned with the behavior of solutions to the initial-value problem for the modified Camassa-Holm (mCH) equation on the real line:

$$\begin{cases} m_t + \big((u^2 - u_x^2)m\big)_x = 0, \\ u(0,x) = u_0(x), \end{cases} \qquad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where

$$m = (1 - \partial_x^2)u = u - u_{xx} \tag{1.2}$$

represents the *momentum density* of the system. This nonlinear partial differential equation was derived by applying the method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified Korteweg–deVries (mKdV) equation [13, 21], which implies that the mCH equation (1.1) is an integrable system possessing a bi-Hamiltonian structure. Subsequently, a Lax representation of (1.1) was constructed [23], making the mCH equation amenable to the method of inverse scattering. Equation (1.1) models the unidirectional propagation of surface waves in shallow water over a flat bottom [11], where $u(t, x)$ represents the free surface elevation in non-dimensional variables.

The Camassa-Holm (CH) equation

$$m_t + um_x + 2mu_x = 0, \qquad m = u - u_{xx}, \tag{1.3}$$

was also proposed [1] as a model describing the uni-directional propagation of shallow water waves. It was later shown to model the propagation of axially symmetric waves in hyperelastic rods [8, 9]. The CH equation was originally constructed by using the recursion operator method [14], and, in a similar vein, can also be derived by applying tri-Hamiltonian duality to the bi-Hamiltonian structure of the Korteweg–deVries (KdV) equation [21]. From this it follows that the CH equation

Date: November 4, 2013.

is completely integrable and can also be solved by the inverse scattering method [1, 13]. In contrast to the KdV equation, the CH equation has three remarkable distinctive properties. First, although completely integrable, it can describe wave breaking phenomenon: the solution remains bounded while its slope becomes infinite in finite time. Second is the existence of peakons, which are nonanalytic solitary waves that are global weak solutions, and, moreover, interact cleanly as do solitons. Third is the variety of interesting geometric formulations of the CH equation [2, 7, 16, 19].

Well-posedness and wave breaking of the CH equation were studied in a number of papers. It has been shown [5, 17, 22] that the Cauchy problem is locally well-posed for initial data $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$. Moreover, if the initial momentum density

$$m_0(x) = m(0,x) = (1-\partial_x^2)u_0 = u_0(x) - u_0''(x) \tag{1.4}$$

does not change sign, the Cauchy problem admits global solution for certain initial values [3, 5, 6], whereas solutions may blow up if their initial momentum density m_0 changes sign [3, 4, 5, 6]. The existence of global weak solutions was investigated in [25, 26].

Like the KdV equation, the CH equation has quadratic nonlinear terms. Two CH-type equations with cubic nonlinearities have been proposed: the modified CH equation (1.1) [13, 21], and the Novikov equation [20]

$$m_t + u^2 m_x + 3muu_x = 0, \qquad m = u - u_{xx}. \tag{1.5}$$

Both equations have peaked solitons and can be used to model wave breaking. The geometric formulation, integrability, local well-posedness, blow-up criteria and wave breaking, existence of peaked solitons (peakons), and the stability of single peakons and periodic peakons to the modified CH equation (1.1) were studied recently in [12, 15, 23, 24]. It is shown that even when the initial momentum density $m_0(x)$ does not change sign, the solutions to the Cauchy problem (1.1) may blow up in finite time, in contrast to the CH equation, the Degasperis–Procesi (DP) equation [10, 18], or the Novikov equation (1.5).

The goal of the present paper is to derive sharp sufficient conditions on the initial data that guarantees the formulation of singularities in the resulting solution in finite time. We take a different approach from that used in [15]. The first step is to rewrite the mCH equation (1.1) as a transport equation for the momentum density (1.2):

$$m_t + (u^2 - u_x^2)m_x = -2m^2 u_x. \tag{1.6}$$

The theory of transport equations implies that the solution m will remain regular and not blow up as long as the slope

$$(u^2 - u_x^2)_x = 2mu_x = 2u_x(u - u_{xx}) \tag{1.7}$$

remains bounded, while the solution blows up in finite time when the slope (1.7) is unbounded from below [12, 15]. The fact that the momentum density satisfies a transport equation also implies that, provided it initially does not change sign, then $m(t,x)$ and, in view of the Green's function formula — see (3.4) below — the solution $u(t,x)$ will maintain the same sign where defined.

Notation. For convenience, in the following, given a Banach space X, we denote its norm by $\|\cdot\|_X$. If there is no ambiguity, we omit the domain of function spaces.

The two basic conserved quantities

$$H_0[u]=\int_{\mathbb{R}} u\,dx, \qquad H_1[u]=\int_{\mathbb{R}} mu dx=\int_{\mathbb{R}}(u^2+u_x^2)\,dx, \tag{1.8}$$

are well-known and play an important role in all analysis of the solutions. When the momentum density m changes sign, the following novel conservation law

$$\|m\|_{L^1}=\int_{\mathbb{R}}|m|\,dx \tag{1.9}$$

will play a crucial role in controlling the solution u and its slope u_x. This observation allows us to establish sharper results than were found in [15]; see Remarks 3.4 and 3.6 below for comparisons.

Let us now state the main results in the present paper. We begin with a blow-up criteria that applies when the initial momentum density $m_0(x)$ is of one sign.

Theorem 1.1. *Suppose $u_0\in H^s(\mathbb{R})\cap L^1(\mathbb{R})$ with $s>\frac{5}{2}$. Assume that $m_0(x)\geq 0$ for all $x\in\mathbb{R}$. Assume that there exists $x_1\in\mathbb{R}$ such that*

$$m_0(x_1)=\sup_{x\in\mathbb{R}} m_0(x)>0 \qquad \textit{and} \qquad u_0'(x_1)<-\frac{H_0[u_0]}{\sqrt{12}}. \tag{1.10}$$

Then the corresponding solution $u(t,x)$ to the initial value problem (1.1) *blows up at a finite time T bounded by*

$$0<T\leq T_1:=-\frac{1}{2\,m_0(x_1)\,u_0'(x_1)}. \tag{1.11}$$

Our second blow-up criterion applies to the more challenging case when the initial momentum density $m_0(x)$ changes sign.

Theorem 1.2. *Suppose $m_0\in H^s(\mathbb{R})\cap L^1(\mathbb{R})$ with $s>\frac{1}{2}$. Assume that there exists $x_2\in\mathbb{R}$ such that*

$$m_0(x_2)=\sup_{x\in\mathbb{R}} m_0(x)>0 \qquad \textit{and} \qquad u_0'(x_2)<-\sqrt{\frac{11\,\|m_0\|_{L^1}^3}{24\,m_0(x_2)}}\,. \tag{1.12}$$

Then the solution $u(t,x)$ blows up in finite time T bounded by

$$0<T\leq T_2:=\frac{24}{11\,\|m_0\|_{L^1}^3}\left(-u_0'(x_2)-\sqrt{u_0'(x_2)^2-\frac{11\,\|m_0\|_{L_1}^3}{24\,m_0(x_2)}}\right). \tag{1.13}$$

It is not hard to see that the blow-up time bound in Theorem 1.2 exceeds that in Theorem 1.1, so $T_1\leq T_2$, and hence, when the initial momentum m_0 does not change sign, the bound (1.11) is stronger than that in (1.13).

Finally, if we drop the assumption that $u_0\in L^1(\mathbb{R})$, then, provided the initial momentum density is of one sign, we can establish the following alternative blow-up criterion, which is significantly stronger than that found in [15]; see Remark 3.6 for details.

Theorem 1.3. *Suppose $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Assume that $m_0(x) \geq 0$ for all $x \in \mathbb{R}$. If there exists x_3 such that*

$$m_0(x_3) = \sup_{x\in\mathbb{R}} m_0(x) > 0 \qquad \textit{and} \qquad u_0'(x_3) < -\sqrt{\frac{\sqrt{2}\,\|u_0\|_{H^1}^3}{12\,m_0(x_3)}}\,. \tag{1.14}$$

Then the solution $u(t,x)$ blows up in finite time T bounded by

$$0 < T \leq T_3 := \frac{-6\sqrt{2}\,u_0'(x_3)}{\|u_0\|_{H^1}^3} - \sqrt{\left(\frac{6\sqrt{2}\,u_0'(x_3)}{\|u_0\|_{H^1}^3}\right)^2 - \frac{6\sqrt{2}}{m_0(x_3)\|u_0\|_{H^1}^3}}. \tag{1.15}$$

Remark 1.4. *Invariance of the mCH equation under the change of sign $u \mapsto -u$ and, hence, $m \mapsto -m$ allows one to easily establish additional blow-up criteria by reversing the signs of the relevant quantities in the hypotheses in Theorems 1.1, 1.2, and 1.3. The resulting criteria and Theorems are left to the reader to explicitly formulate.*

The remainder of the paper is organized as follows. In Section 2, some preliminary estimates and results are recalled and presented. Section 3 is devoted to the proofs to our main results.

2. Preliminaries

We begin by recalling some basic results concerning the mCH equation (1.1), and refer the reader to [12, 15] for details and proofs. We begin with local well-posedness.

Lemma 2.1. *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Then there exists a time $T > 0$ such that the initial-value problem* (1.1) *has a unique strong solution $u \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$. Moreover, the map $u_0 \mapsto u$ is continuous from a neighborhood of the initial data u_0 in $H^s(\mathbb{R})$ into $C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$. Furthermore, if the solution u has maximum time of existence $0 < T < \infty$, then*

$$\int_0^T \|(m\,u_x)(t)\|_{L^\infty} dt = \infty.$$

Using the preceding criterion, the following blow up condition was obtained [15].

Lemma 2.2. *Suppose that $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Then the corresponding solution u to the initial value problem* (1.1) *blows up in finite time $T > 0$ if and only if*

$$\lim_{t\to T} \inf_{x\in\mathbb{R}} \big\{ m(t,x)\,u_x(t,x) \big\} = -\infty.$$

A particular conservative property of m plays a key role in establishing our new blow-up criteria. First, note that application of the method of characteristics to the transport equation (1.6) for m requires analyzing the flow governed by the effective wave speed $u^2 - u_x^2$, namely the solution $q(t,x)$ to the parametrized family of ordinary differential equations

$$\begin{cases} \dfrac{dq(t,x)}{dt} = u(t,q(t,x))^2 - u_x(t,q(t,x))^2, & x \in \mathbb{R}, \quad t \in [0,T), \\ q(0,x) = x, \qquad x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Lemma 2.3. *Suppose $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$, and let $T > 0$ be the maximal existence time of the strong solution u to the corresponding initial value problem* (1.1)*. Then* (2.1) *has a unique solution $q \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$ such that $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = \exp\left(2\int_0^t m(s,q(s,x))\, u_x(s,q(s,x))\, ds\right) > 0$$

for all $(t,x) \in [0,T) \times \mathbb{R}$. Furthermore,

$$m(t,q(t,x))\, q_x(t,x) = m_0(x) \quad \textit{for all} \quad (t,x) \in [0,T) \times \mathbb{R}. \tag{2.2}$$

Thus, by integrating the absolute value of (2.2) over $x \in \mathbb{R}$, we are led to the following unusual conservation law.

Proposition 2.1. *Assume $m_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s > \frac{1}{2}$. Suppose u is the corresponding solution to* (1.1) *with the initial data u_0 with maximal existence time $T > 0$. Then*

$$\|m(t)\|_{L^1(\mathbb{R})} = \|m_0\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |m_0(x)|\, dx, \qquad 0 \le t < T, \tag{2.3}$$

is a constant, independent of t.

3. Proofs of Theorems

The proofs of our main results are based on a series of lemmas. The first lemma plays a crucial role in deriving differential inequalities that enable us to improve earlier blow-up results.

Lemma 3.1. *Let $u_0 \in H^s(\mathbb{R})$, $s \ge 3$. Then the quantity*

$$M(t,x) = m(t,x)\, u_x(t,x) \tag{3.1}$$

satisfies the following integro-differential equation:

$$\begin{aligned} &M_t + (u^2 - u_x^2)M_x \\ &\quad = -2M^2 + m\left(\tfrac{2}{3}u^3 - uu_x^2\right) - \tfrac{1}{6}\, me^{-x}\int_{-\infty}^{x} e^{y}\left(2u^3 + 3uu_y^2 - u_y^3\right) dy \\ &\qquad\qquad - \tfrac{1}{6}\, me^{x}\int_{x}^{\infty} e^{-y}\left(2u^3 + 3uu_y^2 + u_y^3\right) dy \\ &\quad = -2M^2 + m\left(\tfrac{1}{3}u^3 - uu_x^2\right) \\ &\qquad\qquad - \tfrac{1}{6}\, m\left(\int_{-\infty}^{x} e^{y-x}(u-u_y)^3 dy + \int_{x}^{\infty} e^{x-y}(u+u_y)^3 dy\right). \end{aligned} \tag{3.2}$$

Proof. We begin with a straightforward differentiation of (3.1) and use of (1.1) to establish that the quantity M satisfies

$$\begin{aligned} &M_t + (u^2 - u_x^2)M_x \\ &\qquad = -\,2M^2 - 2m(1-\partial_x^2)^{-1}(u_x^2 m) - 2m\partial_x(1-\partial_x^2)^{-1}(uu_x m). \end{aligned} \tag{3.3}$$

We evaluate the second and third terms on the right hand side of (3.3) using the Green's function $G(x,y) = e^{-|x-y|}$ for the differential operator $1-\partial_x^2$ on the line, whereby

$$u(t,x) = (1-\partial_x^2)^{-1} m(t,x) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} m(t,y)\,dy, \tag{3.4}$$

Thus,

$$(1-\partial_x^2)^{-1}(u_x^2 m) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} u_y(t,y)^2 m(t,y)\,dy = I_{11} + I_{12},$$

where, applying integration by parts,

$$\begin{aligned}
I_{11} &= \tfrac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}(uu_y^2 - u_y^2 u_{yy})\,dy = \tfrac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y} uu_y^2 dy - \tfrac{1}{6}e^{-x}\int_{-\infty}^{x} e^{y} du_y^3 \\
&= \tfrac{1}{6}e^{-x}\int_{-\infty}^{x} e^{y}(3uu_y^2 + u_y^3)dy - \tfrac{1}{6}u_x^3(t,x), \\
I_{12} &= \tfrac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}(uu_y^2 - u_y^2 u_{yy})\,dy = \tfrac{1}{2}e^{x}\int_{x}^{\infty} e^{-y} uu_y^2 dy - \tfrac{1}{6}e^{x}\int_{x}^{\infty} e^{-y} du_y^3 \\
&= \tfrac{1}{6}e^{x}\int_{x}^{\infty} e^{-y}(3uu_y^2 - u_y^3)dy + \tfrac{1}{6}u_x^3(t,x).
\end{aligned}$$

Similarly, the third term in (3.3) is

$$\begin{aligned}
\partial_x(1-\partial_x^2)^{-1}(uu_x m) &= \partial_x\Big(\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} u(t,y)u_y(t,y)m(t,y)\,dy\Big) \\
&= -\tfrac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y} uu_y m\,dy + \tfrac{1}{2}e^{x}\int_{x}^{\infty} e^{-y} uu_y m\,dy = I_{21} + I_{22} + I_{23} + I_{24},
\end{aligned}$$

where, by a similar computation,

$$\begin{aligned}
I_{21} &= -\tfrac{1}{6}e^{-x}\int_{-\infty}^{x} e^{y}\,du^3 = -\tfrac{1}{6}u^3(t,x) + \tfrac{1}{6}e^{-x}\int_{-\infty}^{x} e^{y}u^3\,dy, \\
I_{22} &= \tfrac{1}{4}e^{-x}\int_{-\infty}^{x} e^{y}u\,d(u_y^2) = \tfrac{1}{4}u(t,x)u_x^2(t,x) - \tfrac{1}{4}e^{-x}\int_{-\infty}^{x}\big(e^{y}u + e^{y}u_y\big)u_y^2\,dy \\
&= \tfrac{1}{4}u(t,x)u_x^2(t,x) - \tfrac{1}{4}e^{-x}\int_{-\infty}^{x} e^{y}\left(uu_y^2 + u_y^3\right)\,dy, \\
I_{23} &= \tfrac{1}{6}e^{x}\int_{x}^{\infty} e^{-y}\,du^3 = -\tfrac{1}{6}u^3(t,x) + \tfrac{1}{6}e^{x}\int_{x}^{\infty} e^{-y}u^3\,dy, \\
I_{24} &= -\tfrac{1}{4}e^{x}\int_{x}^{\infty} e^{-y}u\,d(u_y^2) = \tfrac{1}{4}u(t,x)u_x^2(t,x) + \tfrac{1}{4}e^{x}\int_{x}^{\infty}\left(-e^{-y}u + e^{-y}u_y\right)u_y^2\,dy \\
&= \tfrac{1}{4}u(t,x)u_x^2(t,x) - \tfrac{1}{4}e^{x}\int_{x}^{\infty} e^{-y}\left(uu_y^2 - u_y^3\right)dy.
\end{aligned}$$

Combining the preceding expressions, we arrive at

$$
\begin{aligned}
&2m(1-\partial_x^2)^{-1}(u_x^2 m) + 2m\partial_x(1-\partial_x^2)^{-1}(uu_x m)\\
&= \tfrac{1}{3} m \Bigg(e^{-x}\int_{-\infty}^{x} e^{y}\left(3uu_y^2+u_y^3\right)dy + e^{x}\int_{x}^{\infty} e^{-y}\left(3uu_y^2-u_y^3\right)dy\\
&\qquad + \tfrac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\left(2u^3-3uu_y^2-3u_y^3\right)dy\\
&\qquad + \tfrac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}\left(2u^3-3uu_y^2+3u_y^3\right)dy + (3uu_x^2-2u^3)(t,x)\Bigg)\\
&= m\left(uu_x^2-\tfrac{2}{3}u^3\right) + \tfrac{1}{6}\, m e^{-x}\int_{-\infty}^{x} e^{y}\left(2u^3+3uu_y^2-u_y^3\right)dy\\
&\qquad + \tfrac{1}{6}\, m e^{x}\int_{x}^{\infty} e^{-y}\left(2u^3+3uu_y^2+u_y^3\right)dy.
\end{aligned}
\tag{3.5}
$$

Substituting back into (3.3) completes the proof of the first formula on the right hand side of (3.2). The second formula follows from the further manipulations of a similar nature:

$$
\begin{aligned}
e^{-x}\int_{-\infty}^{x} e^{y}\big(2u^3+3uu_y^2-u_y^3\big)dy &= e^{-x}\int_{-\infty}^{x} e^{y}\left(u^3+(u-u_y)^3+3u^2u_y\right)dy\\
&= e^{-x}\int_{-\infty}^{x} e^{y}\left(u^3+(u-u_y)^3\right)dy + e^{-x}\int_{-\infty}^{x} e^{y}du^3\\
&= e^{-x}\int_{-\infty}^{x} e^{y}(u-u_y)^3dy + u(t,x)^3,\\
e^{x}\int_{x}^{\infty} e^{-y}\big(2u^3+3uu_y^2+u_y^3\big)dy &= e^{x}\int_{x}^{\infty} e^{-y}\left(u^3+(u+u_y)^3+3u^2u_y\right)dy\\
&= e^{x}\int_{x}^{\infty} e^{-y}\left(u^3+(u+u_y)^3\right)dy + e^{x}\int_{x}^{\infty} e^{-y}du^3\\
&= e^{x}\int_{x}^{\infty} e^{-y}(u+u_y)^3dy + u(t,x)^3.
\end{aligned}
\tag{3.6}
$$

This completes the proof of (3.2). □

Lemma 3.2. *Assume that the initial data* $u_0 \in H^s(\mathbb{R})$, $s > \frac{5}{2}$. *Let* $T > 0$ *be the maximal existence time of the resulting solution* $u(t,x)$ *to the initial value problem* (1.1). *If* $m_0(x) \geq 0$, *then*

$$|u_x(t,x)| \leq u(t,x), \qquad (t,x) \in [0,T) \times \mathbb{R}. \tag{3.7}$$

Moreover, if $m_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, *then*

$$u(t,x) \leq \tfrac{1}{2}\, H_0[u_0], \qquad (t,x) \in [0,T) \times \mathbb{R}. \tag{3.8}$$

Proof. Lemma 2.3 implies that m and u are nonnegative for any $0 \le t < T$, $x \in \mathbb{R}$. Using the Green's function formula (3.4), we have

$$\begin{aligned} u(t,x) &= \tfrac{1}{2}\, e^{-x} \int_{-\infty}^{x} e^{y} m\, dy + \tfrac{1}{2}\, e^{x} \int_{x}^{\infty} e^{-y} m\, dy, \\ u_x(t,x) &= -\tfrac{1}{2}\, e^{-x} \int_{-\infty}^{x} e^{y} m\, dy + \tfrac{1}{2}\, e^{x} \int_{x}^{\infty} e^{-y} m\, dy. \end{aligned} \tag{3.9}$$

Thus,

$$\begin{aligned} 0 \le u(t,x) - u_x(t,x) &= e^{-x} \int_{-\infty}^{x} e^{y} m\, dy \le \int_{-\infty}^{x} m\, dy \\ 0 \le u(t,x) + u_x(t,x) &= e^{x} \int_{x}^{\infty} e^{-y} m\, dy \le \int_{x}^{\infty} m\, dy, \end{aligned} \tag{3.10}$$

which imply (3.7). Moreover, assuming $m_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, (3.10) implies that

$$2u(t,x) \le \int_{-\infty}^{\infty} m\, dy = \int_{-\infty}^{\infty} u\, dy = H_0[u_0],$$

which proves (3.8). $\square$

Remark 3.3. *The proof of Lemma 3.2 implies the following further estimates:*

$$u(t,x) + |u_x(t,x)| \le H_0[u_0], \qquad u(t,x) \le \tfrac{1}{2}\, H_0[u_0] \le u_x(t,x) + H_0[u_0].$$

Proof of Theorem 1.1. Let us abbreviate

$$\begin{aligned} &\widehat{u}(t) := u(t, q(t,x_1)), && \widehat{v}(t) := u_x(t, q(t,x_1)), \\ &\widehat{m}(t) := m(t, q(t,x_1)), && \widehat{M}(t) := M(t, q(t,x_1)) = \widehat{m}(t)\, \widehat{v}(t), \end{aligned} \tag{3.11}$$

where x_1 is defined in (1.10). Since

$$\widehat{m}(0) > 0 \quad \text{we have} \quad \widehat{m}(t) > 0 \quad \text{where defined.} \tag{3.12}$$

Then equations (1.1) and (2.1) imply that

$$\frac{d\,\widehat{m}}{dt} = -2\, \widehat{m}(t)\, \widehat{M}(t) = -2\, \widehat{m}(t)^2\, \widehat{v}(t). \tag{3.13}$$

Moreover, combining (3.2), (3.10), and the fact that $m \ge 0$, we see that

$$\frac{d\,\widehat{M}}{dt} \le -2\, \widehat{M}(t)^2 + \widehat{m}(t) \left(\tfrac{1}{3} \widehat{u}(t)^3 - \widehat{v}(t)^2\, \widehat{u}(t) \right). \tag{3.14}$$

Thus, using (3.11), (3.13), (3.14), and Lemma 3.2, we find

$$\begin{aligned} \frac{d\,\widehat{v}}{dt} = \frac{d}{dt} \left(\frac{\widehat{M}(t)}{\widehat{m}(t)} \right) &= \frac{\widehat{m}(t)\, \widehat{M}'(t) - \widehat{m}'(t)\, \widehat{M}(t)}{\widehat{m}(t)^2} \\ &= \frac{1}{\widehat{m}(t)} \left(\frac{d\,\widehat{M}}{dt} + 2\, \widehat{M}(t)^2 \right) \\ &\le \tfrac{1}{3} \widehat{u}(t)^3 - \widehat{v}(t)^2\, \widehat{u}(t) \le \widehat{u}(t) \left(\tfrac{1}{12} H_0^2[u_0] - \widehat{v}(t)^2 \right). \end{aligned} \tag{3.15}$$

Moreover, our assumption (1.10) implies that

$$\widehat{v}(0) < -\frac{H_0[u_0]}{\sqrt{12}} < 0, \tag{3.16}$$

and hence the right hand side of (3.15) is negative at $t = 0$. A straightforward ordinary differential equation argument allows us to conclude that $\widehat{v}(t)$ must be everywhere decreasing, with

$$\widehat{v}(t) < \widehat{v}(0) < -\frac{H_0[u_0]}{\sqrt{12}} \qquad \text{for all} \qquad t \in [0, T). \tag{3.17}$$

Combining (3.17) with (3.13), we have

$$\frac{d}{dt}\left(\frac{1}{\widehat{m}(t)}\right) = 2\,\widehat{v}(t) < 2\,\widehat{v}(0) \qquad \text{for all} \qquad t \in [0, T). \tag{3.18}$$

Integrating from 0 to t and using (3.12) produces

$$0 < \frac{1}{\widehat{m}(t)} < \frac{1}{\widehat{m}(0)} + 2\,\widehat{v}(0)\,t \qquad \text{for all} \qquad t \in [0, T).$$

Thus (3.16) implies that there exists $0 < T \le T_1$, where T_1 is given in (1.11), such that

$$\widehat{m}(t) \to +\infty, \quad \text{as} \quad t \to T \le T_1 = -\frac{1}{2\,\widehat{m}(0)\,\widehat{v}(0)} = -\frac{1}{2\,m_0(x_1)u_0'(x_1)}.$$

Furthermore, since

$$\widehat{M}(t) = \widehat{v}(t)\,\widehat{m}(t) < \widehat{v}(0)\,\widehat{m}(t) \le 0,$$

we conclude that

$$\inf_{x\in\mathbb{R}} M(t,x) \le \widehat{M}(t) \to -\infty, \qquad \text{as} \qquad t \to T \le T_1, \tag{3.19}$$

which demonstrates that the solution $u(t, x)$ blows up at a time $0 < T \le T_1$. This completes the proof of Theorem 1.1. □

Remark 3.4. *The blow-up time bound in Theorem 1.1 is an improvement over the bound in Theorem 5.2 of* [15], *which is*

$$\begin{aligned}\widetilde{T}_1 &:= \frac{1}{m_0(x_1)} \cdot \frac{1}{-u_0'(x_1) + \sqrt{u_0'(x_1)^2 - \|u_0\|_{H^1}^3/\big(\sqrt{2}\,m_0(x_1)\big)}} \\ &> -\frac{1}{2m_0(x_1)u_0'(x_1)} = T_1.\end{aligned}$$

Proof of Theorem 1.2. We employ the same notation (3.11) as in the preceding proof, but now with x_2, as given in (1.12), replacing x_1. Using the Green's function formulae (3.9), we have

$$\begin{aligned}|u(t,x)| &\le \frac{1}{2}\int_{-\infty}^{\infty} |m(t,x)| dx = \frac{1}{2}\int_{-\infty}^{\infty} |m_0(x)|\,dx = \frac{1}{2}\|m_0\|_{L^1}, \\ |u_x(t,x)| &\le \frac{1}{2}\int_{-\infty}^{\infty} |m(t,x)|\,dx = \frac{1}{2}\int_{-\infty}^{\infty} |m_0(x)|\,dx = \frac{1}{2}\|m_0\|_{L^1}.\end{aligned} \tag{3.20}$$

Plugging these estimates into (3.2) produces

$$\frac{d\,\widehat{M}(t)}{dt} \le -2\,\widehat{M}(t)^2 - \widehat{m}(t)\Big[\big(\widehat{u}(t)\,\widehat{v}(t)^2 - \tfrac{1}{3}\widehat{u}(t)^3\big) - \tfrac{1}{3}\|m_0\|_{L^1}^3\Big]. \tag{3.21}$$

As in (3.15), but now using the inequalities (3.20), (3.21), we find

$$\frac{d\,\widehat{v}(t)}{dt} \leq \tfrac{1}{3}\widehat{u}(t)^3 - \widehat{v}(t)^2\,\widehat{u}(t) + \tfrac{1}{3}\,\|m_0\|^3_{L^1} \leq \tfrac{11}{24}\,\|m_0\|^3_{L^1}. \tag{3.22}$$

Integrating twice from 0 to t produces, first,

$$\frac{1}{2}\,\frac{d}{dt}\left(\frac{1}{\widehat{m}(t)}\right) = \frac{\widehat{M}(t)}{\widehat{m}(t)} = \widehat{v}(t) \leq \tfrac{11}{24}\,\|m_0\|^3_{L^1}\,t + \widehat{v}(0), \tag{3.23}$$

and then

$$\begin{aligned} \frac{1}{\widehat{m}(t)} &\leq \tfrac{11}{24}\,\|m_0\|^3_{L^1}\,t^2 + 2\,\widehat{v}(0)\,t + \frac{1}{\widehat{m}(0)} \\ &= \tfrac{11}{24}\,\|m_0\|^3_{L^1}\,t^2 + 2\,u_0'(x_2)\,t + \frac{1}{m_0(x_2)}. \end{aligned} \tag{3.24}$$

The assumption (1.12) implies that the quadratic polynomial on the right hand side of (3.24) starts out positive at $t = 0$, and becomes zero at time $t = T_2$, as prescribed in (1.13). Since $0 < \widehat{m}(t) = m(t, q(t, x_2))$, cf. (3.12), it follows that there exists $0 < T \leq T_2$ such that $\widehat{m}(t) \to +\infty$ as $t \to T$. On the other hand, (3.23) also implies that

$$\widehat{M}(t) \leq \widehat{m}(t)\big(\tfrac{11}{24}\,\|m_0\|^3_{L^1}\,t + \widehat{v}(0)\big) = \widehat{m}(t)\big(\tfrac{11}{24}\,\|m_0\|^3_{L^1}\,t + u_0'(x_2)\big), \tag{3.25}$$

which, in turn, implies that

$$\inf_{x\in\mathbb{R}} M(t,x) \leq \widehat{M}(t) \to -\infty, \quad \text{as} \quad t \to T \leq T_2.$$

We conclude that the solution $\widehat{u}(t)$ blows up at the finite time $T \leq T_2$, and completes the proof of Theorem 1.2. □

Remark 3.5. *When the initial momentum m_0 does not change sign, the blow-up time bound given in Theorem 1.1 is located before the blow up time bound given in Theorem 1.2, that is, $T_1 \leq T_2$. In the other words, in this situation, the result given in Theorem 1.1 is an improvement over that in Theorem 1.2.*

Finally, we remark that the proof of Theorem 1.3 is completely analogous to that of Theorem 1.2, and can be carried out by replacing x_2 by x_3, while the key inequality (3.22) becomes

$$\frac{d\,\widehat{v}}{dt} \leq \frac{1}{12\sqrt{2}}\,\|u_0\|^3_{H^1}. \tag{3.26}$$

Remark 3.6. *In Theorem 5.2 in* [15]*, the blow-up time bound of*

$$\widetilde{T}_3 = -\frac{u_0'(x_3)}{\sqrt{2}\,\|u_0\|^3_{H^1}} - \frac{1}{2}\sqrt{\left(\frac{\sqrt{2}\,u_0'(x_3)}{\|u_0\|^3_{H^1}}\right)^2 - \frac{2\sqrt{2}}{m_0(x_3)\|u_0\|^3_{H^1}}}$$

was established. We can easily see that the bound in (1.15) *satisfies $T_3 < \widetilde{T}_3$, and hence Theorem 1.3 is a stronger result. To establish the inequality, note that*

$$g(x) = u_0'(x_3) + \sqrt{u_0'(x_3)^2 - \frac{2x}{m_0(x_3)}}$$

is a decreasing function on the interval $0 \le x \le \frac{1}{2} u_0'(x_3)^2 m_0(x_3)$. *Thus,*

$$f(x) = \frac{1}{m_0(x_3)\, g(x)} = \frac{1}{2}\left(\frac{u_0'(x_3)}{x} - \sqrt{\frac{u_0'(x_3)^2}{x^2} - \frac{2}{x\, m_0(x_3)}}\right)$$

is an increasing function on the same interval and satisfies

$$T_3 = f\big(\tfrac{\sqrt{2}}{12}\, \|u_0\|_{H^1}^3\big) < f\big(\tfrac{1}{\sqrt{2}}\, \|u_0\|_{H^1}^3\big) = \widetilde{T}_3.$$

Finally, we note that the preceding proofs imply that there is at least one point where $M = m u_x$ becomes infinite exactly at the time of blow up.

Corollary 3.7. *Suppose that the assumptions of Theorems 1.1, 1.2, or 1.3 hold for the solution* $u(t,x)$ *of* (1.1) *with initial data satisfying* $m_0(x_\star) = \sup_{x\in\mathbb{R}} m_0(x) > 0$, *for some point* $x_\star \in \mathbb{R}$. *If* $T < \infty$ *is the finite blow-up time of the corresponding solution* $u(t,x)$, *then* $\lim_{t\to T} M(t, q(t,x_\star)) = -\infty$.

Acknowledgments. The work of Liu is partially supported by the NSF grant DMS-1207840 and the NSF-China grant-11271192. The work of Olver is partially supported by NSF grant DMS-1108894. The work of Qu is supported in part by the NSF-China for Distinguished Young Scholars grant-10925104. The work of Zhang is partially supported by the NSF of China under the grant 11101337, Doctoral Foundation of Ministry of Education of China grant-20110182120013, Fundamental Research Funds for the Central Universities grant-XDJK2011C046, and China Scholarship Council.

References

[1] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71** (1993), 1661-1664.

[2] K. S. Chou, C. Z. Qu, Integrable equations arising from motions of plane curves I, *Physica D*, **162** (2002), 9-33.

[3] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, *Ann. Inst. Fourier (Grenoble)*, **50** (2000), 321-362.

[4] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.*, **181** (1998), 229-243.

[5] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)*, **26** (1998), 303-328.

[6] A. Constantin and J. Escher, Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation, *Comm. Pure Appl. Math.*, **51** (1998), 475-504.

[7] A. Constantin and H. Kolev, Geodesic flow on the diffeomorphism group of the circle, *Comment. Math. Helv.*, **78** (2003), 787-804.

[8] A. Constantin and W. A. Strauss, Stability of a class of solitary waves in compressible elastic rods, *Phys. Lett. A*, **270** (2000), 140-148.

[9] H. Dai, Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod, *Acta Mech.*, **127** (1998), 193-207.

[10] A. Degasperis and M. Procesi, Asymptotic integrability, in *Symmetry and perturbation theory* (ed. A. Degasperis & G. Gaeta), pp 23-37, World Scientific, Singapore, 1999.

[11] A. S. Fokas, On a class of physically important integrable equation, *Physica D*, **87** (1995), 145-150.

[12] Y. FU, G. L. GUI, Y. LIU AND C. Z. QU, On the Cauchy problem for the integrable modified Camassa-Holm equation with cubic nonlinearity, *J. Differential Equations*, **255** (2013), 1905-1938.
[13] B. FUCHSSTEINER, Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation, *Physica D*, **95** (1996), 229-243.
[14] B. FUCHSSTEINER AND A. FOKAS, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Physica D*, **4** (1981/1982), 47-66.
[15] G. L. GUI, Y. LIU , P. OLVER AND C. Z. QU, Wave-breaking and peakons for a modified Camassa-Holm equation, *Comm. Math. Phy.*, **319** (2013), 731-759.
[16] S. KOURANBAEVA, The Camassa-Holm equation as a geodesic flow on the diffeomorphism group, *J. Math. Phys.*, **40** (1999), 857-868.
[17] Y. LI AND P. OLVER, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, *J. Differential Equations*, **162** (2000), 27-63.
[18] Y. LIU, Z. YIN, Global existence and blow-up phenomena for the Degasperis-Procesi equation, *Comm. Math. Phys.*, **267** (2006), 801-820.
[19] G. MISOŁEK, A shallow water equation as a geodesic flow on the Bott-Virasoro group, *J. Geom. Phys.*, **24** (1998), 203-208.
[20] V. NOVIKOV, Generalizations of the Camassa-Holm equation, *J. Phys. A*, **42** (2009), 342002.
[21] P. OLVER AND P. ROSENAU, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, *Phys. Rev. E*, **53** (1996), 1900-1906.
[22] G. RODRIGUEZ-BLANCO, On the Cauchy problem for the Camassa-Holm equation, *Nonlinear Anal.*, **46** (2001), 30-327.
[23] Z. QIAO, A new integrable equation with cuspons and W/M-shape-peaks solitons, *J. Math. Phys.*, **47** (2006), 112701.
[24] C. Z. QU, X. C. LIU, Y. LIU, Stability of peakons for an integrable modified Camassa-Holm equation with cubic onlinearity, *Comm. Math. Phys.*, **322** (2013), 967-997.
[25] Z. XIN AND P. ZHANG, On the weak solutions to a shallow water equation, *Comm. Pure Appl. Math.*, **53** (2000), 1411-1433.
[26] Z. XIN AND P. ZHANG, On the uniqueness and large time behavior of the weak solutions to a shallow water equation, *Comm. Partial Differential Equations*, **27** (2000), 1815-1844.

(Yue Liu) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, ARLINGTON, TX 76019-0408, USA
E-mail address: yliu@uta.edu

(Peter Olver) SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455, USA
E-mail address: olver@math.umn.edu

(Changzheng Qu) DEPARTMENT OF MATHEMATICS, NINGBO UNIVERSITY, NINGBO 315211, P.R. CHINA
E-mail address: quchangzheng@nbu.edu.cn

(Shuanghu Zhang) DEPARTMENT OF MATHEMATICS, SOUTHWEST UNIVERSITY, CHONGQING 400715, P. R. CHINA
E-mail address: zhangshh@swu.edu.cn