Higher Order Models for Water Waves

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In this note I would like to discuss some new perspectives on the construction of model equations for physical systems, with particular emphasis on the role of Hamiltonian structure and solitary waves. The ideas, which have a wide applicability, will be presented in the specific context of higher order model equations for water waves, valid in the shallow water regime. Details can be found in the references [5], [8], [9].

We begin with the standard free boundary problem for incompressible, irrotational fluid flow in a channel. We restrict attention to two-dimensional motions, taking x as the horizontal and y as the vertical coordinate, the (flat — for simplicity) bottom at $y = 0$, and the free surface at $y = h + \eta(x,t)$, where h is the undisturbed fluid depth. In terms of the velocity potential $\varphi(x, y, t)$, the full water wave problem takes the well-known form

$$\varphi_{xx} + \varphi_{yy} = 0, \qquad 0 < y < h + \eta(x, t), \tag{1}$$

$$\varphi_y = 0, \qquad y = 0, \tag{2}$$

$$\left.\begin{aligned} \varphi_t + \tfrac{1}{2}\varphi_x^2 + \tfrac{1}{2}\varphi_y^2 + g\,\eta &= 0, \\ \eta_t &= \varphi_y - \eta_x\,\varphi_x, \end{aligned}\right\} \qquad y = h + \eta(x, t), \tag{3}$$

where g is the gravitational constant. For simplicity, I have omitted surface tension, although this can be readily incorporated in both the full equations as well as the models discussed below. In the standard Boussinesq (shallow water) approximation to the water wave problem, one begins by introducing the small parameters

$$\varepsilon = \frac{a}{h} \qquad \kappa = \frac{h^2}{\ell^2} = O(\varepsilon),$$

where a is the wave amplitude, ℓ the wave length. The equations (1-3) are rescaled according to $(x, y, t, \eta, \varphi) \longmapsto (\ell x, h\,y, a\,\eta, c^{-1} g\,a\,\ell\,\varphi)$, where $c = \sqrt{g\,h}$ is the (linearized) wave speed. The boundary value problem (1-2) (in the rescaled variables) is then solved for the potential, and the resulting series expansion substituted into the free surface conditions (3). The resulting bidirectional system of equations is typically expressed in terms of the surface elevation $\eta(x, t)$ and the horizontal velocity $u(x, t) = \varphi_x(x, \theta\,h, t)$ at a fraction $0 \le \theta \le 1$ of the undisturbed depth. Truncating to some specified order, one finds a variety of Boussinesq-type systems of model equations for waves propagating in both directions. To specialize to waves moving in a single direction, one resticts to an "approximate" unidirectional function surface, and re-expands the system. The result, to first order, is the celebrated Korteweg-deVries approximation

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon \eta \eta_x - \frac{1}{2}\kappa \eta_{xxx} = 0, \qquad (4)$$

which we have written in terms of η, although the horizontal velocity u satisfies the same equation to first order. (The higher order approximations, though, are different, [8], [9].)

The water wave problem was shown by Zakharov, [11], to be a Hamiltonian system with the total energy serving as the required Hamiltonian functional. Also, as is well known, [10; Chapter 7], the Korteweg-deVries equation (4) has two distinct Hamiltonian structures — indeed, the fact that it is a biHamiltonian system implies, by Magri's theorem, that it is, in fact, a completely integrable Hamiltonian system in the sense that it has an infinite sequence of independent conservation laws and associated (generalized) symmetries. On the other hand, in collaboration with Benjamin, [2], [7], the full water wave problem (1–3) was shown to possess precisely eight (seven if surface tension is included) local conservation laws, corresponding to nine (eight if surface tension is included) independent one-parameter symmetry groups. (The "extra" scaling group is not being "canonical", and thus does not lead to a conserved quantity.) In the course of trying to understand which of the Korteweg-deVries conservation laws correspond to true water wave laws, I found, much to my surprise, that *neither* of the Hamiltonian structures for the Korteweg-deVries equation arises directly from the Hamiltonian structure for the full water wave problem. The crucial feature is that the Boussinesq expansion is *not* canonical, and so cannot lead to a first order Hamiltonian approximation. Indeed, the first order truncation of the water wave energy functional is *not* one of the conserved quantities for the Korteweg-deVries model (4).

The easiest way to appreciate this phenomenon is through an appeal to a simple form of "noncanonical perturbation theory". Consider a Hamiltonian system

$$\frac{dv}{dt} = J(v)\,\nabla H(v), \qquad (5)$$

which, in our application, would represent the full water wave problem. In standard perturbation theory, which ignores any additional structure the model may possess — such as Hamiltonian structure, conservation laws, etc., one derives approximate models by substituting the physically motivated perturbation expansion $v = u + \varepsilon\,\varphi(u) + \varepsilon^2\,\psi(u) + \ldots$ into the system, and then truncating the resulting system to some desired order in ε. However, this procedure must now be correlated with the Hamiltonian structure of (5). Indeed, if the expansion is not canonical then we must not only expand the Hamiltonian function(al) $H(v) = H_0(u) + \varepsilon\,H_1(u) + \varepsilon^2\,H_2(u) + \ldots$, but also the Hamiltonian operator $J(v) \longmapsto J_0(u) + \varepsilon\,J_1(u) + \varepsilon^2\,J_2(u) + \ldots$. If we truncate to just first order, the resulting perturbed system

$$\frac{du}{dt} = J_0\,\nabla H_0 + \varepsilon\,\{\,J_1\,\nabla H_0 + J_0\,\nabla H_1\,\} \qquad (6)$$

is not Hamiltonian in any obvious way. Indeed, as was remarked above in the context of the water wave problem, the first order truncation of the Hamiltonian, $H_0(u) + \varepsilon\,H_1(u)$, is not a conserved quantity for (6). A *Hamiltonian* first order approximation to (5) can be given by retaining some (but not all) of the second order terms:

$$\frac{du}{dt} = [\,J_0 + \varepsilon J_1\,]\,\nabla\,[\,H_0 + \varepsilon H_1\,]$$
$$= J_0\,\nabla H_0 + \varepsilon\,\{\,J_1\,\nabla H_0 + J_0\,\nabla H_1\,\} + \varepsilon^2 J_1\,\nabla H_1\,. \tag{7}$$

(Technically, since the Jacobi identity imposes a quadratic constraint on the Hamiltonian operator, the combination $J_0 + \varepsilon J_1$ is not guaranteed to be Hamiltonian; however, in many cases, including the Korteweg-deVries approximation, this is not a problem.)

In certain situations, the first order model (6) may turn out to be Hamiltonian "by accident". One way in which this can occur is if the two terms in braces are constant multiples of each other, so $J_1\,\nabla H_0 = \lambda\, J_0\,\nabla H_1$. If this happens, the associated first order approximation (6) is in fact biHamiltonian, and hence completely integrable. This observation, which does apply to the Korteweg-deVries equation, can be offered as an explanation of the surprising prevalence of completely integrable soliton equations appearing as models for a wide variety of complicated nonlinear physical systems — it is because they arise from non-canonical perturbation expansions of Hamiltonian systems, while, at the same time, retaining some form of Hamiltonian structure.

Both the Hamiltonian models constructed using the preceding non-canonical perturbation theory, as well as the complete second order models for unidirectional shallow water waves, are evolution equations of the general form

$$u_t + cu_x + \varepsilon\,\{\mu u_{xxx} + 2quu_x\} + \varepsilon^2\,\{\alpha u_{xxxxx} + \beta uu_{xxx} + \delta u_x u_{xx} + 3ru^2 u_x\} = 0. \tag{8}$$

Such models arise in a wide variety of other physical situations, including wave interactions, elastic media with microstructure, and soliton. The precise formulas for the coefficients vary, and I refer the reader to [5] for a survey of (most of the) models of this form in current use, including all water wave models, both first order Hamiltonian, and second order, with and without surface tension. Some analytical results are known for such fifth order models, although much remains unknown. In particular, numerical solutions have not, as far as I know, been implemented. Here I would like to comment on some recent results concerning the existence of solitary wave solutions. For small ε, equation (8) should be regarded as a perturbation of the Korteweg-deVries model, cf. [4], obtained by omitting the $O(\varepsilon^2)$ terms entirely. Therefore, one would expect that the model admits a one-parameter family of solitary wave solutions which would look like small perturbations of the standard sech^2 solitons of the Korteweg-deVries equation. This point of view would be additionally bolstered by the fact that the full water wave model also admits a family of solitary wave solutions, up to a wave of maximal height which satisfies the Stokes' phenomena of exhibiting a 120° corner. Indeed, Kunin, [6], introduces models, using only $\varepsilon^2\,\beta\,u\,u_{xxx}$ in the second order terms, which *do* have solitary waves of maximal height, although these waves have a 0° cusp. Remarkably, the expectation of solitary wave solutions is not correct, and, indeed, most of the fifth order models (8) do not have the expected property. Indeed, in joint work with S. Kichenassamy, [5], it was proved that, subject to a technical analyticity hypothesis, the only models (8) which admit a one-parameter family of solitary wave solutions which, in the $\varepsilon \longrightarrow 0$ limit, reduce to Korteweg-deVries solitons, are the models which admit a one-parameter family of exact sech^2 solitary wave solutions!

In the case of the water wave models, only at the "magic depth" $\theta = \sqrt{\frac{11}{12} - \frac{3}{4}\tau}$ do the higher order models possess solitary wave solutions. In this case, the Hamiltonian model is, in fact a fifth order Korteweg-deVries equation, having soliton solutions. (The many remakable properties of the models at this depth has been noted before, [8], [9], but no explanation is as yet forthcoming.) This fact brings into sharp focus our preconceived notions concerning the construction of model equations for solitary wave phenomena. According to work of Friedrichs and Hyers, [4], and Amick and Toland, [1], the full water wave problem possesses a one-parameter family of exact solitary wave solutions, up to a wave of maximal height. The Korteweg-deVries equation also has a one-parameter family of exact sech^2 solitary wave solutions (of all amplitudes), which, for small amplitudes, are fairly good approximations to the exact solitary water waves, [3]. However, if one tries to improve the approximation by including higher order terms, or maintaining Hamiltonian structure, one in fact does much worse, destroying the solitary wave solutions entirely. At first glance, this is very surprising. However, what should really be surprising is that the models to a physical system have solitary wave solutions in the first place! Indeed, since the $O(\varepsilon^k)$ model is (presumably) only valid for time $O(\varepsilon^{-k})$, the fact that it has a solitary wave solution valid for all time is certainly not guaranteed, even if the full physical system has solitary wave solutions. In fact, all we have a right to expect is a solution which looks like a solitary wave for a long time, but then, possibly, has some completely different behavior, e.g. dissipation, break-up, blow-up, or something else, which is irrelevant for the physical system being modelled. The fact that almost all popular models for wave phenomean *do* have solitary wave solutions is, therefore, an accident that has lulled us into a false sense of security.

The details of the proof of this result are to be found in [5]. The method is to first determine which of the models have exact sech^2 solitary wave solutions. Substituting the explicit formula $u(x,t) = a\,\mathrm{sech}^2 \lambda(x - c\,t)$ into the model (8), we find that the coefficients $a, \lambda, c,$ must satisfy the compatibility conditions

$$\alpha\rho^2 + \mu\rho + (p - c) = 0, \quad 15\,\alpha\rho\sigma + 2(\beta+\gamma)\rho + 3\mu\sigma + 2q = 0, \qquad 15\,\alpha\sigma^2 + (3\beta + 2\gamma)\sigma + 2\rho = 0, \tag{9}$$

where $\rho = 4\lambda^2$, $\sigma = -4\lambda^2/a$. Note that $\sigma < 0$ gives a wave of elevation, $\sigma > 0$ a wave of depression. Analysis of the algebraic system (9) proves that a general fifth order model (8) possesses either $0, 1, 2, \infty$, or $\infty + 1$ explicit sech^2 solitary wave solutions, where ∞ denotes a one-parameter family of such solutions. In particular, the model admits a one-parameter family of explicit sech^2 solitary wave solutions if and only if the coefficients satisfy the two algebraic relations

$$(\beta+\gamma)\mu = 5\,q\,\alpha, \qquad 15\,\alpha\,r = \beta(\beta+\gamma). \tag{10}$$

It should be remarked that these are not enough to guarantee that the model is completely integrable! Interestingly, there may be more than one sech^2 solitary wave solution for subcritical wave speeds if $\alpha\mu < 0$.

In order to prove non-existence, we first construct a suitable solitary wave tail (i.e. for $|x| \longrightarrow \infty$) by proving the convergence of the appropriate formal series solution. On the other hand, there exists a formal expansion of any solitary wave solutions in a series in power of sech, which, if it converged, would actually give a solitary wave solution. However, except when the coefficients of the equation satisfy the algebraic constraints (10) guaranteeing a family of sech^2 solitary wave solutions, the recurrence relations for the formal series solutions introduce poles in the coefficients in the complex ε-plane converging to $\varepsilon = 0$, which serve to violate our underlying analyticity hypothesis. This gives a brief outline of the essence of the proof — the reader can find the full details in [5].

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