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# HAMILTONIAN PERTURBATION THEORY AND WATER WAVES

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ABSTRACT. A general theory of noncanonical perturbations of Hamiltonian systems, both finite dimensional and continuous, is proposed. The results determine a general formula for the deformation of a Poisson structure on a manifold. The theory is applied to the Boussinesq expansion for the free boundary problem for water waves, which leads to the Korteweg-de Vries equation. New Hamiltonian model equations for both uni- and bi-directional propagation of long waves in shallow water are found. An explanation of the complete integrability (soliton property) of the KdV equation,as a consequence of the expansion,is determined.

1. INTRODUCTION. In 1895 Korteweg and deVries first derived their celebrated equation as a model for the unidirectional propagation of long waves in shallow water. Their method proceeded by first applying the perturbation expansion introduced by Boussinesq, and then restricting the resulting bidirectional Boussinesq system to a "submanifold" of approximately unidirectional waves. Hamiltonian methods entered the subject when Zakharov found the Hamiltonian form of the water wave problem. Subsequently, the Korteweg-de Vries equation was shown to be Hamiltonian, in fact in two distinct ways.

In earlier work with Benjamin, [2], [12], symmetry group techniques used in conjunction with Zakharov's Hamiltonian structure proved that the two-dimensional water wave problem without surface tension has precisely eight nontrivial conservation laws. The present work arose in an ongoing investigation as to how these laws behave under the perturbation expansion leading to the KdV equation. This project came to a temporary halt, however, with the surprising discovery that the Hamiltonian structures of these two equations do not match up in any natural way. Indeed, this is first evidenced by the fact that almost all versions of the Boussinesq system, which is the essential halfway point in the derivation, are not Hamiltonian, in particular do not conserve energy. Even more striking is the elementary, but apparently unnoticed observation that the perturbation expansion of the energy for the water wave

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problem does not agree to the requisite order with either of the Hamiltonians for the KdV equation. Alternative models such as the BBM or Regularised Long Wave equation, [1], suffer from the same problem.

In order to better understand this state of affairs, a general theory of noncanonical perturbation expansions of Hamiltonian systems must be developed. In outline, the theory proceeds as follows. Consider a Hamiltonian system

$$\dot{x} = J(x,\epsilon)\nabla H(x,\epsilon) \ , \tag{1.1}$$

in which $\epsilon$ is a small parameter, $H(x,\epsilon)$ is the Hamiltonian function and $J(x,\epsilon)$ the skew-adjoint Hamiltonian (or cosymplectic) operator. Since the operator $J$ appears in the cosymplectic two-vector $\Theta = \frac{1}{2}\,\partial_x^T \wedge J\partial_x$, defining a Poisson structure, we call (1.1) the cosymplectic form of Hamilton's equations, to be distinguished from the symplectic form

$$K(x,\epsilon)\dot{x} = \nabla H(x,\epsilon) \ , \tag{1.2}$$

corresponding to the symplectic two-form $\Omega = -\frac{1}{2}\,dx^T \wedge K dx$, $K = J^{-1}$. (At first sight, this distinction appears trivial, but the two forms lead to very different types of perturbation equations.)

Consider a perturbation expansion

$$x = y + \epsilon\,\varphi(y) + \ldots \ . \tag{1.3}$$

In standard perturbation theory, one substitutes (1.3) into (1.1) or (1.2), expands in powers of $\epsilon$ and truncates to some required order. The resulting system, as simple examples easily show, is not in general Hamiltonian. In order to preserve the Hamiltonian structure we must expand both the Hamiltonian

$$H(x,\epsilon) = H_0(y) + \epsilon H_1(y) + \epsilon^2 H_2(y) + \ldots$$

and the cosymplectic operator

$$J(x,\epsilon) = J_0(y) + \epsilon\, J_1(y) + \epsilon^2 J_2(y) + \ldots$$

and truncate at the required order. (We ignore for the moment the additional complication that the truncated series for $J$ is not in general a true cosymplectic operator - see section 2B.) To first order,

$$\begin{aligned}\dot{y} &= (J_0(y) + \epsilon\, J_1(y))(\nabla H_0(y) + \epsilon\, \nabla H_1(y)) \\ &= J_0\nabla H_0 + \epsilon(J_1\nabla H_0 + J_0\nabla H_1) + \epsilon^2 J_1\nabla H_1 \ ,\end{aligned} \tag{1.4}$$

called the cosymplectic perturbation of (1.1). It agrees with the ordinary perturbation expansion

$$\dot{y} = J_0\nabla H_0 + \epsilon(J_1\nabla H_0 + J_0\nabla H_1) \tag{1.5}$$

to first order but includes some additional terms in $\epsilon^2$ so as to maintain the Hamiltonian structure. Note that (1.4) is not the second order ordinary perturbation of (1.1) - this would include the terms $\epsilon^2(J_0\nabla H_2 + J_2\nabla H_0)$ , which would again destroy the Hamiltonian form of the system. The symplectic perturbation proceeds along the same lines, leading to

$$(K_0(y) + \epsilon\ K_1(y))\dot{y} = \nabla H_0(y) + \epsilon\ \nabla H_1(y)\ , \qquad (1.6)$$

which is always Hamiltonian. For evolution equations, as the examples in section 4 bear out, the cosymplectic form is usually the more desirable because in (1.6) the symplectic operator, which may very well be nonlinear, is applied to temporal derivatives of $y$ .

This Hamiltonian perturbation theory falls between the two main schools of perturbation theory - on the one hand standard perturbation methods, [6], pay no regard to any Hamiltonian structure in the systems under investigation, whereas in classical and celestial mechanics, [15], all perturbations are canonical and the problems discussed here never arise. Nevertheless, the present theory should prove to be of importance in a wide range of physical applications in which the perturbations are more or less prescribed, but one still wishes to maintain some form of Hamiltonian structure.

In the water wave problem, there are two small parameters $\alpha$ and $\beta$ but the expansions take the same form. If (1.1) represents the original free boundary problem, then the non-Hamiltonian Boussinesq systems are of the form (1.5). To make these Hamiltonian, we must add certain quadratic terms in $\alpha^2, \alpha\beta, \beta^2$ , as in (1.4); see (4.15) for the resulting system. Similar remarks apply to the subsequent derivative of the KdV equation (coming from the cosymplectic form of the expansion) or the BBM equation (coming from the symplectic form). In terms of the surface elevation $\eta(x,t)$ , the non-Hamiltonian perturbation equation (1.5) is the familiar KdV equation

$$\eta_t + \eta_x + \frac{3}{2}\alpha\ \eta\eta_x + \frac{1}{6}\beta\ \eta_{xxx} = 0\ . \qquad (1.7)$$

To retain the correct Hamiltonian structure according to the general theory, one must include quadratic terms as in (1.4), leading to the "Hamiltonian version" of the KdV equation

$$\eta_t + \eta_x + \frac{3}{2}\alpha\ \eta\eta_x + \frac{1}{6}\beta\ \eta_{xxx} + \frac{1}{16}\alpha\beta(\eta^2)_{xxx} + \frac{15}{32}\alpha^2\eta^2\eta_x = 0\ . \quad (1.8)$$

This model has Hamiltonian functional

$$H[\eta] = \int_{-\infty}^{\infty}\left(\frac{1}{2}\ \eta^2 + \frac{1}{8}\alpha\ \eta^3\right)dx\ , \qquad (1.9)$$

which is the correct first order expansion of the energy (Hamiltonian) of the water wave problem, and cosymplectic operator

$$J = -[D_x + \frac{1}{4}\alpha(\eta\ D_x + D_x\eta) + \frac{1}{6}\beta\ D_x^3]\ . \qquad (1.10)$$

Note that (1.9) does not agree with either of the usual Hamiltonians for the KdV equation. (Segur, [14], gives a completely different derivation of the

KdV equation using two time scales. His expansion of the energy leads to a linear combination of the two KdV Hamiltonians. It remains to be seen how the two methods can be reconciled.)

There remains the question of why, in spite of the general theory, the KdV equation is Hamiltonian. Note that the operator (1.10) appearing in the Hamiltonian perturbation resembles a linear combination of the two cosymplectic operators for the KdV equation. Under special circumstances, the non-Hamiltonian perturbation (1.5) can inherit two compatible Hamiltonian structures (corresponding to $J_0$ and $J_1$ ), and hence, by a theorem of Magri, [9], is automatically completely integrable. This may offer an explanation for the remarkable fact that completely integrable Hamiltonian systems (soliton equations) such as the KdV, sine-Gordon, and nonlinear Schrödinger equations appear so often as model equations in the perturbation expansions to a wide variety of physical systems.

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2. FINITE DIMENSIONAL HAMILTONIAN PERTURBATION THEORY. The aim is to set up a Hamiltonian perturbation theory for evolution equations, but to keep things simple we begin with the finite dimensional case. One lesson gleaned from the evolutionary case is that one should not rely on the existence of Darboux coordinates in general, so we take a Hamiltonian structure to be defined by either a symplectic two-form, or, more generally, a cosymplectic two-vector field à la Lichnerowicz. To perturb the Hamiltonian structure, it then suffices to perturb either the symplectic form (which is straight forward ) or the cosymplectic two-vector (which is less so); in fact, the correct form of the perturbation of the cosymplectic two-vector requires the full theory of Poisson manifolds, which we develop in a form amenable to be immediately generalized to the infinite-dimensional case of evolution equations.

A. POISSON STRUCTURES. In the usual theory, Hamiltonian mechanics takes place on a manifold $M$ equipped with a symplectic two-form $\Omega$ . One immediate complication is that in local (non-Darboux) coordinates, if

$$\Omega = -\frac{1}{2}\, dx^T \wedge K(x)dx = -\frac{1}{2} \sum K_{ij} dx_i \wedge dx_j ,$$

then both Hamilton's equations

$$\dot{x} = J\, \nabla H(x) , \tag{2.1}$$

and the Poisson bracket

$$\{F,G\} = \nabla F^T J\, \nabla G ,$$

require the inverse $J = K^{-1}$ of the matrix appearing in $\Omega$ . In the infinite-dimensional version, $J$ is a differential operator, so trying to use the symplectic form usually introduces unnecessary complications.

These can be avoided by introducing a Poisson structure, as detailed in the paper by Weinstein in these proceedings. For our purposes, however, it is expedient to adopt the viewpoint of Lichnerowicz, [8], and regard the cosymplectic two-vector field

$$\Theta = \frac{1}{2}\partial_x^T \wedge J(x)\partial_x = \frac{1}{2} \Sigma J_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \qquad (2.2)$$

as the fundamental object determing a Poisson structure, rather than the Poisson bracket, which is easily reccvered from $\Theta$ :

$$\{F,G\} = \langle dF \wedge dG, \Theta\rangle \ . \qquad (2.3)$$

The requirement that the Poisson bracket satisfy the Jacobi identity translates into a system of nonlinear differential equations for the coefficients $J_{ij}(x)$ of $\Theta$ . These are most easily expressed using the Schouten-Nijenhuis bracket.

We begin by describing a new invariant definition of this important bracket between multi-vector fields which will readily generalize to the case of evolution equations. A k-vector field is a section of $\wedge_k TM$ , the bundle of contravariant alternating k-tensors. Note that if $\alpha$ is a k-vector field and $\omega$ a differential (k-1)-form, then the interior product $v = \omega \lrcorner \alpha$ is an ordinary vector field. Thus $v(\theta) = (\omega \lrcorner \alpha)\theta$ , will denote the Lie derivative of another differential form $\theta$ with respect to this vector field.

DEFINITION 2.1 Let $\alpha$ be a k-vector field and $\beta$ an $\ell$-vector field. The <u>Schouten-Nijenhuis bracket</u> $[\alpha,\beta]$ is the following uniquely determined $(k+\ell-1)$ - vector field: For every $k+\ell-1$ <u>closed</u> differential one-forms $\omega_1,\dots,\omega_{k+\ell-1}$ ,

$$\langle [\alpha,\beta], \omega_1\wedge\dots\wedge\omega_{k+\ell-1}\rangle = \frac{(-1)^{k\ell+\ell}}{\ell} \sum_I \text{sign } I \langle \alpha, (\omega_I \lrcorner \beta)\omega_{I'}\rangle + \frac{(-1)^k}{k} \sum_J \text{sign } J \langle \beta, (\omega_J \lrcorner \alpha)\omega_{J'}\rangle \ . \qquad (2.4)$$

In this formula, the first sum is over all multi-indices $I = (i_1,\dots,i_{\ell-1})$ , $1 \le i_1 < \dots < i_{\ell-1} \le k+\ell-1$ , with complement $I' = (i_1',\dots,i_k')$ such that $1 \le i_1' < \dots < i_k' \le k+\ell-1$ and $(i_1,\dots,i_{\ell-1},i_1',\dots,i_k') = \pi(1,\dots,k+\ell-1)$ for some permutation $\pi$ , and sign $I \equiv$ sign $\pi$ . Similarly, the second sum is over all $J = (j_1,\dots,j_{k-1})$ , $1 \le j_1 < \dots < j_{k-1} \le k+\ell-1$ with $J'$ , sign $J$ defined similarly.

In the special case $k = 1$ , so $\alpha = v$ is an ordinary vector field, (2.4) still holds with the understanding that in the second summation there is one term, corresponding to $I = \emptyset$ , $\omega_I = 1$ (constant). It is easily seen that in this case the Schouten-Nijenhuis bracket $[v,\beta]$ is just the Lie derivative of $\beta$ with respect to $v$ . Checking that definition 2.1 agrees with both that of Nijenhuis, [10],and the invariant definition favored by Lichnerowicz, [8], is a useful exercise. We have chosen this definition because it appears to be the only one that readily generalizes to the infinite dimensional formulation needed to treat evolution equations.

Let $\alpha,\tilde{\alpha}$ be k-vector fields, $\beta$ an $\ell$-vector field and $\gamma$ an m-vector field. The basic properties of the bracket follow from (2.4):

a) Bilinearity

$$[c\alpha+\tilde{c}\tilde{\alpha},\beta] = c[\alpha,\beta]+\tilde{c}[\tilde{\alpha},\beta] \ , \ c,\tilde{c}\in \mathbb{R}, \tag{2.5}$$

b) Super-symmetry

$$[\alpha,\beta] = (-1)^{k\ell}[\beta,\alpha] \ , \tag{2.6}$$

c) Jacobi identity

$$(-1)^{km}[[\alpha,\beta],\gamma]+(-1)^{\ell m}[[\gamma,\alpha],\beta]+(-1)^{k\ell}[[\beta,\gamma],\alpha] = 0 \ , \tag{2.7}$$

d) Pseudo-derivation

$$[\alpha,\beta\wedge\gamma] = [\alpha,\beta]\wedge\gamma+(-1)^{\ell m+m}\beta\wedge[\alpha,\gamma] \ . \tag{2.8}$$

These properties, especially (2.8) which does not appear to be as well known, are vital for determining the local coordinate formulae for this bracket.

DEFINITION 2.2 A two-vector field $\Theta$ is <u>cosymplectic</u> if

$$[\Theta,\Theta] = 0 \ . \tag{2.9}$$

A cosymplectic two-vector $\Theta$ determines a Poisson structure on $M$ in the sense of Weinstein, [16], via (2.3) and conversely. For a Hamiltonian function $H:M\to\mathbb{R}$, the associated Hamiltonian vector field is

$$v_H = F_\Theta(dH) \equiv dH\lrcorner\Theta \ , \tag{2.10}$$

with flow given by (2.1) in local coordinates.

THEOREM 2.3 Let $\Theta$ have constant rank $2m\leq n$. Then there is a foliation of $M$ with 2m-dimensional leaves so that on each leaf $L$, $\Theta|_x\in\Lambda_2TL|_x$ and is of maximal rank for each $x\in L$. Thus $\Theta$ defines a symplectic structure on $L$. Each leaf is invariant under the flow of any Hamiltonian vector field on $M$, in fact

$$TL|_x = F_\Theta(T^*M|_x)$$

for any $x\in L\subset M$.

See Lichnerowicz, [8], for a proof and Weinstein, [16], for a discussion of the non-constant rank case. The cosymplectic two-vector $\Theta$ sets up a complex

$$\delta_\Theta = \delta : \Lambda_k TM \to \Lambda_{k+1}TM \ ,$$

with $\delta(\alpha) = [\Theta,\alpha]$. The condition (2.9) implies, using the Jacobi identity (2.7), that the complex is closed: $\delta\circ\delta = 0$. However, unless $\Theta$ is of maximal rank, this complex is not locally exact.

THEOREM 2.4 Let $\Theta$ be cosymplectic, of constant rank. Let $\alpha$ be a k-vector field on $M$. Then $[\Theta,\alpha] = 0$ if and only if in any coordinate cube

$\alpha = [\Theta,\beta] + \alpha_0$ for $\beta$ a (k-1)-vector field and $\alpha_0$ a k-vector field which, in the given coordinates, is constant on the leaves of the symplectic foliation induced by $\Theta$. ($\alpha_0$ will in general depend on the choice of local coordinates.)

The proof of this result, as well as a discussion of the global cohomology, can be found in Lichnerowicz, [8].

B. PERTURBATION THEORY.

We now consider perturbation theory for a system of ordinary differential equations in Hamiltonian form. Throughout this section $\epsilon$ will be a small parameter, and we allow the possibility of both the Hamiltonian and the cosymplectic form depending on $\epsilon$. The basic system is

$$\dot{x} = J(x,\epsilon)\nabla H(x,\epsilon) = F(x,\epsilon) \ . \tag{2.11}$$

Given a perturbation expansion

$$x = y + \epsilon\varphi(y) + \epsilon^2\psi(y) + \dots , \tag{2.12}$$

following standard perturbation methods, we substitute (2.12) into (2.11) and expand the series in $\epsilon$ to first order:

$$(1+\epsilon\nabla\varphi)\dot{y} = F_o(y) + \epsilon F_1(y) \ . \tag{2.13}$$

Here $F_o, F_1$ can easily be evaluated from (2.11) using the chain rule:

$$F_o(y) = F(y,0) = J_o(y)\nabla H_o(y) \ , \ F_1(y) = F_\epsilon(y,0) + \nabla F(y,0)\varphi(y) \ .$$

We can also invert $1+\epsilon\nabla\varphi$ in (2.13) to obtain the alternative system

$$\dot{y} = F_o(y) + \epsilon \tilde{F}_1(y) \ , \tag{2.14}$$

where $\tilde{F}_1 = F_1 - \nabla\varphi \cdot F_o$. Unless the expansion (2.12) happens to be canonical, neither (2.13) nor (2.14) will be in general Hamiltonian. If we expand the Hamiltonian

$$H(x,\epsilon) = H_o(y) + \epsilon H_1(y) + \epsilon^2 H_2(y) + \dots , \tag{2.15}$$

we find that the first order truncation $H_o + \epsilon H_1$ is <u>not</u> in general a constant of the motion.

In order to maintain some form of Hamiltonian structure under perturbation, we must investigate how the symplectic or cosymplectic forms themselves are being perturbed. First we look at the easier case when the system is in symplectic form

$$K(x,\epsilon)\dot{x} = \nabla H(x,\epsilon) \ .$$

The symplectic two-form has the perturbation expansion

$$\Omega(x,\epsilon) = \Omega_o(y) + \epsilon \Omega_1(y) + \epsilon^2\Omega_2(y) + \dots , \tag{2.16}$$

or, in coordinates,

$$-\frac{1}{2}dx^T \wedge K(x,\epsilon)dx = -\frac{1}{2}dy^T \wedge (K_o(y) + \epsilon K_1(y) + \dots)dy \ ,$$

using (2.12). Since the closure condition $d\Omega = 0$ for a symplectic two form is linear, we can truncate the expansion (2.16) at any order and (provided $\epsilon$ is sufficiently small to ensure nondegeneracy) be assured the truncated form, $\Omega_0 + \epsilon\, \Omega_1$ say, remains symplectic. This, together with (2.15), yields the first order symplectic perturbation

$$(K_0(y) + \epsilon\; K_1(y))\dot{y} = \nabla H_0(y) + \epsilon\; \nabla H_1(y) \;, \tag{2.17}$$

which is a Hamiltonian system. Note that (2.17) is not the same as (2.13) or (2.14), but does agree with them up to terms of first order in $\epsilon$. This is because to lowest order $\dot{y} = F_0(y) + O(\epsilon)$, so whenever we see a term like $\epsilon\dot{y}$ we can replace it by $\epsilon F_0(y)$ and still maintain first order agreement. Note also that it is not permissible to invert $K_0 + \epsilon\, K_1$ in (2.17) and truncate and expect to have a Hamiltonian system.

As for the cosymplectic form (2.11), we can similarly expand the two-vector field

$$\Theta(x,\epsilon) = \Theta_0(y) + \epsilon\; \Theta_1(y) + \epsilon^2\Theta_2(y) + \dots \;, \tag{2.18}$$

or

$$\frac{1}{2}\; \partial_x^T \wedge J(x,\epsilon)\partial_x = \frac{1}{2}\; \partial_y^T \wedge (J_0(y) + \epsilon\; J_1(y) + \epsilon^2 J_2(y) + \dots)\partial_y \,.$$

However, owing to the basic nonlinearity of the cosymplectic condition (2.9) one cannot expect in general to be able to truncate the series (2.18) and have the resulting two-vector field be cosymplectic. Thus the first order perturbation

$$\begin{aligned}\dot{y} &= (J_0(y) + \epsilon\; J_1(y))(\nabla H_0(y) + \epsilon\; \nabla H_1(y)) \\ &= J_0\nabla H_0 + \epsilon(J_1\nabla H_0 + J_0\nabla H_1) + \epsilon^2 J_1\nabla H_1\end{aligned} \tag{2.19}$$

will not in general be Hamiltonian. However, since $J_0 + \epsilon\, J_1$ is still skew-symmetric, the perturbed Hamiltonian $H_0 + \epsilon\, H_1$ will always be a constant of the motion of (2.19).

LEMMA 2.5 The perturbed two-vector $\Theta_0 + \epsilon\, \Theta_1$ is cosymplectic if and only if $\Theta_1$ itself is:

$$[\Theta_1, \Theta_1] = 0 \,. \tag{2.20}$$

PROOF.

The full series (2.18) is certainly cosymplectic. (Indeed, the perturbation expansion (2.12) is in essence just a change of coordinates.) Expanding (2.9) in powers of $\epsilon$, and using (2.5,6), we find the infinite series of relations

$$[\Theta_0,\Theta_0] = 0 \;,\; 2[\Theta_0,\Theta_1] = 0 \;,\; 2[\Theta_0,\Theta_2] + [\Theta_1,\Theta_1] = 0, \dots \;, \tag{2.21}$$

resulting from the fact that (2.18) is cosymplectic for all $\epsilon$. On the other hand, the conditions that $\Theta_0 + \epsilon\, \Theta_1$ be cosymplectic are the first two of (2.21), which are automatically fulfilled, plus (2.20). This proves the lemma. (Note, by (2.21) we can replace (2.20) by $[\Theta_o,\Theta_2]=0$ .)

More generally, if (2.20) fails to hold, yet we still wish to retain the Hamiltonian property of the perturbation, we are required to include certain higher order terms in $\epsilon$ in the cosymplectic two-vector agreeing with (2.15) to first order, i.e. of the form

$$\Theta_o+\epsilon\,\Theta_1+\epsilon^2\tilde{\Theta}_2+\dots\ .$$

To accomplish this, we simplify matters by working locally to avoid global integrability conditions.

THEOREM 2.6 Let $\Theta_o,\Theta_1$ be two-vector fields satisfying (2.21) for some $\Theta_2$ . Then there exists a vector field $v_1$ and a two-vector field $\Psi_1$ constant on the leaves of the foliation induced by $\Theta_o$ such that

$$\Theta_1 = [v_1,\Theta_o] + \Psi_1\ . \tag{2.22}$$

Moreover, the two-vector field

$$\Theta^* = \exp(\epsilon v_1)_*(\Theta_o+\epsilon\,\Psi_1) \tag{2.23}$$

is cosymplectic, with expansion

$$\Theta^* = \Theta_o+\epsilon\,\Theta_1+O(\epsilon^2)\ . \tag{2.24}$$

PROOF

The existence of $v_1,\Psi_1$ follows directly from theorem 2.4. In (2.23) the $*$ refers to the action of the one-parameter (local) group of diffeomorphisms $\exp(\epsilon\, v_1)$ on the space of two-vector fields. Since the Schouten-Nijenhuis bracket is invariant under diffeomorphisms it suffices to check that $\Theta_o+\epsilon\,\Psi_1$ is cosymplectic. Clearly $[\Theta_o,\Psi_1]=0$ , so we need only check that $[\Psi_1,\Psi_1]=0$ . Using the Jacobi identity (2.7), and the third equation in (2.21),

$$\begin{aligned}-2[\Theta_o,\Theta_2] &= [\Theta_1,\Theta_1]\\ &= [[v_1,\Theta_o]\ ,\ [v_1,\Theta_o]]+2[[v_1,\Theta_o],\Psi_1]+[\Psi_1,\Psi_1]\\ &= [\Theta_o,-[v_1,[v_1,\Theta_o]]-2[v_1,\Psi_1]]+[\Psi_1,\Psi_1]\ .\end{aligned}$$

Therefore

$$[\Psi_1,\Psi_1] = [\Theta_o,\Gamma]$$

for some well defined $\Gamma$ . But since $\Psi_1$ is constant on the leaves induced by $\Theta_o$ , this latter identity is impossible unless both sides vanish. Finally, to establish (2.24) we need only notice that

$$\exp(\epsilon\, v_1)_*(\alpha) = \alpha+\epsilon[v_1,\alpha]+O(\epsilon^2)$$

for any k-vector field $\alpha$ , using the identification of the bracket with the Lie derivative in this case.

C. SOME QUALITATIVE COMPARISONS. What are some of the advantages of the Hamiltonian theory over standard perturbation methods? The most important is certainly that the Hamiltonian perturbation equations conserves energy, whereas the standard perturbation equation does not in general. (This is also true when one truncates the cosymplectic form without worrying about the bracket condition; however in this case there is no Poisson bracket.) It is easy to find two-dimensional examples in which the orbits of the unperturbed system are closed curves surrounding a fixed point. The Hamiltonian perturbation has the same orbit structure, its orbits just being perturbations of the closed curves, whereas the solutions of the standard perturbation equations slowly spiral into or away from the fixed point. In higher dimensions, KAM theory shows that "most" solutions of a small Hamiltonian perturbation of a completely integrable system remain quasi-periodic, whereas the standard perturbation can again result in spiralling behavior. At the other extreme, only Hamiltonian perturbations of an ergodic system stand a chance of being ergodic in the right way as the standard perturbation will mix up the different energy levels. Of course, both the Hamiltonian and non-Hamiltonian expansions are valid to the same order, and hence give equally valid approximations to the short-time behavior of the system. Based on the above observations, the Hamiltonian perturbation appears to do a better job modelling long-time and qualitative behavior of the system. It remains to see whether any rigorous theorem to this effect can be proved.

3. EVOLUTION EQUATIONS. The Hamiltonian theory of evolution equations is most easily developed using the formal variational calculus introduced in [5], [11]. Here we present a brief outline of the theory, including an extended discussion of multi-vectors and the Schouten-Nijenhuis bracket, the latter being new. For simplicity, we work in Euclidean space, with $x = (x_1,\dots,x_p) \in X \approx \mathbb{R}^p$ and $u = (u^1,\dots,u^q) \in U \approx \mathbb{R}^q$ denoting independent and dependent variables. The infinite jet space $J_\infty = X \times U_\infty$ is the inverse limit of the spaces $J_n = X \times U_n$ with coordinates $(x,u^{(n)}) = (x,\dots,u^i_J,\dots)$, where $u^i_J$ represents the partial derivative $\partial_J u^i = \partial_{j_1} \cdots \partial_{j_m} u^i$, $m \le n$, $\partial_j = \partial/\partial x_j$. Let $\mathcal{G}$ denote the space of smooth functions $P(x,u^{(n)})$, $n$ arbitrary, and $\wedge^k = \wedge_k T^* J_\infty$ the space of vertical k-forms, i.e. finite sums of the form

$$\omega = \Sigma\, P_{\mathcal{J}}(x,u^{(n)})\, du^{i_1}_{J_1} \wedge \dots \wedge du^{i_k}_{J_k} .$$

Vector fields are formal infinite sums

$$v = \Sigma\, Q_j \frac{\partial}{\partial x_j} + \Sigma\, Q^i_J \frac{\partial}{\partial u^i_J} \quad ,$$

with $Q_j, Q^i_J \in \mathcal{G}$. The standard formulae relating Lie derivatives, exterior derivatives and interior products extend readily to this set-up. In particular the total derivatives $D_j$ can be viewed as vector fields, hence act on $\Lambda^k$ by Lie derivatives.

The space of functionals $\mathcal{F}$ is the quotient space of $\mathcal{G}$ by the image of the total divergence, $\mathrm{Div}\, Q = D_1Q_1 + \cdots + D_pQ_p$ , $Q_j \in \mathcal{G}$ . The projection $\mathcal{G} \to \mathcal{F}$ is denoted by an integral sign: $\int P dx \in \mathcal{F}$ for $P \in \mathcal{G}$ . Similarly, the space of functional k-forms is $\Lambda_*^k = \Lambda^k / \mathrm{Div}(\Lambda^k)$ , with projection $\int \omega\, dx$ , $\omega \in \Lambda^k$ . The deRham complex $d: \Lambda^k \to \Lambda^{k+1}$ projects to a locally exact complex $d: \Lambda_*^k \to \Lambda_*^{k+1}$ . The dual space to $\Lambda_*^1$ is the space $T_0$ of <u>evolutionary vector fields</u>

$$v = Q \cdot \partial_u = \Sigma\, D^J Q_i \frac{\partial}{\partial u_J^i} \quad , \quad Q = (Q_1, \ldots, Q_q) \ ,$$

uniquely characterized (except for the trivial translational fields $\partial/\partial x_j$) by the fact that they commute with all total derivatives. Hence they act by Lie derivatives on $\Lambda_*^k$ , and again the standard differential-geometric formulae can be readily established. The exponential $\exp(\epsilon v)$ of an evolutionary vector field can be found by solving the system of evolution equations

$$\frac{\partial u}{\partial \epsilon} = Q \qquad u(x,0) = u_0(x) \ ,$$

with flow $u(x,\epsilon) = \exp(\epsilon v)[u_0]$ , in some appropriate space of functions.

The spaces of multi-vectors, dual to functional forms, are more interesting; they are <u>not</u> images of the spaces $\Lambda_k T J_\infty$ under any projection! Part of the problem is that there is no well-defined exterior product on $\Lambda_*^k$ : $\int \omega\, dx \wedge \int \theta dx \neq \int (\omega \wedge \theta) dx$ . In particular, $\Lambda_k(\Lambda_*^1) \neq \Lambda_*^k$ . We are interested in multi-linear, alternating maps on $\Lambda_*^1$ . First, recall that every functional one form is uniquely equivalent to one of the form

$$\omega_P = \int (P \cdot du) dx = \int (\Sigma P_i du^i) dx \ ,$$

(just integrate by parts). Moreover, by the exactness of the d-complex on $\Lambda_*^1$ , a function one-form $\omega_P$ is closed: $d\omega_P = 0$ , if and only if $\omega_P = d(\int Q dx)$ for some functional, which means that $P = E(Q)$ where $E$ is the Euler operator, or variational derivative, [11].

EXAMPLE 3.1. A <u>functional one-vector</u> will be determined by q-tuple of differential operators $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_q)$ , $\mathcal{D}_i = \Sigma\, Q_J^i D^J$ (finite sums, $D^J = D_{j_1} \cdots D_{j_m}$) with $Q_J^i \in \mathcal{G}$ . Given $\mathcal{D}$ , consider the linear map

$$\mathcal{D} \cdot \partial_u = \Sigma\, \mathcal{D}_i \partial_{u^i} : \Lambda_*^1 \to \mathcal{F} \ ,$$

given by $\mathcal{D} \cdot \partial_u [\int (P \cdot du) dx] = \int \mathcal{D} P\, dx = \int [\Sigma\, \mathcal{D}_i P_i] dx$ . A simple integration by parts shows that

$$\mathcal{D} \cdot \partial_u [\omega_*] = \tilde{Q} \cdot \partial_u [\omega_*] \ , \ \omega_* \in \Lambda_*^1$$

where

$$\tilde{Q}_i = \sum_J (-D)^J Q_J^i \ ,$$

so the space $\Lambda_1^*$ of functional one-vectors can be identified with $T_o$, the space of evolutionary vector fields. (Note that in the above notation we are regarding $\{\partial_{u^i}\}$ as the basis of $\Lambda_1^*$ dual to the "basis" $\{du^i\}$ of $\Lambda_*^1$.)

DEFINITION 3.2 A functional k-vector is a finite, constant coefficient linear combination of the basic k-vectors, defined as follows. Given differential operators $\mathcal{D}_1,\ldots,\mathcal{D}_k$,

$$\alpha = \mathcal{D}_1 \frac{\partial}{\partial u^{m_1}} \wedge \ldots \wedge \mathcal{D}_k \frac{\partial}{\partial u^{m_k}}, \quad 1 \le m_j \le q, \tag{3.1}$$

is defined so that for any

$$\omega_j = \int (P^j\, du)\,dx = \int (\Sigma\, P_i^j du^i)\,dx \in \Lambda_*^1, \quad j = 1,\ldots,k,$$

we have

$$\alpha(\omega_1 \wedge \ldots \wedge \omega_k) = \int \det[\mathcal{D}_i P_{m_i}^j]\,dx,$$

the determinant being of a $k \times k$ matrix with the $(i,j)$-entry indicated.

EXAMPLE 3.3 Suppose $q = 1$. A functional two vector is of the form

$$\alpha = \mathcal{D}_1 \partial_u \wedge \mathcal{D}_2 \partial_u,$$

with

$$\alpha(\omega_P \wedge \omega_Q) = \int (\mathcal{D}_1 P\, \mathcal{D}_2 Q - \mathcal{D}_2 P\, \mathcal{D}_1 Q)\,dx = \int (P \mathcal{D} Q)\,dx,$$

where $\mathcal{D} = \mathcal{D}_1^* \mathcal{D}_2 - \mathcal{D}_2^* \mathcal{D}_1$ is skew adjoint ($\mathcal{D}^* = -\mathcal{D}$). Thus every functional two vector is uniquely equivalent to one of the form $\frac{1}{2}\, \partial_u \wedge \mathcal{D}\, \partial_u$ for $\mathcal{D}$ skew-adjoint. This integration by parts argument easily generalizes to functional k-vectors.

Once the basic definition of a functional multi-vector has been properly presented, the definition and properties of a Poisson structure readily adapt to this infinite dimensional situation. In particular, the definition 2.1 of the Schouten-Nijenhuis bracket carries over with no change, as it does not rely on the exterior derivative $d$. (This is the definition used by Gel'fand and Dorfman, [5], in the special case $k = \ell = 2$, although they appear to omit the vital assumption that the one-forms $\omega_j$ be closed.) Thus a skew-adjoint differential operator $\mathcal{D}$ is cosymplectic if and only if the two-vector $\Theta = \partial_u \wedge \mathcal{D}.\partial_u$ satisfies $[\Theta,\Theta] = 0$. In particular, if $\mathcal{D}$ does not depend on $u$, it is automatically cosymplectic.

EXAMPLE 3.4 Consider the KdV equation in the form

$$u_t = u_{xxx} + uu_x.$$

This is Hamiltonian in two ways:

$$u_t = J_0 \nabla H_1 = J_1 \nabla H_0,$$

in which $\nabla$ denotes the variational derivative with respect to $u$,

$$H_0 = \int \frac{1}{2} u^2 dx \ , \quad H_1 = \int ( \frac{1}{6} u^3 - \frac{1}{2} u_x^2) dx \ ,$$

and

$$J_0 = D_x \qquad , \quad J_1 = D_x^3 + \frac{2}{3} u D_x + \frac{1}{3} u_x \ .$$

The first operator is cosymplectic since it does not depend on $u$ ; the proof that $J_1$ is cosymplectic is not difficult and can be found in [5], [9], [11].

The only part of the theory that has not so far been adapted to this context is the exactness result of the $\delta$-complex in theorem 2.4. We still have $\delta \circ \delta = 0$ , and I strongly suspect that some version of this theorem is true, but do not have a proof. Thus in the perturbation theorem 2.6, one cannot at present be guaranteed the existence of a vector field $v_1$ and two-vector $\Psi_1$ , but in all the simple examples I have looked at, $v_1$ is easy to find and $\Psi_1$ is invariably zero.

Finally, we need to discuss change of variables. For simplicity, assume $p = q = 1$ , but the result readily generalizes. Given a change of variables $u = F(v, v_x, \ldots)$ (e.g. the Miura transformation $u = v^2 + v_{xx}$ for the KdV) define the differential operator

$$D_F^* = \frac{\partial F}{\partial v} - D_x \frac{\partial F}{\partial v_x} + D_x^2 \frac{\partial F}{\partial v_{xx}} - \cdots \ ,$$

so $D_F^*$ is the adjoint of the Frechet derivative of $F$ . Then the functional multi-vectors transform according to the basic rule

$$\frac{\delta}{\delta v} = D_F^* \frac{\delta}{\delta u} \qquad (3.2)$$

applied to (3.1). For example,

$$\delta_v \wedge \mathcal{D}\, \delta_v = D_F^* \delta_u \wedge \mathcal{D}\, D_F^* \delta_u = \delta_u \wedge D_F \mathcal{D}\, D_F^* \delta_u \ .$$

To see this, a one-form clearly transforms by

$$\omega_P = \int (P(u, u_x, \ldots) du] dx = \int [P(F, D_x F, \ldots) dF] dx$$
$$= \int P(\frac{\partial F}{\partial v} dv + \frac{\partial F}{\partial v} dv_x + \ldots) dx = \int [D_F^*(P) dv] dx \ .$$

From this, (3.2) follows by duality. (Often, as $D_F^*$ depends on $v$ , (3.2) is not directly useful except in conjuction with some perturbation expansion!)

4. WATER WAVES. The water wave problem means the free boundary problem of irrotational, inviscid, incompressible, ideal fluid flow with gravity. We also omit surface tension effects, although this is not essential - see [13]. The model equations are for long, small amplitude, two-dimensional waves over a shallow horizontal bottom. The basic equations, and subsequent derivation of the KdV equation, are given in Whitham, [17, pp. 464-6], whose notation we use here. After rescaling, the problem takes the form

$$\beta \varphi_{xx} + \varphi_{yy} = 0 \ , \qquad 0 < y < 1 + \alpha \eta \ , \qquad (4.1)$$

$$\varphi_y = 0 , \qquad y = 0 , \tag{4.2}$$

$$|\nabla\varphi| \to 0 , \qquad |x| \to \infty , \tag{4.3}$$

$$\left.\begin{aligned} &\varphi_t + \tfrac{1}{2}\alpha\varphi_x^2 + \tfrac{1}{2}\alpha\beta^{-1}\varphi_y^2 + \eta = 0 , \quad (4.4)\\ &\eta_t = \beta^{-1}\varphi_y - \alpha\eta_x\varphi_x . \quad (4.5)\end{aligned}\right\} y = 1 + \alpha\eta$$

Here $x$ is the horizontal and $y$ the vertical coordinate, $\varphi(x,y,t)$ the velocity potential, $1+\alpha\eta(x,t)$ the surface elevation. The two small parameters are $\alpha = a/h$, the ratio of wave amplitude to undisturbed water depth, and $\beta = h^2/\ell^2$, the square of the ratio between depth and wave length.

A. NON-HAMILTONIAN PERTURBATIONS. In Boussinesq's method, the first step is to solve the elliptic boundary value problem (4.1-3) in terms of the potential $\psi = \psi^\theta(x,t) = \varphi(x,\theta,t)$ at depth $0 \le \theta \le 1$, giving the series solution

$$\varphi = \psi + \frac{1}{2}\beta(\theta^2 - y^2)\psi_{xx} + \frac{1}{24}\beta^2(5\theta^4 - 6\theta^2y^2 + y^4)\psi_{xxxx} + \dots . \tag{4.6}$$

(We will not worry about problems concerning the precise domains of definition of the functions - see Lebovitz, [7].) Substituting the series (4.6) into (4.4,5), differentiating the former with respect to $x$ and truncating to first order leads to the following version of the Boussinesq system:

$$\begin{aligned} 0 &= u_t + \eta_x + \alpha uu_x + \tfrac{1}{2}\beta(\theta^2 - 1)u_{xxt} , \\ 0 &= \eta_t + u_x + \alpha(\eta u)_x + \tfrac{1}{6}\beta(3\theta^2 - 1)u_{xxx} , \end{aligned} \tag{4.7}$$

in which $u = u^\theta(x,t) = \varphi_x(x,\theta,t)$ is the horizontal velocity at depth $\theta$. The basic system (4.7) can be modified by resubstituting, expanding and truncating again; for instance since to leading order $u_t = -\eta_x$, the term $u_{xxt}$ in the first equation can be replaced by $-\eta_{xxx}$ to yield a purely evolutionary system. See Bona and Smith, [3], for a complete discussion of the possibilities, and the companion paper [13] for the second order terms in the expansion.

To specialize to unidirectional waves, one looks for an expansion of the form $\eta = u + \alpha A + \beta B + \dots$ such that the two equations in (4.7) become the same up to the requisite order. To first order,

$$\eta = u + \frac{1}{4}\alpha u^2 + \frac{1}{6}\beta(3\theta^2 - 2)u_{xx} , \tag{4.8}$$

leading to the KdV equation

$$u_t + u_x + \frac{3}{2}\alpha uu_x + \frac{1}{6}\beta u_{xxx} = 0 , \tag{4.9}$$

independent of depth $\theta$. Alternatively, one can express $u$ in terms of $\eta$, leading to the same equation for $\eta$, (1.7). Again one can play the same games as with the Boussinesq system, so, for instance, since $u_t = -u_x$ to leading order, we can replace $u_{xxx}$ by $-u_{xxt}$, yielding the BBM equation, [1], whose dispersion relation offers some advantages over the KdV model.

B. HAMILTONIAN MODELS. In Zakharov's Hamiltonian formulation of the water wave problem, the basic variables are the surface elevation $\eta$ and the potential on the surface $\varphi_S(x,t) = \varphi(x,1+\alpha\eta(x,t),t)$, the values of $\varphi$ within the fluid being determined from $\varphi_S$ by solving the auxilliary boundary value problem (4.1-3), cf. [2]. The Hamiltonian is the energy

$$H = \int_S \{ \tfrac{1}{2} \varphi(\beta^{-1}\varphi_y - \alpha\eta_x\varphi_x) + \tfrac{1}{2} \eta^2 \}dx . \qquad (4.10)$$

(The S on the integral means all terms are evaluated on the free surface $y = 1+\alpha\eta$.) The water wave probelm (4.1-5) is now in canonical form

$$\frac{\partial\varphi_S}{\partial t} = -\frac{\delta H}{\delta\eta} , \quad \frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\varphi_S} . \qquad (4.11)$$

First consider bidirectional Boussinesq systems. Substituting (4.6) into (4.10), and truncating, we get the first order expansion

$$H^{(1)} = \int_{-\infty}^{\infty} [ \tfrac{1}{2} u^2 + \tfrac{1}{2} \eta^2 + \tfrac{1}{2}\alpha\eta u^2 + \tfrac{1}{6} \beta(2-3\theta^2)u_x^2]dx \qquad (4.12)$$

for the energy. For the symplectic version of the Boussinesq system, we expand the two form $\Omega = d\eta \wedge d\varphi_S$ appropriate to (4.11), leading to

$$\Omega^{(1)} = d\eta \wedge (d\varphi + \tfrac{1}{2} \beta(\theta^2-1)d\varphi_{xx}) = d\eta \wedge (D_x^{-1} + \tfrac{1}{2} \beta(\theta^2-1)D_x)du . \quad (4.17)$$

(We omit the integral sign from $\Omega^{(1)}$ for simplicity.) This yields

$$0 = u_t + \eta_x + \alpha uu_x + \tfrac{1}{2} \beta(\theta^2-1)u_{xxt} , \qquad (4.13)$$
$$0 = \eta_t + u_x + \alpha(u\eta)_x + \tfrac{1}{2} \beta(\theta^2-1)\eta_{xxt} + \beta(\theta^2 - \tfrac{2}{3})u_{xxx} .$$

(We have differentiated both equations with respect to $x$ here.). Note that the "symplectic Boussinesq" system (4.13) agrees to first order with (4.7) after manipulations similar to those discussed earlier.

Alternatively, we can perturb the cosymplectic two-vector $\Theta = \delta_\eta \wedge \delta_{\varphi_S}$. Using (4.6) again, from (3.2) we find

$$\Theta^{(1)} = \delta_\eta \wedge \{D_x + \tfrac{1}{2} \beta(1-\theta^2)D_x^3\}\delta_u , \qquad (4.14)$$

which is cosymplectic since the underlying operator is constant coefficient. This yields the "cosymplectic Boussinesq" system

$$0 = u_t + \eta_x + \alpha uu_x + \tfrac{1}{2} \beta(1-\theta^2)\eta_{xxx} + \tfrac{1}{4}\alpha\beta(1-\theta^2)(u^2)_{xxx} , \qquad (4.15)$$
$$0 = \eta_t + u_x + \alpha(\eta u)_x + \tfrac{1}{6} \beta(3\theta^2-1)u_{xxx} + \tfrac{1}{2}\alpha\beta(1-\theta^2)(\eta u)_{xxx} - \tfrac{1}{3} \beta(3\theta^4-5\theta^2+2)u_{xxxxx},$$

differing from (4.7) by the inclusion of quadratic terms. The special case $\theta = 1$ is of special note, as remarked by Broer [4], since to first order the expansion (4.6) is equivalent to a canonical expansion in the variables $\eta, \varphi_S$:

the Hamiltonian systems (4.13,14) reduce to versions of the usual system (4.7). The more general ($\theta \neq 1$) Hamiltonian Boussinesq systems are new.

As for unidirectional models, since we are still matching the two equations to first order in the Boussinesq system, the definition (4.8) of the submanifold of unidirectional solutions remains the same. Thus we need only substitute (4.8) into the energy and the (co-) symplectic form and expand to first order. The appropriate Hamiltonian is

$$\tilde{H}^{(1)} = \int_{-\infty}^{\infty} [u^2 + \tfrac{3}{4}\alpha u^3 + (\tfrac{2}{3} - \theta^2)\beta u_x^2] dx \,, \qquad (4.16)$$

where we have integrated one term by parts. For the cosymplectic model, note first that from (4.8)

$$\frac{\delta}{\delta u} = [1 + \tfrac{1}{2}\alpha u + (\tfrac{1}{2}\theta^2 - \tfrac{1}{3})\beta D_x^2] \frac{\delta}{\delta \eta} \,,$$

cf. (3.2), hence to first order

$$\frac{\delta}{\delta \eta} = (1 - \tfrac{1}{2}\alpha u + (\tfrac{1}{3} - \tfrac{1}{2}\theta^2)\beta D_x^2) \frac{\delta}{\delta u} \,.$$

Therefore, substituting into (4.14), we find

$$\tilde{\Theta}^{(1)} = \partial_u \wedge (D_x - \tfrac{1}{4}\alpha(uD_x + D_x u) + (\tfrac{5}{6} - \theta^3)\beta D_x^3)\partial_u \,, \qquad (4.17)$$

which can be proved to be cosymplectic, [5], [9]. Combining (4.16,17) we find the following "cosymplectic version" of the Korteweg-de Vries equation.

$$u_t + [D_x - \tfrac{1}{4}\alpha(uD_x + D_x u) + (\tfrac{5}{6} - \theta^2)\beta D_x^3][u + \tfrac{9}{8}\alpha u^2 + (\theta^2 - \tfrac{2}{3})\beta u_{xx}] = 0 \,,$$

or, in detail,

$$u_t + u_x + \tfrac{3}{2}\alpha u u_x + \tfrac{1}{6}\beta u_{xxx} + \tfrac{1}{18}\beta^2(-18\theta^4 + 27\theta^2 - 10)u_{xxxxx} + \qquad (4.18)$$
$$+ (\tfrac{53}{24} - \tfrac{11}{4}\theta^2)\alpha\beta u u_{xxx} + (\tfrac{139}{24} - 7\theta^2)\alpha\beta u_x u_{xx} - \tfrac{45}{32}\alpha^2 u^2 u_x = 0 \,.$$

(In deriving (4.18) we have multiplied by $\frac{1}{4}$ - this is rigorously justified since we are restricting the system to a submanifold.)

The symplectic form, which resembles more closely the BBM equation, is more complicated. We find

$$\tilde{\Omega}^{(1)} = du \wedge [D_x^{-1} + \tfrac{1}{4}\alpha(uD_x^{-1} + D_x^{-1} u) + (\theta^2 - \tfrac{5}{6})\beta D_x^3] du \,,$$

hence, formally,

$$[D_x^{-1} + \tfrac{1}{4}\alpha(uD_x^{-1} + D_x^{-1} u) + (\theta^2 - \tfrac{5}{6})D_x](u_t) + u + \tfrac{9}{8}\alpha u^2 + (\theta^2 - \tfrac{2}{3})\beta u_{xx} = 0 \,.$$

To convert this into a bona-fide differential equation, recall $u = \partial_x \psi$, and differentiate:

$$\psi_{xt} + \tfrac{1}{2}\alpha\psi_x\psi_{xt} + \tfrac{1}{4}\alpha\psi_{xx}\psi_t + (\theta^2 - \tfrac{5}{6})\beta\psi_{xxxt} + \psi_{xx} + \tfrac{9}{4}\alpha\psi_x\psi_{xx} + (\theta^2 - \tfrac{2}{3})\beta\psi_{xxxx} = 0.$$

This example illustrates well the previous remarks that the symplectic perturbation is easier to handle theoretically, but the resulting equations are much more unpleasant.

There are a lot of open questions concerning these models, most of which are probably only amenable to numerical investigation. What are their solitary-wave solutions like, and how do they interact? (Only the $\eta$-equation (1.8) can be solved "explicitly" in terms of a hyperelliptic integral.) How do the solutions compare with those of the KdV or BBM equation? In particular, do they give any truer indication of the qualitative or long time behavior of water waves? Does the dependence of (4.18) on the depth $\theta$ have any relevance to the breaking of water waves, in that solitary waves of the same amplitude may move a different speeds at different depths, thereby setting up some kind of shearing instability? (See also [13].) All those questions must await further research.

5. COMPLETE INTEGRABILITY. We now turn to the question of why the KdV equation happens to be Hamiltonian. Returning to the general set-up, as summarized in (1.4,5), we see that one possibility for (1.5) to be Hamiltonian is if the first order terms are multiples of each other:

$$J_1 \nabla H_o = \sigma\, J_o \nabla H_1 . \tag{5.1}$$

This of course cannot be expected in general, but if it does happen, the situation can be handled by the theorem of Magri on complete integrability of bi-Hamiltonian systems, [9], [5].

THEOREM 5.1 Suppose a system $\dot{x} = K_1(x)$ can be written in Hamiltonian form in two distinct ways: $K_1 = J_o \nabla H_1 = J_1 \nabla H_o$. Assume also that the two Hamiltonian structures are compatible, meaning that $J_o + \mu\, J_1$ is cosymplectic for all constant $\mu$. Then the recursion relation $K_n = J_o \nabla H_n = J_1 \nabla H_{n-1}$ defines an infinite sequence of commuting bi-Hamiltonian flows $\dot{x} = K_n(x)$, with mutually conserved Hamiltonians $H_n(x)$ in involution (with respect to either the $J_o$ or $J_1$ Poisson bracket). (One also needs to assume that $J_o$ in the recursion relation always invertible, but this usually holds.)

Thus, in this special case, both the noncanonical perturbation equation (1.5) and the cosymplectic version (1.6) are linear combinations of the completely integrable flows $K_o$, $K_1$, $K_2$, and hence, provided "enough" of the Hamiltonians $H_n$ are independent, are both completely integrable Hamiltonian systems.

For the water wave expansion, in the Korteweg-de Vries model the $O(\alpha,\beta)$-terms are in the right ratio only at the "magic" depth $\theta = \sqrt{11/12}$, and for this depth (4.18) is a linear combination of a fifth, third and first order KdV equation. For more general $\theta$, one must fudge the condition (5.1) slightly to obtain complete integrability.

Nevertheless, this leads to an intriguing speculation. Does condition (5.1) often hold in the perturbational derivation of model equations from conservative physical systems? If true, it would provide a good explanation of the common feature of many systems that in the zeroth order perturbation one has linear equations, and in the first order perturbation the equations are nonlinear, but completely integrable soliton equations. Presumably the second order expansion leads to nonintegrable models with some chaotic components. A good place to check this is in Zakharov's derivation of the nonlinear Schrodinger equation as the modulational equation for periodic water waves, [18].

REFERENCES

1. T.B. Benjamin, J.E. Bona and J.J. Mahony, "Model equations for long waves in nonlinear dispersive systems", Phil. Trans Roy. Soc. London A 272 (1972) 47-78.

2. T.B. Benjamin and P.J. Olver, "Hamiltonian structure, symmetries and conservation laws for water waves", J. Fluid Mech. 125 (1982) 137-185.

3. J.L. Bona and R. Smith, "A model for the two-way propagation of water waves in a channel," Math. Proc. Camb. Phil. Soc. 79 (1976) 167-182.

4. L.J.F. Broer, "Apprximate equations for long water waves", Appl. Sci. Res. 31 (1975) 377-395.

5. I.M. Gelfand and I. Ya. Dorfman, "Hamiltonian operators and related algebraic structures", Func. Anal. Appl. 13 (1979) 13-30.

6. J. Kevorkian and J.D. Cole, Perturbation Methods in Applied Mathematics, Springer-Verlag, New York, 1981.

7. N. Lebovitz, "Perturbation expansions on perturbed domains", SIAM Rev. 24 (1982) 381-400.

8. A.Lichnerowicz, "Les varietes de Poisson et leurs algebres de Lie Associees" J. Diff. Geom. 12 (1977) 253-300.

9. F. Magri, "A simple model of the integrable Hamiltonian equation", J. Math. Phys. 19 (1978) 1156-1162.

10. A. Nijenhuis, "Jacobi-type identities for bilinear differential concomitants of certain tensor fields", Proc. Kon. Ned. Akad. Wet. 58A (1955) 390-405

11. P.J. Olver, "On the Hamiltonian structure of evolution equations", Math. Proc. Camb. Phil. Soc. 88 (1980) 71-88.

12. P.J. Olver, "Conservation laws of free boundary problems and the classification of conservation laws for water waves", Trans. Amer. Math. Soc. 277 (1983) 353-380.

13. P.J. Olver, "Hamiltonian and non-Hamiltonian models for water waves", Proceedings, 5th Symposium on Trends in Applications of Pure Mathematics to Mechanics, to appear.

14. H. Segur, "Solitons and the inverse scattering transform", Topics in Ocean Physics 80 (1982) 235-277.

15. C.L. Siegel and J.K. Moser, Lectures on Celestial Mechanics, Springer-Verlag, New York, 1971.

16. A. Weinstein, "The local structure of Poisson manifolds", CPAM preprint #97, Berkeley, 1982.

17. G.B. Whitham, Linear and Nonlinear Waves, Wiley-Interscience, New York, 1974.

18. V.E. Zakharov, "Stability of periodic waves of finite amplitude on the surface of a deep fluid", J. Appl. Mech. Tech. Phys. 2 (1968) 190-194.

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