Liouville correspondences for integrable hierarchies

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Abstract

This chapter contains a survey on Liouville correspondences between integrable hierarchies. A Liouville transformation between the corresponding isospectral problems induces a Liouville correspondence between their flows and Hamiltonian functionals. As prototypical examples, we construct Liouville correspondences for the modified Camassa-Holm, the Novikov, the Degasperis-Procesi, the two-component Camassa-Holm and the two-component Novikov (Geng-Xue) hierarchies. In addition, a new Liouville correspondence for a certain dual Schrödinger integrable hierarchy is presented.

1 Introduction

Due to the pioneering works [1] and [43], it is known that the spatial isospectral problem in the Lax-pair formulation of an integrable system plays a fundamental role when constructing the soliton solutions using the inverse scattering transform, as well as analyzing the long-time behavior of solutions based on the Riemann-Hilbert approach. The transition from one isospectral problem to another via a change of variables can usually be identified as a form of *Liouville transformation*; see [61] and [64] for this terminology. It is then expected that such a correspondence based on a Liouville transformation, which we call a *Liouville correspondence* for brevity, can be used to establish an inherent correspondence between associated integrability properties including symmetries, conserved quantities, soliton solutions, Hamiltonian structures, etc. Indeed, this is a basic idea for investigating the integrability of a new system by establishing its relation to a known integrable system through some kind of transformation, which, besides Liouville transformations, can include Bäcklund transformations, Miura transformations, gauge transformations, Darboux transformations, hodograph transformations, etc. [12, 15, 41, 59, 62, 66, 70, 74]. Applying an appropriate transformation enables one to adapt known solutions and integrable structures in order to derive explicit solutions and investigate the integrability properties for the transformed system.

In recent years, a great deal of attention has been devoted to integrable systems of the Camassa-Holm type, following the discovery of their novel properties, including the structure of nonlinear dispersion, which (as a rule) supports non-smooth soliton solutions, such as peakons, cuspons, compactons, etc. [10, 52], and the ability of such systems to model wave-breaking phenomena. Previous investigations demonstrate that many integrable hierarchies of Camassa-Holm type admit a Liouville correspondence with certain classical integrable hierarchies. In the following, recent advances in the study of Liouville correspondences for the integrable hierarchies of Camassa-Holm type and non-Camassa-Holm type are summarized.

Among integrable systems of Camassa-Holm type, the best-studied example is the Camassa-Holm (CH) equation

$$n_t + 2v_x n + v n_x = 0, \qquad n = v - v_{xx}, \tag{1.1}$$

that has a quadratic nonlinearity [8, 9, 17, 27, 35], while the modified Camassa-Holm (mCH) equation [67]

$$m_t + \left((u^2 - u_x^2)\, m \right)_x = 0, \qquad m = u - u_{xx}, \tag{1.2}$$

is a prototypical integrable model with cubic nonlinearity, which presents several novel properties, as described, for instance, in [7, 30, 53, 54, 58, 69]. The CH equation (1.1) appears as an integrable generalization of the Korteweg–de Vries (KdV) equation possessing infinitely many symmetries [27], and is shown to correspond to the first negative flow of the KdV integrable hierarchy using the Liouville transformation [26]; see also [5, 6]. A novel link between the mCH equation (1.2) and the modified KdV (mKdV) equation was found and used to obtain the multisoliton solutions of (1.2) from the known multisoliton solutions of the mKdV equation in [58]. In addition, all the equations in the CH, mCH, KdV, and mKdV hierarchies have a classical bi-Hamiltonian form (see [67] for example). The interesting feature here is that the two Hamiltonians for the CH and mCH integrable hierarchies can be constructed from those of the KdV and mKdV hierarchies, respectively, using the approach of *tri-Hamiltonian duality* [24, 26, 67]. This approach is based on the observation that, by applying an appropriate scaling argument, many standard integrable equations that possess a bi-Hamiltonian structure in fact admit a compatible triple of Hamiltonian operators. Different combinations of the members of the compatible Hamiltonian triple can generate different types of integrable bi-Hamiltonian systems, which admit a *dual* relation. In [72], another kind of duality by exploiting the zero curvature formulations of the CH and mCH hierarchies and the KdV and mKdV hierarchies was also discussed.

As a consequence of these connections between the CH equation (1.1) and the KdV equation, and between the mCH equation (1.2) and the mKdV equation, it is anticipated that the KdV and mKdV hierarchies should be related to their respective dual counterparts, the CH and mCH hierarchies, in a certain manner. A relationship between the KdV and CH hierarchies provided by the approach of loop groups is explored in [73]. Note that the Lax-pair formulations of equations in the CH, mCH, KdV, and mKdV hierarchies are all based on a second order isospectral problem. Using the Liouville transformation between the spatial isospectral problems of the CH hierarchy and the KdV hierarchy, the Liouville correspondence between these two integrable hierarchies is established in [44] and [60], and in addition gives rise to a correspondence between the Hamiltonian functionals of the two hierarchies. The analysis for the KdV-CH setting depends strongly on the subtle relation between one of the original KdV Hamiltonian operators and one of the dual CH Hamiltonian operators. Furthermore, the relation between the smooth traveling-wave solutions of the CH equation and the KdV equation under the Liouville transformation was investigated in [45]. In [36], we established the Liouville correspondence between the integrable mCH hierarchy induced by (1.2) and the mKdV hierarchy, including the explicit relations between their equations and Hamiltonian functionals. In contrast to the CH-KdV setting, the analysis in [36] was based on the interrelation between the respective recursion operators and the conservative structure of all the equations in the mCH hierarchy. It is worth pointing out that the tri-Hamiltonian duality relationships helps establish corresponding Liouville correspondences in both the CH-KdV and mCH-mKdV cases as shown in [44] and [36]. As a by-product, we constructed in [36] a novel transformation mapping the mCH equation (1.2) to the CH equation (1.1) in terms of the respective Liouville correspondences between the CH-KdV hierarchies and between the mCH-mKdV hierarchies.

In [67], the nonlinear Schrödinger equation

$$u_t = \mathrm{i}\left(u_{xx} + u|u|^2\right),$$

was investigated using the method of tri-Hamiltonian duality, and the dual integrable version

$$u_t + \mathrm{i}\, u_{xt} = |u|^2\,(\mathrm{i}\, u - u_x), \tag{1.3}$$

was derived (see also [21]).The dual Schrödinger equation (1.3) has attracted much interest in recent years for its soliton solutions, well-posedness, physical relevance, etc. [23, 46, 47, 48, 75]. In this article, we present a new result of Liouville correspondence for the integrable hierarchy induced by equation (1.3). We prove that this dual integrable hierarchy can be related to the mKdV integrable hierarchy by a different Liouville transformation compared to the transformation used to establish the Liouville correspondence between the mCH hierarchy and the mKdV hierarchy.

The tri-Hamiltonian duality theory is a fruitful approach for deriving new dual integrable hierarchies from known integrable soliton hierarchies, provided that the bi-Hamiltonian formulation of the latter hierarchy admits a compatible triple of Hamiltonian operators through appropriate rescalings. For integrable soliton hierarchies with generalized bi-Hamiltonian structures, i.e., compatible pairs of Dirac structures [20], the corresponding Hamiltonian operators usually do not support a decomposition as linear combination of different parts using the scaling argument, and so constructing the associated dual integrable hierarchy in this case is unclear. On the other hand, other interesting CH-type integrable hierarchies without tri-Hamiltonian duality structure can be found using some particular classification procedure. Two representative examples are the Degasperis-Procesi (DP) and Novikov integrable hierarchies.

The DP integrable equation with quadratic nonlinearity

$$n_t = 3v_x n + v n_x, \qquad n = v - v_{xx}, \tag{1.4}$$

was derived in [19] as a result of the asymptotic integrability method for classifying (a class of) third-order nonlinear dispersive evolution equations. In such a classification framework, the CH equation and the KdV equation are two only other integrable candidates. The DP equation (1.4) is integrable with a Lax pair involving a 3×3 isospectral problem as well as a bi-Hamiltonian structure [18]. In [18, 31], it was pointed out that using the Liouville transformation, equation (1.4) is related to the first negative flow of the Kaup-Kupershmidt (KK) hierarchy, which is initiated from the classical KK integrable equation [40, 42]

$$P_\tau + P_{yyyyy} + 10PP_{yyy} + 25P_yP_{yy} + 80P^2P_y = 0. \tag{1.5}$$

The Novikov integrable equation with cubic nonlinearity

$$m_t = 3uu_x m + u^2 m_x, \qquad m = u - u_{xx}, \tag{1.6}$$

was discovered as a consequence of the symmetry classification of nonlocal partial differential equations involving both cubic and quadratic nonlinearities in [63]. The Lax pair formulation with 3×3 isospectral problem and bi-Hamiltonian structure were established in [32]. It was shown in [32] that the Novikov equation (1.6) is related by the Liouville transformation to the first negative flow of the Sawada-Kotera (SK) hierarchy, which is initiated from the classical SK integrable equation [11, 71]

$$Q_\tau + Q_{yyyyy} + 5QQ_{yyy} + 5Q_yQ_{yy} + 5Q^2Q_y = 0. \tag{1.7}$$

Although the DP and Novikov hierarchies are bi-Hamiltonian, the corresponding Hamiltonian operators are not amenable to the tri-Hamiltonian duality construction, in particular one that is related to the Hamiltonian operators of the

KK and SK equations. In addition, the KK equation (1.5) and the SK equation (1.7) both have generalized bi-Hamiltonian formulations with the corresponding hierarchy generated by the respective recursion operators, which are obtained by composing symplectic and implectic operators that fail to satisfy the conditions of non-degeneracy or invertibility [28]. Therefore, studying a Liouville correspondence for the DP or Novikov integrable hierarchies requires a more delicate analysis. In [37], using the Liouville transformations and conservative structures in both the DP and Novikov settings, we found the corresponding operator identities relating the recursion operator of the DP/Novikov hierarchy and the adjoint operator of the recursion operator of the KK/SK hierarchy, and then were able to establish the Liouville correspondences between the DP and KK integrable hierarchies, as well as the Novikov and SK integrable hierarchies. In particular, a nontrivial operator factorization for the recursion operator of the SK equation discovered in [14] plays a key role in constructing the Liouville correspondence for the Novikov hierarchy. Note that the SK equation (1.5) and the KK equation (1.7) are related to the so-called Fordy-Gibbons-Jimbo-Miwa equation via certain Miura transformations [25]. Exploiting such a relation, we also obtained in [37] a nontrivial link between the Novikov equation (1.6) and the DP equation (1.4).

One can also investigate Liouville correspondences for multi-component integrable hierarchies. The integrable two-component CH hierarchy and the integrable two-component Novikov (Geng-Xue) hierarchy are two typical examples. The well-studied two-component CH (2CH) system

$$m_t + 2u_x m + u m_x + \rho\rho_x = 0, \qquad \rho_t + (\rho u)_x = 0, \qquad m = u - u_{xx}, \tag{1.8}$$

arises as the dual version for the integrable two-component Ito system

$$u_t = u_{xxx} + 3uu_x + vv_x, \qquad v_t = (uv)_x,$$

introduced in [33] using the method of tri-Hamiltonian duality [67]. Such a duality structure ensures that the system (1.8) is integrable, with bi-Hamiltonian formulation and compatible Hamiltonians, which thereby recursively generate the entire 2CH integrable hierarchy, with (1.8) forming the second flow in the positive direction.

In [16], the 2CH system (1.8) was derived as a model describing shallow water wave propagation. In [13], the Lax pair of system (1.8) was converted into a Lax pair of the integrable system

$$P_\tau = \rho_y, \quad Q_\tau = \frac{1}{2}\rho P_y + \rho_y P, \quad \rho_{yyy} + 2\rho_y Q + 2(\rho Q)_y = 0, \tag{1.9}$$

by a Liouville-type transformation introduced in [2, 3, 4]. In [13], this system was found to be the first negative flow of the Ablowitz-Kaup-Newell-Segur hierarchy [1] in terms of the spectral structure of its Lax-pair formulation. Although the 2CH and Ito integrable systems are related by tri-Hamiltonian duality, in contrast to the

CH-KdV and mCH-mKdV cases, the Liouville correspondence between these two hierarchies is unexpected because the transformation between the corresponding isospectral problems is not obvious.

It is anticipated that one can establish a Liouville correspondence between the 2CH hierarchy and a second integrable hierarchy involving the integrable system (1.9) as one of flows in the negative direction. Nevertheless, the integrable structures including the recursion operator and Hamiltonians for such a hierarchy are not clear. The required integrability information was also not presented in [13]. In [38], we elucidated this entire integrable hierarchy, which we call the hierarchy associated with system (1.9) or the *associated 2CH (a2CH) hierarchy* for brevity. We show that it has a bi-Hamiltonian structure and establish a Liouville correspondence between the 2CH and a2CH hierarchies. As in the scalar case, verifying the Liouville correspondence relies on analyzing the underlying operators, which have a matrix form in the multi-component case, and a more careful calculation of the nonlinear interplay among the various components is hence required. Furthermore, we find in [38] that the second positive flow of the a2CH integrable hierarchy is closely related to shallow water models studied in [34] and [39].

The Novikov equation (1.6) has the following two-component integrable generalization

$$m_t+3vu_xm+uvm_x=0,\ n_t+3uv_xn+uvn_x=0,\ m=u-u_{xx},\ n=v-v_{xx},\quad (1.10)$$

which was introduced by Geng and Xue [29], and is referred to as the GX system; see [55] and references therein. As a prototypical multi-component integrable system with cubic nonlinearity, the GX system (1.10) supports special multi-peakon dynamics and has recently attracted much attention [49, 50, 51, 55, 56]. In [49], it was shown that there exists a certain Liouville transformation converting the Lax pair of GX system (1.10) into the Lax pair of the integrable system

$$\begin{aligned} Q_\tau &= \frac{3}{2}(q_y+p_y)-(q-p)P, \qquad & p_{yy}+2p_yP+pP_y+pP^2-pQ+1=0,\\ P_\tau &= \frac{3}{2}(q-p), & q_{yy}-2q_yP-qP_y+qP^2-qQ+1=0, \end{aligned}\quad (1.11)$$

where $q=v\,m^{2/3}n^{-1/3}$ and $p=u\,m^{-1/3}n^{2/3}$. The system (1.11) is bi-Hamiltonian, whose bi-Hamiltonian structure is derived in [49]. In [38], the entire *associated GX (aGX)* integrable hierarchy is investigated, in which (1.11) is the first negative flow. The Liouville correspondence between the integrable GX hierarchy generated by (1.10) and the aGX hierarchy is also established.

We conclude this section by outlining the rest of the survey. Section 2 is devoted to the Liouville correspondence for the mCH integrable hierarchy. The Liouville correspondences between the Novikov and SK hierarchies, and between the DP and KK hierarchies are discussed in Section 3. In Section 4, multi-component cases are discussed. Finally, we present the new result for the Liouville correspondence of the dual Schrödinger hierarchy generated by (1.3).

2 Liouville correspondences for the CH and mCH integrable hierarchies

In this section, we present our procedure to establish the Liouville correspondence between the mCH hierarchy and the (defocusing) mKdV hierarchy in details. A novel transformation mapping the mCH equation (1.2) to the CH equation (1.1) in terms of the respective Liouville correspondences between the CH-KdV hierarchies and between the mCH-mKdV hierarchies found in [36] is also addressed.

2.1 Liouville correspondence for mCH integrable hierarchy

Let us begin by presenting the basic CH-KdV case. For the CH equation (1.1), the bi-Hamiltonian structure takes the following form

$$n_t = \mathcal{J}\,\frac{\delta E_2}{\delta n} = \mathcal{L}\,\frac{\delta E_1}{\delta n},$$

where $E_1 = E_1(n)$, $E_2 = E_2(n)$ are Hamiltonian functionals, and the compatible Hamiltonian operators are given by

$$\mathcal{J} = -\left(\partial_x - \partial_x^3\right), \qquad \mathcal{L} = -\left(\partial_x n + n\,\partial_x\right).$$

These are related to the Hamiltonian pair

$$\overline{\mathcal{D}} = \partial_y, \qquad \overline{\mathcal{L}} = \tfrac{1}{4}\partial_y^3 - \tfrac{1}{2}\left(P\partial_y + \partial_y P\right) \tag{2.1}$$

of the KdV equation

$$P_\tau + P_{yyy} - 6\,PP_y = 0. \tag{2.2}$$

It was proved in [44] and [60] that the corresponding Liouville transformation relating their isospectral problems transforms between the CH and KdV hierarchies. The following identities

$$\mathcal{L}^{-1} = -\frac{1}{2\sqrt{n}}\,\overline{\mathcal{D}}^{-1}\frac{1}{n}, \qquad \overline{\mathcal{L}} = \frac{1}{4n}\,\mathcal{J}\,\frac{1}{\sqrt{n}}$$

relating the Hamiltonian operators under the Liouville transformation play an important role in the analysis used in [44] and [60].

The mCH equation (1.2) can be written in bi-Hamiltonian form [67]

$$m_t = \mathcal{J}\,\frac{\delta \mathcal{H}_2}{\delta m} = \mathcal{K}\,\frac{\delta \mathcal{H}_1}{\delta m}, \qquad m = u - u_{xx},$$

where

$$\mathcal{J} = -\left(\partial_x - \partial_x^3\right), \qquad \mathcal{K} = -\partial_x\, m\, \partial_x^{-1}\, m\, \partial_x \tag{2.3}$$

are compatible Hamiltonian operators, while the corresponding Hamiltonian functionals are given by

$$\mathcal{H}_1(m) = \int m\,u\,\mathrm{d}x, \qquad \mathcal{H}_2(m) = \frac{1}{4}\int \Big(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\Big)\mathrm{d}x.$$

In general, for an integrable bi-Hamiltonian equation with two compatible Hamiltonian operators $\mathcal{K}$ and $\mathcal{J}$, Magri's theorem [57, 66] establishes the formal existence of an infinite hierarchy

$$m_t = K_n(m) = \mathcal{J}\,\frac{\delta\mathcal{H}_n}{\delta m} = \mathcal{K}\,\frac{\delta\mathcal{H}_{n-1}}{\delta m}, \qquad n = 1, 2, \ldots, \tag{2.4}$$

of higher-order commuting bi-Hamiltonian systems, based on the higher-order Hamiltonian functionals $\mathcal{H}_n = \mathcal{H}_n(m)$, $n = 0, 1, 2, \ldots$, common to all members of the hierarchy. The members in the hierarchy (2.4) are obtained by applying successively the recursion operator $\mathcal{R} = \mathcal{K}\,\mathcal{J}^{-1}$ to a seed symmetry [65], which in the mCH setting takes the following form:

$$m_t = K_1(m) = -2m_x, \qquad \text{with} \qquad \mathcal{H}_0(m) = \int m\,\mathrm{d}x.$$

The positive flows in the mCH hierarchy (2.4) are hence

$$m_t + \left(\mathcal{K}\mathcal{J}^{-1}\right)^n (2\,m_x) = 0, \qquad n = 0, 1, \ldots. \tag{2.5}$$

Clearly, the mCH equation (1.2) appears in this hierarchy as

$$m_t = K_2(m) = -\left((u^2 - u_x^2)\,m\right)_x = \mathcal{R}K_1(m).$$

Similarly, one obtains an infinite number of higher-order commutative bi-Hamiltonian systems in the negative direction:

$$m_t = K_{-n}(m) = \mathcal{J}\,\frac{\delta\mathcal{H}_{-n}}{\delta m} = \mathcal{K}\,\frac{\delta\mathcal{H}_{-(n+1)}}{\delta m}, \qquad n = 1, 2, \ldots,$$

starting from the Casimir functional $\mathcal{H}_C(m) = \int 1/m\,\mathrm{d}x$ of $\mathcal{K}$. Then, the first equation $m_t = K_{-1}(m)$ in the negative direction of the mCH hierarchy is

$$m_t = \mathcal{J}\,\frac{\delta\mathcal{H}_{-1}}{\delta m} = \mathcal{J}\,\frac{\delta\mathcal{H}_C}{\delta m} = \left(\frac{1}{m^2}\right)_x - \left(\frac{1}{m^2}\right)_{xxx}, \tag{2.6}$$

which is called the Casimir equation in [67]. It was noted in [67] that equation (2.6) has the form of a Lagrange transformation, modulo an appropriate complex transformation, of the mKdV equation. Successively applying $\mathcal{J}\mathcal{K}^{-1}$ produces the hierarchy of negative flows, in which the n-th member is

$$m_t = K_{-n}(m) = -\left(\mathcal{J}\mathcal{K}^{-1}\right)^{n-1}\mathcal{J}\frac{1}{m^2}, \quad n = 1, 2, \ldots. \tag{2.7}$$

As the original soliton equation in the duality relationship with the mCH equation (1.2), the (defocusing) mKdV equation

$$Q_\tau + Q_{yyy} - 6\,Q^2 Q_y = 0 \tag{2.8}$$

also admits a hierarchy consisting of an infinite number of integrable equations in both the positive and negative directions. Each member in the positive direction takes the form

$$Q_\tau = \overline{K}_n(Q) = \overline{\mathcal{J}}\,\frac{\delta \overline{\mathcal{H}}_n}{\delta Q} = \overline{\mathcal{K}}\,\frac{\delta \overline{\mathcal{H}}_{n-1}}{\delta Q}, \qquad n = 1, 2, \ldots, \tag{2.9}$$

where

$$\overline{\mathcal{K}} = -\frac{1}{4}\partial_y^3 + \partial_y Q \partial_y^{-1} Q \partial_y, \qquad \overline{\mathcal{J}} = -\partial_y, \tag{2.10}$$

are the compatible Hamiltonian operators, and $\overline{\mathcal{H}}_n = \overline{\mathcal{H}}_n(Q)$, $n = 0, 1, 2, \ldots$, are the corresponding Hamiltonian functionals. Using the recursion operator $\overline{\mathcal{R}} = \overline{\mathcal{K}}\,\overline{\mathcal{J}}^{-1}$, the positive flows in (2.9) are

$$Q_\tau + \left(\overline{\mathcal{K}}\,\overline{\mathcal{J}}^{-1}\right)^n (4\,Q_y) = 0, \qquad n = 0, 1, \ldots. \tag{2.11}$$

The negative flow in the following form

$$\overline{\mathcal{R}}^n Q_\tau = 0, \qquad n = 1, 2, \ldots,$$

can be rewritten as

$$\partial_y \left(\tfrac{1}{4}\partial_y - Q\partial_y^{-1} Q\right) \left(\overline{\mathcal{K}}\,\overline{\mathcal{J}}^{-1}\right)^{n-1} Q_\tau = 0,$$

due to the forms of the Hamiltonian operators (2.10), and thus, for each $n \geq 1$,

$$\left(\tfrac{1}{4}\partial_y - Q\partial_y^{-1} Q\right) \left(\overline{\mathcal{K}}\,\overline{\mathcal{J}}^{-1}\right)^{n-1} Q_\tau = \overline{C}_{-n}, \tag{2.12}$$

with $\overline{C}_{-n}$ being the corresponding constant of integration.

The zero curvature formulation [68, 72] for the mCH equation (1.2) is

$$\mathbf{\Psi}_x = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\lambda m \\ -\frac{1}{2}\lambda m & \frac{1}{2} \end{pmatrix} \mathbf{\Psi}, \qquad \mathbf{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2.13}$$

and

$$\mathbf{\Psi}_t = \begin{pmatrix} \lambda^{-2} + \frac{1}{2}(u^2 - u_x^2) & -\lambda^{-1}(u - u_x) - \frac{1}{2}\lambda m (u^2 - u_x^2) \\ \lambda^{-1}(u + u_x) + \frac{1}{2}\lambda m(u^2 - u_x^2) & -\lambda^{-2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix} \mathbf{\Psi}.$$

On the other hand, the zero curvature formulation for (2.8) comes from the compatibility condition between

$$\mathbf{\Phi}_y = \begin{pmatrix} -\mathrm{i}\,\mu & \mathrm{i}\,Q \\ -\mathrm{i}\,Q & \mathrm{i}\,\mu \end{pmatrix} \mathbf{\Phi}, \qquad \mathbf{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{2.14}$$

and

$$\mathbf{\Phi}_\tau = \begin{pmatrix} -4\,\mathrm{i}\,\mu^3 - 2\,\mathrm{i}\,\mu Q^2 & 4\mu^2\,\mathrm{i}\,Q + 2\,\mathrm{i}\,Q^3 - 2\mu Q_y - iQ_{yy} \\ -4\mu^2\,\mathrm{i}\,Q - 2\,\mathrm{i}\,Q^3 - 2\mu Q_y + iQ_{yy} & 4\,\mathrm{i}\,\mu^3 + 2\,\mathrm{i}\,\mu Q^2 \end{pmatrix} \mathbf{\Phi}.$$

One can verify that the following Liouville transformation

$$\mathbf{\Phi} = \begin{pmatrix} -1 & \mathrm{i} \\ -\,\mathrm{i} & 1 \end{pmatrix} \mathbf{\Psi}, \qquad y = \int^x m(\xi)\,\mathrm{d}\xi,$$

will convert the isospectral problem (2.13) into the isospectral problem (2.14), with

$$Q = \frac{1}{2m}, \qquad \lambda = -2\mu.$$

We hence introduce the following transformation:

$$y = \int^x m(t,\xi)\,\mathrm{d}\xi, \qquad \tau = t, \qquad Q(\tau, y) = \frac{1}{2\,m(t,x)}, \tag{2.15}$$

and investigate how it affects the underlying correspondence between the flows of the mCH and (defocusing) mKdV hierarchies.

Hereafter, for a non-negative integer n, we denote the n-th equation in the positive and negative directions of the mCH hierarchy by $(\mathrm{mCH})_n$ and $(\mathrm{mCH})_{-n}$, respectively, while the n-th positive and negative flows in the (defocusing) mKdV hierarchy are denoted by $(\mathrm{mKdV})_n$ and $(\mathrm{mKdV})_{-n}$, respectively. Applying the Lemmas 3.2 and 3.3 in [36], we are able to establish a Liouville correspondence between the (defocusing) mKdV and mCH hierarchies.

Theorem 1. *Under the transformation* (2.15)*, for each* $n \in \mathbb{Z}$*, the* $(\mathrm{mCH})_{n+1}$ *equation is related to the* $(\mathrm{mKdV})_{-n}$ *equation. More precisely, for each integer* $n \geq 0$*,*

(i). *m solves equation* (2.5) *if and only if Q satisfies $Q_\tau = 0$ for $n = 0$ or* (2.12) *for $n \geq 1$, with $\overline{C}_{-n} = 1/(-4)^n$;*

(ii). *For $n \geq 1$, the function m is a solution of the following rescaled version of* (2.7)*:*

$$m_t = K_{-n}(m) = \frac{(-1)^{n+1}}{2^{2n-1}} \left(\mathcal{J}\mathcal{K}^{-1}\right)^{n-1} \mathcal{J}\,\frac{1}{m^2}, \qquad n = 1, 2, \ldots,$$

if and only if Q satisfies equation (2.11)*. In addition, for $n = 0$, the corresponding equation $m_t = 0$ is equivalent to $Q_\tau + 4Q_y = 0$.*

Magri's bi-Hamiltonian scheme enables one to recursively construct an infinite hierarchy of Hamiltonian functionals of the mCH equation (1.2). The Hamiltonian functionals $\mathcal{H}_n = \mathcal{H}_n(m)$ satisfy the recursive formula

$$\mathcal{J}\,\frac{\delta \mathcal{H}_n}{\delta m} = \mathcal{K}\,\frac{\delta \mathcal{H}_{n-1}}{\delta m}, \qquad n \in \mathbb{Z},$$

where $\mathcal{K}$ and $\mathcal{J}$ are the two compatible Hamiltonian operators (2.3) admitted by the mCH equation. On the other hand, the recursive formula

$$\overline{\mathcal{J}}\,\frac{\delta\overline{\mathcal{H}}_n}{\delta Q}=\overline{\mathcal{K}}\,\frac{\delta\overline{\mathcal{H}}_{n-1}}{\delta Q},\qquad n\in\mathbb{Z}$$

can be used to obtain Hamiltonian functionals of the (defocusing) mKdV equation (2.8). Applying the Lemmas 4.1 and 4.2 in [36], we proved the following relation between the sequences of the Hamiltonian functionals admitted by the mCH and the (defocusing) mKdV equations.

Theorem 2. *For any non-zero integer n, each Hamiltonian functionals $\overline{\mathcal{H}}_n(Q)$ of the (defocusing) mKdV equation yields the Hamiltonian functionals $\mathcal{H}_{-n}(m)$ of the mCH equation, under the Liouville transformation* (2.15)*, according to the following identity*

$$\mathcal{H}_{-n}(m)=(-1)^n 2^{2n-1}\,\overline{\mathcal{H}}_n(Q),\qquad 0\neq n\in\mathbb{Z}.$$

2.2 Relationship between the mCH and CH equations

It is wellknown that the KdV equation and the mKdV equation are linked by the Miura transformation. This leads to a question whether there exists a transformation relating their respective dual counterparts, in other words, a transformation between the CH equation (1.1) and the mCH equation (1.2). From the viewpoint of tri-Hamiltonian duality, the CH equation (1.1) is regarded as the dual integrable counterpart of the KdV equation (2.2). The KdV equation (2.2) is related to the (defocusing) mKdV equation (2.8) via the Miura transformation

$$\mathcal{B}(P,Q)\equiv P-Q^2+Q_y=0. \tag{2.16}$$

Furthermore, Fokas and Fuchssteiner [22] proved that all the positive members of the KdV hierarchy admit the same Miura transformation. In addition, we have the following result.

Proposition 1. *Assume that Q satisfies the equation*

$$\left(\overline{\mathcal{K}}\,\overline{\mathcal{J}}^{-1}\right)Q_\tau=0, \tag{2.17}$$

where $\overline{\mathcal{K}}$ and $\overline{\mathcal{J}}$ are given in (2.10)*. Then $P=Q^2-Q_y$ satisfies*

$$\left(\overline{\mathcal{L}}\,\overline{\mathcal{D}}^{-1}\right)P_\tau=0, \tag{2.18}$$

where $\overline{\mathcal{L}}$ and $\overline{\mathcal{D}}$ are defined by (2.1)*.*

Using Proposition 1, we are able to construct a transformation from the mCH equation (1.2) to the CH equation (1.1). First, it was shown in [26, 44, 60] that the following Liouville transformation

$$P(\tau,y)=\frac{1}{n(t,x)}\Big(\frac{1}{4}-\frac{(n(t,x)^{-1/4})_{xx}}{n(t,x)^{-1/4}}\Big),\ y=\int^{x}\sqrt{n}\mathrm{d}\xi,\ n=v-v_{xx},\ \tau=t, \tag{2.19}$$

relating the respective isospectral problems for the CH hierarchy and the KdV hierarchy, gives rise to the one-to-one correspondence between the CH equation (1.1) and the first negative flow (2.18). On the other hand, from Theorem 1, $m(t,x)$ satisfies the mCH equation (1.2) if and only if

$$Q(\tau,y)=\frac{1}{2m(t,x)},\qquad y=\int^{x}m(t,\xi)\,\mathrm{d}\xi,\qquad \tau=t, \tag{2.20}$$

is the solution of equation (2.17). We deduce that the composite transformation including (2.16), (2.19), and (2.20) defines a map from the mCH equation (1.2) to the CH equation (1.1), albeit not one-to-one.

Proposition 2. *Assume $m(t,x)$ is the solution of the mCH equation* (1.2)*. Then, $n(t,x)$ satisfies the CH equation* (1.1)*, where $n(t,x)$ is determined by the relation* (2.19) *with $P(\tau,y)=Q^2(\tau,y)-Q_y(\tau,y)$ and $Q(\tau,y)$ defined by* (2.20)*.*

3 Liouville correspondences for the Novikov and DP integrable hierarchies

In this section, we first survey the main results on Liouville correspondences for the Novikov and Degasperis-Procesi (DP) hierarchies, and then show the implicit relationship which associates the Novikov and DP equations.

3.1 Liouville correspondences for the Novikov and DP hierarchies

The Novikov equation (1.6) can be expressed in bi-Hamiltonian form [32]

$$m_t=K_1(m)=\mathcal{J}\,\frac{\delta\mathcal{H}_1}{\delta m}=\mathcal{K}\,\frac{\delta\mathcal{H}_0}{\delta m},\qquad m=u-u_{xx},$$

where

$$\mathcal{K}=\frac{1}{2}m^{\frac{1}{3}}\,\partial_x\,m^{\frac{2}{3}}\,(4\partial_x-\partial_x^3)^{-1}\,m^{\frac{2}{3}}\,\partial_x\,m^{\frac{1}{3}},\quad \mathcal{J}=(1-\partial_x^2)\,m^{-1}\,\partial_x\,m^{-1}\,(1-\partial_x^2)$$

are the compatible Hamiltonian operators. The corresponding Hamiltonian functionals are given by

$$\mathcal{H}_0(m)=9\int\left(u^2+u_x^2\right)\,\mathrm{d}x,\quad \mathcal{H}_1(m)=\frac{1}{6}\int um\partial_x^{-1}m(1-\partial_x^2)^{-1}(u^2m_x+3uu_xm)\,\mathrm{d}x.$$

As for the Novikov integrable hierarchy

$$m_t = K_n(m) = \mathcal{J}\,\frac{\delta \mathcal{H}_n}{\delta m} = \mathcal{K}\,\frac{\delta \mathcal{H}_{n-1}}{\delta m}, \qquad n \in \mathbb{Z}, \tag{3.1}$$

the Novikov equation (1.6) serves as the first member in the positive direction of (3.1). While, in the opposite direction, note that

$$K_0(m) = \mathcal{J}\,\frac{\delta \mathcal{H}_0}{\delta m} = 0.$$

The first negative flow is the Casimir equation

$$m_t = K_{-1}(m) = 3\,\mathcal{J}\,m^{-\frac{1}{3}}.$$

In addition, the Lax pair for the Novikov equation (1.6) consists of [32]

$$\boldsymbol{\Psi}_x = \begin{pmatrix} 0 & \lambda m & 1 \\ 0 & 0 & \lambda m \\ 1 & 0 & 0 \end{pmatrix} \boldsymbol{\Psi}, \qquad \boldsymbol{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{3.2}$$

and

$$\boldsymbol{\Psi}_t = \begin{pmatrix} \frac{1}{3}\lambda^{-2} - u u_x & \lambda^{-1} u_x - \lambda u^2 m & u_x^2 \\ \lambda^{-1} u & -\frac{2}{3}\lambda^{-2} & -\lambda^{-1} u_x - \lambda u^2 m \\ -u^2 & \lambda^{-1} u & \frac{1}{3}\lambda^{-2} + u u_x \end{pmatrix} \boldsymbol{\Psi}.$$

It was proved in [32] that by the Liouville transformation

$$y = \int^x m^{\frac{2}{3}}(t,\,\xi)\,\mathrm{d}\xi, \quad \tau = t, \quad Q = \frac{4}{9}\,m^{-\frac{10}{3}} m_x^2 - \frac{1}{3}\,m^{-\frac{7}{3}} m_{xx} - m^{-\frac{4}{3}}, \tag{3.3}$$

the isospectral problem (3.2) is transformed into

$$\Phi_{yyy} + Q\,\Phi_y = \mu\,\Phi, \tag{3.4}$$

with $\Phi = \psi_2$ and $\mu = \lambda^2$. The linear system (3.4) is a third-order spectral problem for the SK equation (1.7), which together with the corresponding time evolution of $\Phi = \psi_2$ yields

$$Q_\tau = W_y, \qquad W_{yy} + Q\,W = T, \qquad T_y = 0. \tag{3.5}$$

As noted in [32], the system (3.5) is the first negative flow of the SK hierarchy. Based on the Liouville transformation between the isospectral problems of Novikov equation and the first negative flow of the SK hierarchy, we are inspired to establish the Liouville correspondence for their entire hierarchies.

As far as the SK equation (1.7) is concerned, it exhibits a generalized bi-Hamiltonian structure, whose corresponding integrable hierarchy is generated by a recursion operator $\overline{\mathcal{R}} = \overline{\mathcal{K}}\,\overline{\mathcal{J}}$, with an implectic (Hamiltonian) operator

$$\overline{\mathcal{K}} = -\left(\partial_y^3 + 2\,Q\,\partial_y + 2\partial_y\,Q\right)$$

and a symplectic operator

$$\overline{\mathcal{J}} = 2\partial_y^3 + 2\partial_y^2\,Q\,\partial_y^{-1} + 2\partial_y^{-1}\,Q\,\partial_y^2 + Q^2\,\partial_y^{-1} + \partial_y^{-1}\,Q^2.$$

More precisely, the SK equation (1.7) can be written as

$$Q_\tau = \overline{K}_1(Q) = \overline{\mathcal{K}}\,\frac{\delta\overline{\mathcal{H}}_0}{\delta Q},$$

where the Hamiltonian functional is

$$\overline{\mathcal{H}}_0(Q) = \frac{1}{6}\int\left(Q^3 - 3Q_y^2\right)\,\mathrm{d}y.$$

In conclusion, the positive flows of the SK hierarchy are given by

$$Q_\tau = \overline{K}_n(Q) = \left(\overline{\mathcal{K}}\,\overline{\mathcal{J}}\right)^{n-1}\overline{K}_1, \qquad n = 1, 2, \ldots.$$

On the other hand, in the negative direction, in view of the fact that the trivial function $f \equiv 0$ satisfies the equation

$$\overline{\mathcal{J}}\cdot f = \frac{\delta\overline{\mathcal{H}}_0}{\delta Q},$$

as proposed in [37], the n-th negative flow has the form

$$\overline{\mathcal{R}}^n Q_\tau = 0, \qquad n = 1, 2, \ldots.$$

The following theorem is taken from [37] to illustrate the underlying one-to-one correspondence between the flows in the Novikov and SK hierarchies. In this section, for a positive integer n, the n-th equation in the positive and negative directions of the Novikov hierarchy are denoted by $(\mathrm{Nov})_n$ and $(\mathrm{Nov})_{-n}$, respectively, while the n-th positive and negative flows of the SK hierarchy are denoted by $(\mathrm{SK})_n$ and $(\mathrm{SK})_{-n}$, respectively. Based on Lemmas 2.3 and 2.4 in [37], we are able to establish the following result.

Theorem 3. *Under the Liouville transformation* (3.3)*, for each positive integer $n \in \mathbb{Z}^+$, the n-th positive flow* $(\mathrm{Nov})_n$ *and negative flow* $(\mathrm{Nov})_{-n}$ *of the Novikov hierarchy are mapped into the n-th negative flow* $(\mathrm{SK})_{-n}$ *and positive flow* $(\mathrm{SK})_n$*, respectively.*

In addition, in the Novikov-SK setting, in order to establish the explicit relationship between the flows in the positive Novikov hierarchy and the flows in the negative SK hierarchy, the following factorization of the recursion operator $\overline{\mathcal{R}} = \overline{\mathcal{K}}\,\overline{\mathcal{J}}$ of the SK equation is necessary to identify the equations transformed from the positive flows in the Novikov hierarchy as the corresponding negative flows in the SK hierarchy exactly. The factorization is based on the following operator identity [14]:

$$\overline{\mathcal{R}} = -2\left(\partial_y^4 + 5Q\partial_y^2 + 4Q_y\partial_y + Q_{yy} + 4Q^2 + 2Q_y\partial_y^{-1}Q\right)\left(\partial_y^2 + Q + Q_y\partial_y^{-1}\right).$$

Based on the Liouville correspondence, as shown in [37], there also exists a one-to-one correspondence between the sequences of the Hamiltonian functionals $\{\mathcal{H}_n(m)\}$ of the Novikov equation and $\{\overline{\mathcal{H}}_n(Q)\}$ of the SK equation. In particular, with their Hamiltonian pairs $\mathcal{K}$, $\mathcal{J}$ and $\overline{\mathcal{K}}$, $\overline{\mathcal{J}}$ in hand, the corresponding Hamiltonian functionals $\{\mathcal{H}_n(m)\}$ and $\{\overline{\mathcal{H}}_n(Q)\}$ are determined by the following two recursive formulae:

$$\mathcal{J}\,\frac{\delta\mathcal{H}_n}{\delta m} = \mathcal{K}\,\frac{\delta\mathcal{H}_{n-1}}{\delta m}, \qquad \overline{\mathcal{J}}\,\overline{\mathcal{K}}\,\frac{\delta\overline{\mathcal{H}}_{n-1}}{\delta Q} = \frac{\delta\overline{\mathcal{H}}_n}{\delta Q}, \qquad n\in\mathbb{Z},$$

respectively. Indeed, in [37], the relationship between the two hierarchies and the effect of the Liouville transformation on the variational derivatives were investigated. Applying Lemmas 2.5 and 2.6 in [37], we can prove the following theorem.

Theorem 4. *Under the Liouville transformation* (3.3)*, for each $n\in\mathbb{Z}$, the Hamiltonian functionals $\overline{\mathcal{H}}_n(Q)$ of the SK equation are related to the Hamiltonian functionals $\mathcal{H}_{-n}(m)$ of the Novikov equation, according to the following identity*

$$\mathcal{H}_n(m) = 18\,\overline{\mathcal{H}}_{-(n+2)}(Q), \qquad n\in\mathbb{Z}.$$

In analogy with the Liouville correspondence between the Novikov and SK hierarchies, there exists a similar correspondence between the DP and KK hierarchies, as well as their respective hierarchies of the Hamiltonian functionals. The main results are presented in the following two theorems. We refer the interested reader to [37] for further details.

Theorem 5. *Under the Liouville transformation*

$$y = \int^x n^{\frac{1}{3}}(t,\,\xi)\,\mathrm{d}\xi, \quad \tau = t, \quad P = \frac{1}{4}\left(\frac{7}{9}n^{-\frac{8}{3}}n_x^2 - \frac{2}{3}n^{-\frac{5}{3}}n_{xx} - n^{-\frac{2}{3}}\right), \tag{3.6}$$

for each positive integer $l\in\mathbb{Z}^+$, the l-th positive flow $(\mathrm{DP})_l$ *and negative flow* $(\mathrm{DP})_{-l}$ *of the DP hierarchy are mapped into the l-th negative flow* $(\mathrm{KK})_{-l}$ *and positive flow* $(\mathrm{KK})_l$ *of the KK hierarchy, respectively.*

Theorem 6. *Under the Liouville transformations* (3.6)*, for each $l\in\mathbb{Z}$, the Hamiltonian functional $\overline{\mathcal{E}}_l(P)$ of the KK equation* (1.5) *is related to that $\mathcal{E}_l(n)$ of the DP equation* (1.4)*, according to the following identity*

$$\mathcal{E}_l(n) = 36\,\overline{\mathcal{E}}_{-(l+2)}(P), \qquad l\in\mathbb{Z}.$$

3.2 Relationship between the Novikov and DP equations

As proposed in [37], a further significant application of the Liouville correspondence between the Novikov-SK and DP-KK hierarchies is to establish the relationship between the Novikov and DP equations, which is motivated by the following issues. Firstly, it has been shown in [25] that under the Miura transformations

$$Q - V_y + V^2 = 0, \qquad P + V_y + \frac{1}{2}V^2 = 0, \tag{3.7}$$

the SK equation (1.7) and the KK equation (1.5) are respectively transformed into the *Fordy-Gibbons-Jimbo-Miwa equation*

$$V_\tau + V_{yyyyy} - 5\big(V_y V_{yyy} + V_{yy}^2 + V_y^3 + 4V V_y V_{yy} + V^2 V_{yyy} - V^4 V_y\big) = 0. \tag{3.8}$$

In [37], this relationship was generalized to first negative flows of the SK and KK hierarchies. Indeed, in view of (3.7), the recursion operator $\overline{\mathcal{R}}$ of the SK equation and the recursion operator $\mathcal{R}^*$ of equation (3.8) satisfy

$$\mathcal{R}^* = T_1 \overline{\mathcal{R}} T_1^{-1}, \qquad \text{with} \qquad T_1 = (2V - \partial_y)^{-1}.$$

Similarly, the recursion operator $\widehat{\mathcal{R}}$ of the KK equation is linked with the recursion operator $\mathcal{R}^*$ according to the identity

$$\mathcal{R}^* = T_2 \widehat{\mathcal{R}} T_2^{-1}, \qquad \text{with} \qquad T_2 = (V + \partial_y)^{-1}.$$

Based on this, one has the following result.

Lemma 1. *Assume that V satisfies the equation*

$$\mathcal{R}^* V_\tau = 0. \tag{3.9}$$

Then $Q = V_y - V^2$ and $P = -V_y - V^2/2$ satisfy the first negative flow of the SK hierarchy $\overline{\mathcal{R}}\, Q_\tau = 0$ and the first negative flow of the KK hierarchy $\widehat{\mathcal{R}}\, P_\tau = 0$, respectively.

Finally, using Lemma 1, combined with the Liouville correspondences between the Novikov-SK and DP-KK hierarchies, we establish the relationship between the Novikov equation and the DP equation, which is summarized in the following theorem.

Theorem 7. *Both the Novikov equation* (1.6) *and the DP equation* (1.4) *are linked with equation* (3.9) *in the following sense. If $V(\tau, y)$ is a solution of equation* (3.9), *then the function $m(t, x)$ determined implicitly by the relation*

$$V_y - V^2 = -m^{-1}(1 - \partial_x^2)\, m^{-\frac{1}{3}}, \qquad y = \int^x m^{\frac{2}{3}}(t, \xi)\, \mathrm{d}\xi, \qquad \tau = t,$$

satisfies the Novikov equation (1.6), *while the function* $n(t,x)$ *determined by*

$$V_y + \frac{1}{2}V^2 = \frac{1}{4}n^{-\frac{1}{2}}\left(1 - 4\partial_x^2\right) n^{-\frac{1}{6}}, \qquad y = \int^x n^{\frac{1}{3}}(t,\,\xi)\,\mathrm{d}\xi, \qquad \tau = t,$$

satisfies the DP equation (1.4).

4 Liouville correspondences for multi-component integrable hierarchies

In this section, we shall survey the main results concerning Liouville correspondences for the 2CH and GX hierarchies.

4.1 Liouville correspondence for 2CH hierarchy

The 2CH system (1.8) is a bi-Hamiltonian integrable system [67]

$$\begin{pmatrix} m \\ \rho \end{pmatrix}_t = \mathcal{J}\,\delta\mathcal{H}_2(m,\,\rho) = \mathcal{K}\,\delta\mathcal{H}_1(m,\,\rho), \quad \delta\mathcal{H}_n(m,\rho) = \left(\frac{\delta\mathcal{H}_n}{\delta m},\,\frac{\delta\mathcal{H}_n}{\delta\rho}\right)^T,\ n = 1,\,2,$$

with compatible Hamiltonian operators

$$\mathcal{K} = \begin{pmatrix} m\partial_x + \partial_x m & \rho\partial_x \\ \partial_x\rho & 0 \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} \partial_x - \partial_x^3 & 0 \\ 0 & \partial_x \end{pmatrix}, \tag{4.1}$$

and the associated Hamiltonian functionals

$$\mathcal{H}_1(m,\,\rho) = -\frac{1}{2}\int (u^2 + u_x^2 + \rho^2)\,\mathrm{d}x, \qquad \mathcal{H}_2(m,\,\rho) = -\frac{1}{2}\int u(u^2 + u_x^2 + \rho^2)\,\mathrm{d}x.$$

The Hamiltonian pair (4.1) induces the hierarchy

$$\begin{pmatrix} m \\ \rho \end{pmatrix}_t = \mathbf{K}_n = \mathcal{J}\delta\mathcal{H}_n = \mathcal{K}\delta\mathcal{H}_{n-1}, \quad \delta\mathcal{H}_n = \left(\frac{\delta\mathcal{H}_n}{\delta m},\,\frac{\delta\mathcal{H}_n}{\delta\rho}\right)^T, \quad n \in \mathbb{Z}, \tag{4.2}$$

of commutative bi-Hamiltonian systems, based on the corresponding Hamiltonian functionals $\mathcal{H}_n = \mathcal{H}_n(m,\rho)$. The positive flows of (4.2) begin with the seed system

$$\begin{pmatrix} m \\ \rho \end{pmatrix}_t = \mathbf{K}_1 = -\begin{pmatrix} m \\ \rho \end{pmatrix}_x,$$

and the 2CH system (1.8) is the second member. On the other hand, the negative flows start from the *Casimir system*

$$\begin{pmatrix} m \\ \rho \end{pmatrix}_t = \mathbf{K}_{-1} = \mathcal{J}\delta\mathcal{H}_C,$$

with the associated Casimir functional $\mathcal{H}_C(m,\rho) = \int m/\rho \,\mathrm{d}x$ for the Hamiltonian operator $\mathcal{K}$. The first negative flow for the hierarchy (4.2) has the explicit form

$$m_t = (\partial_x - \partial_x^3)\left(\frac{1}{\rho}\right), \qquad \rho_t = -\left(\frac{m}{\rho^2}\right)_x, \qquad m = u - u_{xx},$$

which, together with the inverse recursion operator $\mathcal{R}^{-1} = \mathcal{J}\,\mathcal{K}^{-1}$ produces the members in the negative direction of (4.2), namely

$$\begin{pmatrix} m \\ \rho \end{pmatrix}_t = \mathbf{K}_{-n} = (\mathcal{J}\,\mathcal{K}^{-1})^{n-1}\,\mathcal{J}\begin{pmatrix} \rho^{-1} \\ -m\rho^{-2} \end{pmatrix}, \qquad n = 1,\,2,\ldots.$$

In [38], we established the Liouville correspondence between the 2CH hierarchy and another integrable hierarchy, called the associated 2CH (a2CH) hierarchy. The transformation relating these hierarchies is motivated by the Liouville transformation [13]

$$\begin{gathered} \tau = t, \qquad y = \int^x \rho(t,\xi)\,\mathrm{d}\xi, \qquad P(\tau,y) = -m(t,x)\,\rho(t,x)^{-2}, \\ Q(\tau,y) = -\frac{1}{4}\rho(t,x)^{-2} + \frac{3}{4}\rho(t,x)^{-4}\rho_x^2(t,x) - \frac{1}{2}\rho(t,x)^{-3}\rho_{xx}(t,x), \end{gathered} \tag{4.3}$$

which converts the isospectral problem

$$\Psi_{xx} + \left(-\frac{1}{4} - \lambda m + \lambda^2\rho^2\right)\Psi = 0, \quad \Psi_t = \left(\frac{1}{2\lambda} - u\right)\Psi_x + \frac{u_x}{2}\Psi$$

of the 2CH system into

$$\Phi_{yy} + (Q + \lambda P + \lambda^2)\Phi = 0, \qquad \Phi_\tau - \frac{1}{2\lambda}\rho\,\Phi_y + \frac{1}{4\lambda}\rho_y\Phi = 0, \tag{4.4}$$

with $\Phi = \sqrt{\rho}\,\Psi$.

We clarified in [38] some integrability properties of the a2CH hierarchy. This hierarchy is generated by the recursion operator

$$\overline{\mathcal{R}} = \frac{1}{2}\begin{pmatrix} 0 & \partial_y^2 + 4Q + 2Q_y\partial_y^{-1} \\ -4 & 4P + 2P_y\partial_y^{-1} \end{pmatrix},$$

and the corresponding positive flows and the negative flows are given by

$$\begin{pmatrix} Q \\ P \end{pmatrix}_\tau = \overline{\mathbf{K}}_n = \overline{\mathcal{R}}^{n-1}\,\overline{\mathbf{K}}_1, \qquad \overline{\mathcal{R}}^n\begin{pmatrix} Q \\ P \end{pmatrix}_\tau = \overline{\mathbf{K}}_0, \qquad n = 1,\,2,\ldots,$$

respectively, where $\overline{\mathbf{K}}_1 = (-Q_y,\, -P_y)^T$ is the usual seed symmetry and the trivial symmetry $\overline{\mathbf{K}}_0 = (0,\,0)^T$ is determined by $\overline{\mathcal{R}}\,\overline{\mathbf{K}}_0 = \overline{\mathbf{K}}_1$. Furthermore, the a2CH hierarchy can be written in bi-Hamiltonian form

$$\begin{pmatrix} Q \\ P \end{pmatrix}_\tau = \overline{\mathbf{K}}_n = \overline{\mathcal{K}}\delta\overline{\mathcal{H}}_{n-1} = \overline{\mathcal{J}}\delta\overline{\mathcal{H}}_n, \quad \delta\overline{\mathcal{H}}_n = \left(\frac{\delta\overline{\mathcal{H}}_n}{\delta Q},\, \frac{\delta\overline{\mathcal{H}}_n}{\delta P}\right)^T, \quad n \in \mathbb{Z}, \tag{4.5}$$

using the compatible Hamiltonian operators

$$\overline{\mathcal{K}} = \frac{1}{4}\begin{pmatrix} \mathcal{L}\,\partial_y^{-1}\mathcal{L} & 2\mathcal{L}\,\partial_y^{-1}(P\partial_y + \partial_y P) \\ 2(P\partial_y + \partial_y P)\partial_y^{-1}\mathcal{L} & 4(P\partial_y + \partial_y P)\partial_y^{-1}(P\partial_y + \partial_y P) + 2\mathcal{L} \end{pmatrix},$$

$$\overline{\mathcal{J}} = \frac{1}{2}\begin{pmatrix} 0 & \mathcal{L} \\ \mathcal{L} & 2(P\partial_y + \partial_y P) \end{pmatrix}, \qquad \mathcal{L} = \partial_y^3 + 2Q\partial_y + 2\partial_y Q.$$

In particular, as noted in [38], the second positive flow of the 2CH hierarchy takes the explicit form

$$Q_\tau = -\frac{1}{2}P_{yyy} - 2QP_y - Q_yP, \qquad P_\tau = 2Q_y - 3PP_y, \tag{4.6}$$

which can be written as the bi-Hamiltonian form (4.5) with

$$\overline{\mathcal{H}}_1 = -\int P\,\mathrm{d}y, \qquad \overline{\mathcal{H}}_2 = -\int \Big(\frac{1}{2}P^2 + 2Q\Big)\,\mathrm{d}y.$$

Moreover, the system (4.6) can be obtained from the y-component of the Lax-pair formulation (4.4) together with

$$\Phi_\tau + (2\lambda + P)\Phi_y - \frac{1}{2}P_y\Phi = 0.$$

Based on this, system (4.6) is shown to be equivalent to the Kaup-Boussinesq system [39].

The scheme of the Liouville correspondence between the 2CH and a2CH hierarchies as proposed in [38] is as follows, where, for a positive integer n, the $(\mathrm{2CH})_n$, $(\mathrm{2CH})_{-n}$ and $(\mathrm{a2CH})_n$, $(\mathrm{a2CH})_{-n}$ denote the n-th positive and negative flows of 2CH hierarchy and the a2CH hierarchy, respectively.

Theorem 8. *Under the Liouville transformation* (4.3)*, for each integer n, the* $(\mathrm{2CH})_{n+1}$ *equation is mapped into the* $(\mathrm{a2CH})_{-n}$ *equation.*

Furthermore, as a consequence of the Liouville transformation, we are led to the one-to-one correspondence between the Hamiltonian functionals in the 2CH and a2CH hierarchies.

Theorem 9. *Under the Liouville transformation* (4.3)*, for each nonzero integer n, the Hamiltonian functionals $\mathcal{H}_n(m,\rho)$ of the 2CH hierarchy are related to the Hamiltonian functionals $\overline{\mathcal{H}}_n(Q,P)$ of the a2CH hierarchy, according to*

$$\mathcal{H}_n(m,\rho) = \overline{\mathcal{H}}_{-n}(Q,P), \qquad 0 \neq n \in \mathbb{Z}.$$

4.2 Liouville correspondence for the Geng-Xue hierarchy

The GX system (1.10) can be written in a bi-Hamiltonian form [50]

$$\begin{pmatrix} m \\ n \end{pmatrix}_t = \mathcal{K}\delta\mathcal{H}_1(m,\, n) = \mathcal{J}\delta\mathcal{H}_2(m,\, n),$$

where the compatible Hamiltonian operators are

$$\mathcal{K} = \frac{3}{2}\begin{pmatrix} 3m^{\frac13}\partial_x m^{\frac23}\Omega^{-1}m^{\frac23}\partial_x m^{\frac13} + m\partial_x^{-1}m & 3m^{\frac13}\partial_x m^{\frac23}\Omega^{-1}n^{\frac23}\partial_x n^{\frac13} - m\partial_x^{-1}n \\ 3n^{\frac13}\partial_x n^{\frac23}\Omega^{-1}m^{\frac23}\partial_x m^{\frac13} - n\partial_x^{-1}m & 3n^{\frac13}\partial_x n^{\frac23}\Omega^{-1}n^{\frac23}\partial_x n^{\frac13} + 3n\partial_x^{-1}n \end{pmatrix},$$

$$\mathcal{J} = \begin{pmatrix} 0 & \partial_x^2 - 1 \\ 1 - \partial_x^2 & 0 \end{pmatrix}, \qquad \Omega = \partial_x^3 - 4\,\partial_x,$$

while

$$\mathcal{H}_1(m,\, n) = \int un\, \mathrm{d}x, \qquad \mathcal{H}_2(m,\, n) = \int (u_x v - u v_x)un\, \mathrm{d}x$$

are the initial Hamiltonian functionals. The GX integrable hierarchy can be obtained by applying the resulting hereditary recursion operator $\mathcal{R} = \mathcal{K}\,\mathcal{J}^{-1}$ to the particular seed system

$$\begin{pmatrix} m \\ n \end{pmatrix}_t = \mathbf{G}_1(m,n) = \begin{pmatrix} -m \\ n \end{pmatrix}.$$

Hence, the l-th member in the positive direction of the GX hierarchy takes the form

$$\begin{pmatrix} m \\ n \end{pmatrix}_t = \mathbf{G}_l(m,n) = \mathcal{R}^{l-1}\,\mathbf{G}_1(m,n), \quad l = 1,2,\ldots, \tag{4.7}$$

and the GX system (1.10) is exactly the second positive flow. While, the l-th negative flow of the GX hierarchy is

$$\begin{pmatrix} m \\ n \end{pmatrix}_t = \mathbf{G}_{-l}(m,n) = (\mathcal{J}\,\mathcal{K}^{-1})^{l-1}\,\mathcal{J}\delta\mathcal{H}_C, \qquad l = 1,\, 2, \ldots, \tag{4.8}$$

where

$$\mathcal{H}_C(m,\, n) = 3\int \Delta^{\frac13}\, \mathrm{d}x, \quad \text{with} \quad \delta\mathcal{H}_C = \left(m^{-\frac23}n^{\frac13},\, m^{\frac13}n^{-\frac23}\right)^T,$$

is the Casimir functional for the Hamiltonian operator $\mathcal{K}$, and we set $\Delta = mn$ throughout this subsection.

The Lax-pair formulation for the GX system (1.10) takes the form [29]

$$\begin{aligned} \mathbf{\Psi}_x &= \begin{pmatrix} 0 & \lambda m & 1 \\ 0 & 0 & \lambda n \\ 1 & 0 & 0 \end{pmatrix}\mathbf{\Psi}, \qquad \mathbf{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \\ \mathbf{\Psi}_t &= \begin{pmatrix} -u_x v & \lambda^{-1}u_x - \lambda uvm & u_x v_x \\ \lambda^{-1}v & -\lambda^{-2} + u_x v - u v_x & -\lambda uvn - \lambda^{-1}v_x \\ -uv & \lambda^{-1}u & uv_x \end{pmatrix}\mathbf{\Psi}. \end{aligned} \tag{4.9}$$

As is shown in [49], by the Liouville transformation

$$y = \int^x \Delta^{\frac{1}{3}} \mathrm{d}\xi,\ \tau = t,\ Q = \frac{1}{\Delta^{\frac{2}{3}}} + \frac{1}{6}\frac{\Delta_{xx}}{\Delta^{\frac{5}{3}}} - \frac{7}{36}\frac{\Delta_x^2}{\Delta^{\frac{8}{3}}},\ P = \frac{1}{2}\frac{n^{\frac{2}{3}}}{m^{\frac{4}{3}}}\left(\frac{m}{n}\right)_x, \tag{4.10}$$

the isospectral problem (4.9) is converted into

$$\mathbf{\Phi}_y = \begin{pmatrix} 0 & \lambda & Q \\ 0 & P & \lambda \\ 1 & 0 & 0 \end{pmatrix} \mathbf{\Phi}, \qquad \mathbf{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},$$
$$\mathbf{\Phi}_\tau = \frac{1}{2}\begin{pmatrix} A & 2\lambda^{-1}(p_y + p\,P) & p+q \\ 2\lambda^{-1}q & A - 2\lambda^{-2} & 2\lambda^{-1}(Pq - q_y) \\ 0 & 2\lambda^{-1}p & A \end{pmatrix} \mathbf{\Phi},$$

where

$$A = q_y p - q p_y - 2pqP, \qquad q = v m^{\frac{2}{3}} n^{-\frac{1}{3}}, \qquad p = u m^{-\frac{1}{3}} n^{\frac{2}{3}}.$$

The compatibility condition $\mathbf{\Phi}_{y\tau} = \mathbf{\Phi}_{\tau y}$ gives rise to the following integrable system

$$\begin{aligned} Q_\tau &= \frac{3}{2}(q_y + p_y) - (q-p)P, \qquad & p_{yy} + 2p_y P + pP_y + pP^2 - pQ + 1 = 0, \\ P_\tau &= \frac{3}{2}(q-p), & q_{yy} - 2q_y P - qP_y + qP^2 - qQ + 1 = 0, \end{aligned}$$

which can be viewed as a negative flow of an integrable hierarchy, namely the associated Geng-Xue (aGX) integrable hierarchy. In addition, the Hamiltonian pair admitted by the aGX hierarchy are

$$\overline{\mathcal{K}} = \mathbf{\Gamma}\begin{pmatrix} 0 & \Theta \\ -\Theta^* & 0 \end{pmatrix}\mathbf{\Gamma}^* \quad \text{and} \quad \overline{\mathcal{J}} = \frac{1}{2}\begin{pmatrix} \mathcal{E} & 0 \\ 0 & -3\partial_y \end{pmatrix}, \tag{4.11}$$

where the matrix operator $\mathbf{\Gamma}$, and operators Θ, $\mathcal{E}$ are defined by

$$\mathbf{\Gamma} = -\frac{1}{6}\begin{pmatrix} \mathcal{E}\partial_y^{-1} & \mathcal{E}\partial_y^{-1} \\ (3\partial_y^2 - 2\partial_y P)\partial_y^{-1} & -(3\partial_y^2 + 2\partial_y P)\partial_y^{-1} \end{pmatrix}, \quad \begin{aligned} \Theta &= \partial_y^2 + P\partial_y + \partial_y P + P^2 - Q, \\ \mathcal{E} &= \partial_y^3 - 2Q\partial_y - 2\partial_y Q. \end{aligned}$$

Consequently, the l-th positive flow and negative flow of the aGX integrable hierarchy have the form

$$\begin{pmatrix} Q \\ P \end{pmatrix}_\tau = \overline{\mathbf{G}}_l = -\overline{\mathcal{R}}^{l-1}\begin{pmatrix} Q \\ P \end{pmatrix}_y, \qquad \overline{\mathcal{R}}^l\begin{pmatrix} Q \\ P \end{pmatrix}_\tau = \overline{\mathbf{G}}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad l = 1, 2, \ldots, \tag{4.12}$$

respectively.

The main result on the Liouville correspondence between the GX and aGX integrable hierarchies given in [38] is described in the following theorem. Adopting a similar notation as above, the l-th positive and negative flows of the GX and aGX hierarchies are denoted by $(\mathrm{GX})_l$ and $(\mathrm{GX})_{-l}$, and by $(\mathrm{aGX})_l$ and $(\mathrm{aGX})_{-l}$, respectively.

Theorem 10. *Under the Liouville transformation* (4.10)*, for each integer* $l \geq 1$*,*

(i). *If* $\big(m(t,x),\, n(t,x)\big)$ *is a solution of the* $(\mathrm{GX})_l$ *system* (4.7)*, then the corresponding* $\big(Q(\tau,y),\, P(\tau,y)\big)$ *satisfies the* $(\mathrm{aGX})_{-l}$ *system* (4.12)*;*

(ii). *If* $\big(m(t,x),\, n(t,x)\big)$ *is a solution of the* $(\mathrm{GX})_{-l}$ *system* (4.8)*, then the corresponding* $\big(Q(\tau,y),\, P(\tau,y)\big)$ *satisfies the* $(\mathrm{aGX})_{l+1}$ *system* (4.12)*.*

Considering the correspondence between the two hierarchies of Hamiltonian functionals admitted by these two integrable hierarchies, we have the following theorem.

Theorem 11. *For any nonzero integer* l*, each Hamiltonian functionals* $\mathcal{H}_l(m,n)$ *of the GX hierarchy relates the Hamiltonian functionals* $\overline{\mathcal{H}}_l(Q,P)$*, under the Liouville transformation* (4.10)*, according to the following identity*

$$\mathcal{H}_l(m,n) = 6(-1)^{l+1}\overline{\mathcal{H}}_{-(l+1)}(Q,P), \quad 0 \neq l \in \mathbb{Z}.$$

Remark 1. Notably, as claimed in [38] that the recursion operator $\overline{\mathcal{R}}$ for the aGX hierarchy satisfies the following composition identity

$$\overline{\mathcal{R}} = \mathcal{U}\,\mathcal{V}, \tag{4.13}$$

where $\mathcal{U}$ and $\mathcal{V}$ are the matrix operators defined by

$$\mathcal{U} = \begin{pmatrix} \mathcal{E} & \mathcal{E} \\ \mathcal{F} + 3\,\partial_y^2 & \mathcal{F} - 3\partial_y^2 \end{pmatrix}, \qquad \mathcal{F} = -(2P\partial_y + 2P_y),$$
$$\mathcal{V} = \frac{1}{54}\begin{pmatrix} 3\,\partial_y^{-1}\,\Theta\,\partial_y^{-1} & \partial_y^{-1}\,\Theta\,\partial_y^{-1}\,(2P - 3\,\partial_y) \\ -3\,\partial_y^{-1}\,\Theta^*\,\partial_y^{-1} & -\partial_y^{-1}\,\Theta^*\,\partial_y^{-1}\,(2P + 3\,\partial_y) \end{pmatrix}.$$

Formula (4.13) can be viewed as a new operator factorization for $\overline{\mathcal{R}}$, which is different with the decomposition of $\overline{\mathcal{R}} = \overline{\mathcal{K}}\,\overline{\mathcal{J}}^{-1}$ using the Hamiltonian pair given in (4.11). It is worth mentioning that such a novel factorization is a key issue in the proof of Theorem 10, especially in identifying the systems transformed from the negative (positive) flows of the GX hierarchy to be the corresponding positive (negative) flows of the aGX hierarchy.

5 Liouville correspondences for the dual Schrödinger and (defocusing) mKdV hierarchies

The nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\,u_t + u_{xx} + \sigma u\,|u|^2 = 0, \qquad \sigma = \pm 1 \tag{5.1}$$

is a reduction of a bi-Hamiltonian system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \mathcal{L}\,\mathcal{D}^{-1}\begin{pmatrix} u \\ v \end{pmatrix}_x, \tag{5.2}$$

where

$$\mathcal{L} = \begin{pmatrix} \partial_x + u\partial_x^{-1}v & -u\partial_x^{-1}u \\ -v\partial_x^{-1}v & \partial_x + v\partial_x^{-1}u \end{pmatrix}, \qquad \mathcal{D} = \begin{pmatrix} -\,\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$

are compatible Hamiltonian operators. System (5.2) reduces to the NLS equation (5.1) when $v = \sigma\overline{u}$. In spirit of the general approach of tri-Hamiltonian duality, we introduce the following Hamiltonian pair

$$\mathcal{K} = \begin{pmatrix} m\partial_x^{-1}m & -m\partial_x^{-1}n \\ -n\partial_x^{-1}m & n\partial_x^{-1}n \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & \mathrm{i} - \partial_x \\ -(\mathrm{i} + \partial_x) & 0 \end{pmatrix}, \tag{5.3}$$

and define $m = u + \mathrm{i}\,u_x$, $n = v - \mathrm{i}\,v_x$, which leads to the integrable system

$$m_t = \mathrm{i}\,muv, \qquad n_t = -\,\mathrm{i}\,nuv. \tag{5.4}$$

If we let $v = \overline{u}$ and then $n = \overline{m}$, system (5.4) reduces to

$$m_t = u_t + \mathrm{i}\,u_{xt} = |u|^2\,(\mathrm{i}\,u - u_x), \tag{5.5}$$

which is exactly the dual version (1.3) derived in [67]. Hence, we call system (5.4) the dual Schrödinger equation and the integrable hierarchy generated by (5.4) the dual Schrödinger hierarchy.

In general, the bi-Hamiltonian integrable hierarchy initiated with the dual NLS (dNLS) equation takes the form

$$\begin{pmatrix} m \\ n \end{pmatrix}_t = \mathbf{F}_l = \mathcal{K}\,\delta\mathcal{H}_{l-1}(m,\,n) = \mathcal{J}\,\delta\mathcal{H}_l(m,\,n), \quad n = \overline{m}, \quad l = 1, 2, \ldots, \tag{5.6}$$

with $\delta\mathcal{H}_l(m,\,n) = \big(\,\delta\mathcal{H}_l/\delta m,\ \delta\mathcal{H}_l/\delta n\,\big)^T$, which is governed by the usual recursion procedure using the resulting hereditary recursion operator $\mathcal{R} = \mathcal{K}\mathcal{J}^{-1}$. More precisely, the dNLS equation serves as the second member corresponding to $l = 2$ in (5.6), where the Hamiltonian functionals are

$$\mathcal{H}_1 = \mathrm{i}\int m\,\overline{u}_x\,\mathrm{d}x, \qquad \mathcal{H}_2 = \frac{1}{2}\int m\,u|u|^2\,\mathrm{d}x,$$

and the seed Hamiltonian system is

$$\begin{pmatrix} m \\ n \end{pmatrix}_t = \mathbf{F}_1 = \mathcal{J}\,\delta\mathcal{H}_1(m,\,n) = \begin{pmatrix} m \\ n \end{pmatrix}_x.$$

To obtain the negative hierarchy, note that (5.4) admits a conserved functional

$$\mathcal{H}_{-1} = \int (m\,n)^{\frac{1}{2}}\,\mathrm{d}x,$$

which satisfies

$$\mathbf{F}_0 = \mathcal{K}\,\delta\mathcal{H}_{-1}(m,\,n) = \big(\,cm,\ -cn\,\big)^T,$$

where c is the integration constant. This means that one can take

$$\begin{pmatrix} m \\ n \end{pmatrix}_t = \mathbf{F}_0 = \mathcal{J}\,\delta\mathcal{H}_0(m,\,n), \qquad \mathcal{H}_0 = -c\,\mathrm{i}\int (uv - \mathrm{i}\,u_x v)\,\mathrm{d}x,$$

as the initial equation. Then the l-th negative flow of the dNLS hierarchy is given by

$$\mathcal{R}^l \begin{pmatrix} m \\ n \end{pmatrix}_t = \mathbf{F}_0, \qquad l = 1, 2, \ldots. \tag{5.7}$$

We now focus our attention on the isospectral problem associated with the dNLS equation in [47]:

$$\begin{aligned} \mathbf{\Psi}_x &= \begin{pmatrix} \frac{\mathrm{i}}{2} & \lambda m \\ \lambda n & -\frac{\mathrm{i}}{2} \end{pmatrix} \mathbf{\Psi}, \qquad \mathbf{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \\ \mathbf{\Psi}_t &= \begin{pmatrix} \frac{\mathrm{i}}{2}(uv + \frac{1}{2\lambda^2}) & \frac{u}{2\lambda} \\ \frac{v}{2\lambda} & -\frac{\mathrm{i}}{2}(uv + \frac{1}{2\lambda^2}) \end{pmatrix} \mathbf{\Psi}. \end{aligned} \tag{5.8}$$

One can verify that the following Liouville transformation

$$\mathbf{\Phi} = \begin{pmatrix} (n/m)^{\frac{1}{4}} & (n/m)^{-\frac{1}{4}} \\ (n/m)^{\frac{1}{4}} & -(n/m)^{-\frac{1}{4}} \end{pmatrix} \mathbf{\Psi}, \qquad y = \int^x (mn)^{\frac{1}{2}}(\xi)\,\mathrm{d}\xi$$

will convert the isospectral problem (5.8) into the isospectral problem

$$\mathbf{\Phi}_y = \begin{pmatrix} -\mathrm{i}\,\mu & Q \\ Q & \mathrm{i}\,\mu \end{pmatrix} \mathbf{\Phi}, \qquad \mathbf{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{5.9}$$

with

$$\lambda = -\mathrm{i}\,\mu, \quad Q = \frac{1}{4}(mn)^{-\frac{1}{2}} \left(2\,\mathrm{i} + \left(\frac{m}{n}\right)\left(\frac{n}{m}\right)_x\right).$$

It is remarked that (5.9) is the isospectral problem of the (defocusing) mKdV equation

$$Q_\tau + \frac{1}{4}Q_{yyy} - \frac{3}{2}Q^2 Q_y = 0. \tag{5.10}$$

Motivated by these results, we are led to establish the Liouville correspondence between the dNLS hierarchy and the (defocusing) mKdV hierarchy by utilizing the Liouville transformation

$$\begin{aligned} & y = \int^x \Delta(t,\xi)\mathrm{d}\xi, \quad \tau = t, \\ & Q = \frac{1}{4\Delta}\left(2\,\mathrm{i} + \left(\frac{m}{n}\right)\left(\frac{n}{m}\right)_x\right), \quad \Delta = (mn)^{\frac{1}{2}} = |m|. \end{aligned} \tag{5.11}$$

First of all, under the coordinate transformation (5.11), the Hamiltonian pair $\mathcal{K}$ and $\mathcal{J}$ (5.3) of the dNLS hierarchy will yield the Hamiltonian pair of the (defocusing) mKdV hierarchy. The following theorem is thus established to illustrate the preceding claim.

Theorem 12. *By the coordinate transformation* (5.11)*, the Hamiltonian pair $\mathcal{K}$ and $\mathcal{J}$* (5.3) *admitted by the dNLS equation* (5.5) *is related to the Hamiltonian pair*

$$\overline{\mathcal{K}} = -\frac{1}{8}\,\partial_y^3 + \frac{1}{2}\,\partial_y\, Q\, \partial_y^{-1}\, Q\, \partial_y \qquad \text{and} \qquad \overline{\mathcal{J}} = -\frac{1}{4}\,\partial_y. \tag{5.12}$$

Proof. In view of the transformation (5.11), one has

$$\partial_x = \Delta \partial_y, \tag{5.13}$$

and then further concludes that the operator identity:

$$\Delta^{-1}\left(\frac{n}{m}\right)^{-\beta}\left(2\,\mathrm{i} + \frac{1}{\beta}\,\partial_x\right)\left(\frac{n}{m}\right)^{\beta} = 4Q + \frac{1}{\beta}\,\partial_y \tag{5.14}$$

holds for arbitary nonzero constant β. Next, define

$$\begin{aligned} \mathbf{T} &= -\frac{1}{4}\Delta\left((\partial_y + 2Q + 2Q_y\,\partial_y^{-1})\, m^{-1}(-\partial_y + 2Q + 2Q_y\,\partial_y^{-1})\, n^{-1}\right), \\ \mathbf{T}^* &= -\frac{1}{4}\begin{pmatrix} (n/m)^{\frac{1}{2}}\ (2\partial_y^{-1}\, Q\,\partial_y - \partial_y) \\ (n/m)^{-\frac{1}{2}}\ (2\partial_y^{-1}\, Q\,\partial_y + \partial_y) \end{pmatrix}. \end{aligned} \tag{5.15}$$

Hence, the Hamiltonian operators $\overline{\mathcal{K}}$ and $\overline{\mathcal{J}}$ follow from the formulae $\overline{\mathcal{K}} = \Delta^{-1}\mathbf{T}\mathcal{J}\mathbf{T}^*$ and $\overline{\mathcal{J}} = \Delta^{-1}\mathbf{T}\mathcal{K}\mathbf{T}^*$, respectively, where the identities (5.13) and (5.14) are used. This completes the proof of the theorem. ■

It is worth noting that $\overline{\mathcal{K}}$ and $\overline{\mathcal{J}}$, as given in (5.12), are the compatible operators admitted by the hierarchy initiated with the (defocusing) mKdV equation (5.10), whose bi-Hamiltonian structure takes the following form

$$Q_\tau = \overline{\mathcal{K}}\,\frac{\delta\overline{\mathcal{H}}_1}{\delta Q} = \overline{\mathcal{J}}\,\frac{\delta\overline{\mathcal{H}}_2}{\delta Q},$$

with

$$\overline{\mathcal{H}}_1 = \int Q^2\,\mathrm{d}y, \qquad \overline{\mathcal{H}}_2 = -\frac{1}{2}\int (Q_y^2 + Q^4)\,\mathrm{d}y.$$

As for the (defocusing) mKdV hierarchy, each member in the positive direction takes the form

$$Q_\tau = \overline{F}_l = \overline{\mathcal{K}}\,\frac{\delta\overline{\mathcal{H}}_{l-1}}{\delta Q} = \overline{\mathcal{J}}\,\frac{\delta\overline{\mathcal{H}}_l}{\delta Q}, \qquad l = 1, 2, \ldots, \tag{5.16}$$

where, the (defocusing) mKdV equation (5.10) is the second member in (5.16) and the seed equation corresponding to $l = 1$ is

$$Q_\tau = \overline{F}_1 = -\frac{1}{2}\, Q_y = \overline{\mathcal{J}}\, \frac{\delta \overline{\mathcal{H}}_1}{\delta Q}.$$

However, in the negative direction, the l-th negative flow is

$$\overline{\mathcal{R}}^l\, Q_\tau = 0, \qquad l = 1, 2, \ldots, \qquad \text{where} \qquad \overline{\mathcal{R}} = \overline{\mathcal{K}}\, \overline{\mathcal{J}}^{-1} = \frac{1}{2}\partial_y^2 - 2\partial_y\, Q\, \partial_y^{-1}\, Q$$

is the recursion operator of the (defocusing) mKdV hierarchy.

Hereafter, for each positive integer l, we denote the l-th equation in the positive and negative directions of the dNLS hierarchy by $(\mathrm{dNLS})_l$ and $(\mathrm{dNLS})_{-l}$, respectively, while the l-th positive and negative flows in the (defocusing) mKdV hierarchy are denoted by $(\mathrm{mKdV})_l$ and $(\mathrm{mKdV})_{-l}$, respectively. With this notation, we are now in a position to establish the following theorem, illustrating the Liouville correspondence between the two hierarchies.

Theorem 13. *Under the transformation* (5.11)*, for each integer* $l \in \mathbb{Z}^+$*, the* $(\mathrm{dNLS})_{-l}$ *equation is mapped into the* $(\mathrm{mKdV})_l$ *equation, and the* $(\mathrm{dNLS})_l$ *equation is mapped into the* $(\mathrm{mKdV})_{-(l-1)}$ *equation.*

In order to prove Theorem 13, a relation identity with regard to the two recursion operators admitted by the dNLS and (defocusing) mKdV hierarchies is required.

Lemma 2. *Let* $\mathcal{R} = \mathcal{K}\,\mathcal{J}^{-1}$ *and* $\overline{\mathcal{R}} = \overline{\mathcal{K}}\,\overline{\mathcal{J}}^{-1}$ *be the recursion operators admitted by the dNLS and (defocusing) mKdV hierarchies, respectively. Define*

$$\mathbf{D} = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}. \tag{5.17}$$

Then, for each integer $l \geq 1$*,*

$$\overline{\mathcal{R}}^l\, \big(\partial_y \quad -\partial_y\,\big)\, \mathbf{D}^{-1}\, \mathcal{R}^l = -4\Delta^{-1}\, \mathbf{T} \tag{5.18}$$

under the transformation (5.11)*, where* $\mathbf{T}$ *is the matrix differential operator defined in* (5.15)*.*

Proof. Note first, in the case of $l = 1$, equation (5.18) is equivalent to

$$\Delta^{-1}\, \mathbf{T}\, \mathcal{J} = -\frac{1}{4}\, \overline{\mathcal{R}}\, \big(\partial_y \quad -\partial_y\,\big)\, \mathbf{D}^{-1}\, \mathcal{K},$$

which can be directly verified by utilizing the formulae (5.13) and (5.14).

In addition, using the relations between $\overline{\mathcal{K}}$, $\overline{\mathcal{J}}$ and $\mathcal{K}$, $\mathcal{J}$ allows us to deduce that $\overline{\mathcal{R}}$ satisfies

$$\overline{\mathcal{R}}\, \Delta^{-1}\, \mathbf{T}\, \mathcal{K} = \Delta^{-1}\, \mathbf{T}\, \mathcal{J} \qquad \text{and then} \qquad \overline{\mathcal{R}}\, \Delta^{-1}\, \mathbf{T}\, \mathcal{R} = \Delta^{-1}\, \mathbf{T}. \tag{5.19}$$

Hence, for the general case, we assume that (5.18) holds for $l = k$. Then for $l = k + 1$,

$$-\frac{1}{4}\overline{\mathcal{R}}^{l+1}\left(\partial_y \;\; -\partial_y\right)\mathbf{D}^{-1}\mathcal{R}^{l+1} = -\frac{1}{4}\overline{\mathcal{R}}\overline{\mathcal{R}}^{l}\left(\partial_y \;\; -\partial_y\right)\mathbf{D}^{-1}\mathcal{R}^{l}\mathcal{R} = \overline{\mathcal{R}}\Delta^{-1}\,\mathbf{T}\mathcal{R} = \Delta^{-1}\mathbf{T},$$

which completes the induction step and thus establishes (5.18) for $l \geq 1$. We thus verify (5.18) holds in general, proving the lemma. ■

Proof of Theorem 13. As the first step, we deduce from the relation between Q and m given in transformation (5.11) that

$$Q_\tau = \Delta^{-1}\,\mathbf{T}\begin{pmatrix} m \\ n \end{pmatrix}_t, \tag{5.20}$$

where $\mathbf{T}$ is the matrix differential operator given in (5.15).

Next, we consider the $(\mathrm{dNLS})_{-l}$ equation for $l \geq 1$. Note first that the $(\mathrm{dNLS})_{-1}$ equation corresponding to $l = 1$ in (5.7) can be written as

$$\begin{pmatrix} m \\ n \end{pmatrix}_t = \frac{1}{2}\mathcal{J}\begin{pmatrix} (n/m)^{\frac{1}{2}} \\ (n/m)^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} mQ \\ -nQ \end{pmatrix}.$$

Substituting it into (5.20) yields

$$Q_\tau = \Delta^{-1}\,\mathbf{T}\begin{pmatrix} mQ \\ -nQ \end{pmatrix} = -\frac{1}{2}Q_y,$$

which reveals that, under the transformation (5.11), the $(\mathrm{dNLS})_{-1}$ equation is mapped into the $(\mathrm{mKdV})_1$ equation.

Furthermore, it follows from (5.18) that, for each $l \geq 2$, equation (5.20) can be rewritten as

$$Q_\tau = -\frac{1}{4}\overline{\mathcal{R}}^{l}\,\left(\partial_y \quad -\partial_y\,\right)\mathbf{D}^{-1}\,\mathcal{R}^{l}\begin{pmatrix} m \\ n \end{pmatrix}_t,$$

with $\mathbf{D}$ defined in (5.17). Hence, in general, if $m(t,x)$ is a solution of $(\mathrm{dNLS})_{-l}$ equation, and $n(t,x) = \overline{m}(t,x)$, so that (5.7) holds for each $l \geq 2$, then the corresponding $Q(\tau, y)$ satisfies

$$Q_\tau = -\frac{1}{4}\overline{\mathcal{R}}^{l}\,\left(\partial_y \quad -\partial_y\,\right)\mathbf{D}^{-1}\,\mathbf{F}_0 = -\frac{1}{4}\overline{\mathcal{R}}^{l}\,\left(\partial_y \quad -\partial_y\,\right)\begin{pmatrix} c \\ -c \end{pmatrix} = \overline{\mathcal{R}}^{l-1}\left(-\frac{1}{2}Q_y\right),$$

which means that $Q(\tau, y)$ solves the $(\mathrm{mKdV})_l$ equation (5.16), completing the first part.

Finally, concerning the opposite direction, it is worth noting that

$$\Delta^{-1}\,\mathbf{T}\begin{pmatrix} m_x \\ n_x \end{pmatrix} = -(Q\Delta)_y + \frac{1}{4}\partial_y\left(\frac{m}{n}\left(\frac{n}{m}\right)_x\right) = 0.$$

Hence, if $m(t,x)$ is a solution of $(\mathrm{dNLS})_l$ equation (5.6), and $n(t,x)=\overline{m}(t,x)$, i.e.,

$$\begin{pmatrix} m \\ n \end{pmatrix}_t = (\mathcal{K}\,\mathcal{J})^{l-1} \begin{pmatrix} m \\ n \end{pmatrix}_x, \qquad l=1,2,\ldots,$$

then in view of (5.20), the corresponding $Q(\tau,y)$ satisfies

$$Q_\tau = \Delta^{-1}\,\mathbf{T}\,(\mathcal{K}\,\mathcal{J})^{l-1} \begin{pmatrix} m \\ n \end{pmatrix}_x,$$

and then

$$\overline{\mathcal{R}}^{l-1} Q_\tau = \overline{\mathcal{R}}^{l-1}\Delta^{-1}\mathbf{T}\mathcal{R}^{l-1} \begin{pmatrix} m \\ n \end{pmatrix}_x = \Delta^{-1}\mathbf{T} \begin{pmatrix} m \\ n \end{pmatrix}_x = 0.$$

This allows us to draw the conclusion that $Q(\tau,y)$ is a solution of the $(\mathrm{mKdV})_{-(l-1)}$ equation for $l \geq 1$. We thus complete the proof of Theorem 13 in general. ■

In what follows, we investigate the effect of the transformation (5.11) on the two hierarchies of the Hamiltonian functionals $\{\mathcal{H}_l\}$ and $\{\overline{\mathcal{H}}_l\}$ of the dNLS equation and the (defocusing) mKdV equation. With the two pairs of Hamiltonian operators $\mathcal{K}$ and $\mathcal{J}$ admitted by the dNLS equation and $\overline{\mathcal{K}}$ and $\overline{\mathcal{J}}$ admitted by the (defocusing) mKdV equation, $\{\mathcal{H}_l\}$ and $\{\overline{\mathcal{H}}_l\}$ are determined by the recursive formulae

$$\mathcal{K}\,\delta\mathcal{H}_l = \mathcal{J}\,\delta\mathcal{H}_{l+1},\; \delta\mathcal{H}_l = \left(\frac{\delta\mathcal{H}_l}{\delta m}, \frac{\delta\mathcal{H}_l}{\delta n}\right)^T, \; \overline{\mathcal{K}}\,\frac{\delta\overline{\mathcal{H}}_l}{\delta Q} = \overline{\mathcal{J}}\,\frac{\delta\overline{\mathcal{H}}_{l+1}}{\delta Q}, \qquad l\in\mathbb{Z}. \quad (5.21)$$

Lemma 3. *Let $\{\mathcal{H}_n\}$ and $\{\overline{\mathcal{H}}_n\}$ be the hierarchies of conserved functionals determined by the recursive formulae (5.21). Then, for each $l\in\mathbb{Z}$, their respective variational derivatives satisfy the relation*

$$\Delta^{-1}\,\mathbf{T}\,\mathcal{J}\,\delta\mathcal{H}_l = \overline{\mathcal{J}}\,\frac{\delta\overline{\mathcal{H}}_{-l}}{\delta Q}. \qquad (5.22)$$

Proof. We first prove (5.22) for $l\geq 0$ by induction on l. In the case of $l=0$, equation (5.22) follows from the fact $\mathcal{J}\delta\mathcal{H}_0 = \big(cm,\ -cn\big)^T$ and $\Delta^{-1}\,\mathbf{T}\big(cm,\ -cn\big)^T=0$. Assume now (5.22) holds for $l=k$ with $k\geq 1$, say

$$\Delta^{-1}\,\mathbf{T}\,\mathcal{J}\,\delta\mathcal{H}_k = \overline{\mathcal{J}}\,\frac{\delta\overline{\mathcal{H}}_{-k}}{\delta Q}.$$

Then, on the one hand, by the assumption and in view of (5.19)

$$\overline{\mathcal{R}}\,\Delta^{-1}\,\mathbf{T}\,\mathcal{K}\,\delta\mathcal{H}_k = \Delta^{-1}\,\mathbf{T}\,\mathcal{J}\,\delta\mathcal{H}_k = \overline{\mathcal{J}}\,\frac{\delta\overline{\mathcal{H}}_{-k}}{\delta Q} = \overline{\mathcal{K}}\,\frac{\delta\overline{\mathcal{H}}_{-(k+1)}}{\delta Q}, \qquad (5.23)$$

while, on the other hand, by the recursive formula (5.21),

$$\overline{\mathcal{R}}\,\Delta^{-1}\,\mathbf{T}\,\mathcal{K}\,\delta\mathcal{H}_k = \overline{\mathcal{R}}\,\Delta^{-1}\,\mathbf{T}\,\mathcal{J}\,\delta\mathcal{H}_{k+1},$$

which, in comparison with (5.23) produces

$$\overline{\mathcal{J}}^{-1}\,\Delta^{-1}\,\mathbf{T}\,\mathcal{J}\,\delta\mathcal{H}_{k+1}=\frac{\delta\overline{\mathcal{H}}_{-(k+1)}}{\delta Q}.$$

Then

$$\Delta^{-1}\,\mathbf{T}\,\mathcal{J}\,\delta\mathcal{H}_{k+1}=\overline{\mathcal{J}}\,\frac{\delta\overline{\mathcal{H}}_{-(k+1)}}{\delta Q}$$

follows, establishing (5.22) for $l\geq 0$.

Next, in the case of $l=-1$, the fact $\mathcal{J}\delta\overline{\mathcal{H}}_1/\delta Q=-Q_y/2$ and $\mathcal{J}\delta\mathcal{H}_{-1}=\mathbf{D}\big(Q,\ -Q\big)^T$ shows that (5.22) holds for $l=-1$. Finally, we prove (5.22) for all $l\leq -1$ by induction. Assume that it holds for $l=k$, then, for $l=k-1$, according to the assumption and using the formula (5.19) again, we arrive at

$$\Delta^{-1}\,\mathbf{T}\,\mathcal{J}\,\delta\mathcal{H}_{k-1}=\overline{\mathcal{R}}\,\Delta^{-1}\,\mathbf{T}\,\mathcal{K}\,\delta\mathcal{H}_{k-1}=\overline{\mathcal{R}}\,\Delta^{-1}\,\mathbf{T}\,\mathcal{J}\,\delta\mathcal{H}_{k}=\overline{\mathcal{K}}\,\frac{\delta\overline{\mathcal{H}}_{-k}}{\delta Q},$$

which completes the induction step and verifies (5.22) holds for all $l\in\mathbb{Z}$, proving the lemma. ■

In addition, we deduce a formula which reveals the change of the variational derivative under the transformation (5.11).

Lemma 4. *Let* $m(t,x)$, $n(t,x)=\overline{m}(t,x)$ *and* $Q(\tau,y)$ *be related by the Liouville transformation* (5.11). *If the Hamiltonian functionals* $\mathcal{H}(m,n)=\overline{\mathcal{H}}(Q)$, *then*

$$\delta\mathcal{H}(m,n)=\big(\delta\mathcal{H}/\delta m,\ \delta\mathcal{H}/\delta n\big)^T=\mathbf{T}^*\,\frac{\delta\overline{\mathcal{H}}}{\delta Q},$$

where $\mathbf{T}^*$ *is the formal adjoint of* $\mathbf{T}$ *given in* (5.15).

Finally, under the hypothesis of Lemma 4, we define a functional

$$\mathcal{G}_k(Q)\equiv\mathcal{H}_l(m,n),$$

for some $k\in\mathbb{Z}$. Then, it follows from Lemma 3 and Lemma 4 that

$$\overline{\mathcal{K}}\,\frac{\delta\overline{\mathcal{H}}_{-(l+1)}}{\delta Q}=\Delta^{-1}\,\mathbf{T}\,\mathcal{J}\,\delta\mathcal{H}_l(m,n)=\Delta^{-1}\,\mathbf{T}\,\mathcal{J}\,\mathbf{T}^*\,\frac{\delta\mathcal{G}_k}{\delta Q}=\overline{\mathcal{K}}\,\frac{\delta\mathcal{G}_k}{\delta Q},$$

which immediately leads to

$$\frac{\delta\overline{\mathcal{H}}_{-(l+1)}}{\delta Q}=\frac{\delta\mathcal{G}_k}{\delta Q},$$

and then

$$\mathcal{H}_l(m,n) = \mathcal{G}_k(Q) = \overline{\mathcal{H}}_{-(l+1)}(Q)$$

follows. Consequently, we conclude that there also exists a one-to-one correspondence between the Hamiltonian functionals admitted by the dNLS and (defocusing) mKdV equations.

Theorem 14. *Under the transformation* (5.11)*, for each integer l, the Hamiltonian conservation law $\mathcal{H}_l(m,n)$ of the dNLS equation is related to the Hamiltonian conservation law $\overline{\mathcal{H}}_l(Q)$ of the (defocusing) mKdV equation, according to the following identity*

$$\mathcal{H}_l(m,n) = \overline{\mathcal{H}}_{-(l+1)}(Q), \qquad l \in \mathbb{Z}.$$

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