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VII

HOW TO FIND THE SYMMETRY GROUP OF A DIFFERENTIAL EQUATION

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Before applying group-theoretic methods to the construction of the bifurcation equations of some system of partial differential equations, it is of course necessary to know a group of symmetries of the equations in question. In this chapter we describe a useful, systematic computational method for finding the group of symmetries of a given system of partial differential equations. This method essentially dates back to the original investigations of Sophus Lie; other modern treatments of this material may be found in references [2] and [5]. The groups under consideration will be local Lie groups transforming both the independent and dependent variables of the differential equations. Thus, we will leave aside any questions on the discrete symmetries of the equation. The reason for this restriction is to take full advantage of the infinitesimal methods available in Lie group theory. For simplicity, we will work in

Euclidean spaces although many of the results hold equally well for differential equations on manifolds. (See [4] for a rigorous exposition.)

1. Local Transformation Groups

Definition 7.1. A local group of transformations acting on $\mathbb{R}^n$ consists of a Lie group G, an open set V, with $\{e\} \times \mathbb{R}^n \subset V \subset G \times \mathbb{R}^n$, and a smooth ($C^\infty$) map $\Phi : V \to \mathbb{R}^n$, satisfying the conditions

i) $\Phi(e, x) = x$ for $x \in \mathbb{R}^n$

ii) $\Phi(g, \Phi(h, x)) = \Phi(g \cdot h, x)$

whenever $g, h \in G$, $x \in \mathbb{R}^n$, and $(h, x), (g, \Phi(h, x)), (g \cdot h, x) \in V$. (In other words, this equation holds whenever both sides make sense.)

If $V = G \times \mathbb{R}^n$, then the group action of G is global. In general, however, for each $x \in \mathbb{R}^n$, only those group elements in a neighborhood of e in G (depending on x) can transform x. Examples of local transformation groups whose action cannot be globalized arise naturally as symmetries of partial differential equations. Note further that the action of the group is not restricted to be linear.

Associated with a local transformation group are its infinitesimal generators. These are vector fields on $\mathbb{R}^n$ defined as follows: Let $\mathfrak{g}$ denote the Lie algebra of G. Given $\alpha \in \mathfrak{g}$, let $\exp(t\alpha)$ be the one-parameter subgroup of G generated by α. The corresponding infinitesimal generator on $\mathbb{R}^n$ is the vector field $\varphi(\alpha)$ whose value at $x \in \mathbb{R}^n$ is

$$\varphi(\alpha)\Big|_x = \frac{d}{dt}\Big|_{t=0} \Phi(\exp(t\alpha), x) . \tag{7.1}$$

If $x = (x_1, \ldots, x_n)$ are coordinates on $\mathbb{R}^n$, we shall adopt the differential-geometric notation

$$\vec{v} = \xi^1(x) \frac{\partial}{\partial x_1} + \ldots + \xi^n(x) \frac{\partial}{\partial x_n}$$

for vector fields on $\mathbb{R}^n$. Thus if $\Phi(\exp(t\alpha), x) = (\Phi^1(t,x), \ldots, \Phi^n(t,x))$, then the infinitesimal generator has coordinate functions $\xi^i(x) = \frac{d}{dt} \Phi^i(t,x)\Big|_{t=0}$. Conversely, given a vector field $\vec{v}$, as above, the one-parameter local group of transformations generated by $\vec{v}$ is found by solving the system of ordinary differential equations

$$dx_i/dt = \xi^i(x) , \quad i = 1, \ldots, n ,$$

$$x(0) = x .$$

Vector fields on $\mathbb{R}^n$ can also be viewed as first-order partial differential operators (derivations) which act on smooth functions $F: \mathbb{R}^n \to \mathbb{R}$:

$$\vec{v}F(x) = \xi^1(x)\frac{\partial F}{\partial x_1} + \ldots + \xi^n(x)\frac{\partial F}{\partial x_n} .$$

Given two vector fields $\vec{v}$ and $\vec{w}$, the Lie bracket is the vector field

$$[\vec{v}, \vec{w}] = \vec{v}\,\vec{w} - \vec{w}\,\vec{v} ,$$

where we are viewing the vector fields as derivations. The map φ from the Lie algebra to vector fields defined by (7.1) preserves the Lie bracket:

$$\varphi([\alpha, \beta]) = [\varphi(\alpha), \varphi(\beta)] .$$

Thus the infinitesimal generators of a local transformation group form a finite-dimensional Lie algebra of vector fields on $\mathbb{R}^n$. Conversely, given a finite-dimensional Lie algebra of vector fields on $\mathbb{R}^n$, Frobenius' theorem (cf. [3] or [8]) says that there is a local transformation group whose infinitesimal generators are precisely the vector fields in question. We are thus justified in viewing local transformation groups and Lie algebras of vector fields as equivalent concepts. In practice, to find the symmetry group of a differential equation, the infinitesimal generators will in fact be calculated, this being much easier to accomplish.

Example 7.2. Let $G = \mathbb{R}$ with coordinate t, and consider the following action on $\mathbb{R}^2$

$$\Phi(t; x, y) = \left(\frac{x}{1 - ty} , \frac{y}{1 - ty} \right) .$$

Here $V = \{(t; x, y): t < y \text{ for } y > 0 \text{ and } t > y \text{ for } y < 0\}$. The reader should check that Φ does satisfy the conditions of Definition 7.1. Also, this action is not global, i.e., cannot be defined for all $t \in \mathbb{R}$. The infinitesimal generator of the group action is

$$\left.\frac{d}{dt}\right|_{t=0} \Phi(t; x, y) = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} .$$

Indeed, as can easily be checked, the coordinate functions satisfy the ordinary differential equations

$$dx/dy = xy \qquad dy/dt = y^2 .$$

In the sequel we will often use the simplified notation of denoting $\Phi(g, x)$ by gx. Also, an element α of the Lie algebra will be identified with the vector field it defines on $\mathbb{R}^n$, and the map φ will be suppressed.

Now, a function $F: \mathbb{R}^n \to \mathbb{R}^m$ is called a G-invariant function if for all $x \in \mathbb{R}^n$, $F(gx) = F(x)$ whenever gx is defined. Similarly, a subset $S \subset \mathbb{R}^n$ is called a G-invariant subset if for every $x \in S$, $gx \in S$ whenever gx is defined. Note that if F is a G-invariant function, all the level sets of F, $\{x: F(x) = c\}$, are G-invariant sets. However, if a set S is a subvariety given by the vanishing of a function, i.e., $S = \{x: F(x) = 0\}$, and S is G-invariant, it does not necessarily follow that F is a G-invariant function. Thus the symmetry group of a single level set of a function (meaning the "largest group of transformations" leaving the level set invariant) will in general contain more symmetries than the symmetry group of the function. The next theorem gives infinitesimal criteria for the invariance of a function or subvariety.

Theorem 7.3. Suppose G is a connected Lie group of transformations acting on $\mathbb{R}^n$, such that for each $x \in \mathbb{R}^n$, $G_x = \{g: gx \text{ is defined}\}$ is also connected. Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a differential function whose Jacobian matrix is of maximal rank everywhere.

i) F is a G-invariant function if and only if

$$\alpha F(x) = 0 \tag{7.2}$$

for every infinitesimal generator α of G and every $x \in \mathbb{R}^n$.

ii) The subvariety $S = \{x: F(x) = 0\}$ is G-invariant if and only if (7.2) holds for every infinitesimal generator α and for every $x \in S$.

Proof. First the second statement will be proven. The necessity of (7.2) follows from differentiating the equation

$$F(\exp(t\alpha)x) = 0 \qquad \alpha \in \mathcal{G}, \ x \in S,$$

with respect to t and setting $t = 0$. To prove sufficiency, if the Jacobian of F at x_0 has maximal rank then by the implicit function theorem we may locally change coordinates so that F has the form

$$F(x_1, \ldots, x_n) = (x_1, \ldots, x_m).$$

Thus $S = \{(0, \ldots, 0, x_{m+1}, \ldots, x_n)\}$. The infinitesimal condition (7.2) implies that α, when restricted to S, has the form $\alpha_{m+1}(x) \frac{\partial}{\partial x_{m+1}} + \ldots + \alpha_n(x) \frac{\partial}{\partial x_n}$, $x \in S$. Then the one-parameter subgroup $\exp(t\alpha)$ obtained by integrating the requisite system of o.d.e.'s obviously leaves S locally invariant. Hence for each $x \in S$, there is a neighborhood $\tilde{N}_x$ of 0 in $\mathcal{G}$ such that for $\alpha \in \tilde{N}_x$, $\exp(\alpha)x \in S$. Now $\exp: \mathcal{G} \to G$ maps a sufficiently small neighborhood of 0 in $\mathcal{G}$ homeomorphically onto a neighborhood of e in G, [8; page 103]. Thus for each $x \in S$, there is a neighborhood N_x of e in G (depending continously on x) such that whenever $g \in N_x$, $gx \in S$. Finally, to show S is G-invariant, given $x \in S$, let $H_x = \{g \in G : gx \text{ is defined and } gx \in S\}$. It is easy to show that H_x is open and also if $g \in \operatorname{clos} H_x$, then $g \cdot x$ is not defined. This implies, by the connectedness of G_x, that $H_x = G_x$ and hence S is indeed G-invariant. To prove part i), it suffices to note that F is invariant if and only if every level set of F is invariant. Therefore part ii) implies part i).

(In the sequel, when G is a connected group of transformations, it will be implicitly assumed that all the G_x's are connected so as to avoid restating this technical condition.)

In practice, if the vector fields $\alpha_i = \sum \alpha_i^k(x)\, \partial/\partial x_k$, $i = 1, \ldots, r$, form a basis of the Lie algebra and $F(x) = (F^1(x), \ldots, F^m(x))$ then the infinitesimal criterion of (7.2) is just

$$\sum_{k=1}^{n} \alpha_i^k(x) \frac{\partial F^j}{\partial x_k} = 0 \,, \qquad \begin{array}{l} i = 1, \ldots, r \,, \\ j = 1, \ldots, m \,. \end{array} \tag{7.2'}$$

In the second case of the theorem, these equations only need hold when $F^1(x) = \ldots = F^m(x) = 0$. Thus determining the invariance of a subvariety under a connected group reduces to a routine verification of condition (7.2'). For nonconnected Lie groups, one must further check that at least one element in each connected component leaves the subvariety invariant.

Example 7.4. Consider the same local group of transformations as in example 7.2. First consider the function $F(x, y) = x/y$. Then applying the infinitesimal generator to F shows

$$\left(xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}\right) \frac{x}{y} = 0$$

hence F is an invariant function, as may easily be checked. Therefore the sets $\{(x, y) : x = cy\}$ for any constant c are invariant.

Secondly, consider the function $F'(x, y) = xy$. Then

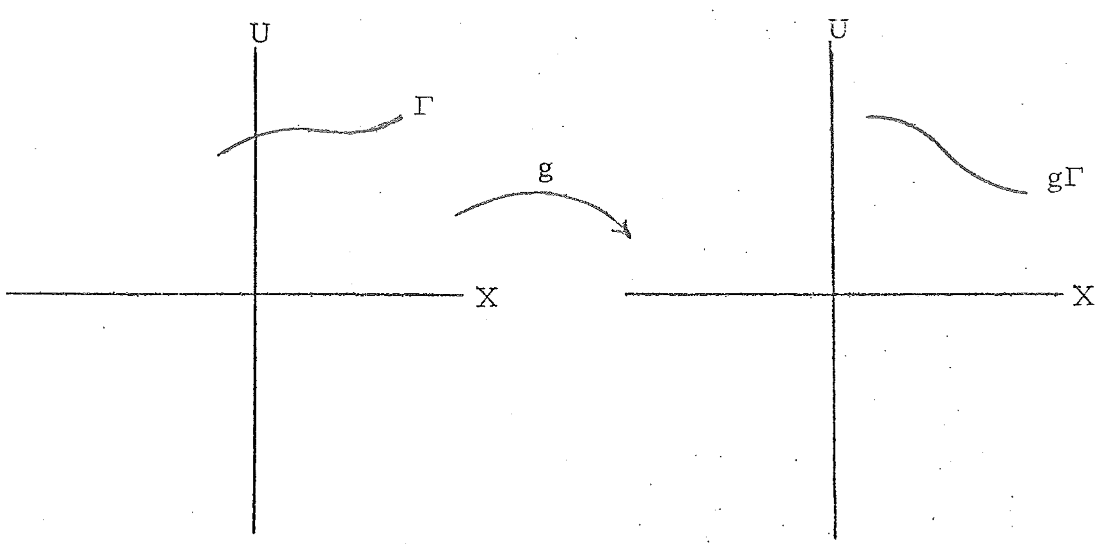
$$\left(xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}\right) xy = 2xy^2 \;;$$

so F' is not an invariant function. However, the subvariety $\{(x, y) : xy = 0\}$ is invariant since if $xy = 0$, then $2xy^2 = 0$. This may again be verified directly from the definition of G.

2. Groups and Differential Equations.

Suppose we are considering a system of partial differential equations, S, in p independent variables $(x_1, \ldots, x_p) = x$ and q dependent variables $(u^1, \ldots, u^q) = u$. Let $X = \mathbb{R}^p$, with coordinates x, be the space representing the independent variables, and let $U = \mathbb{R}^q$, with coordinates u represent the dependent variables. The solutions $u = f(x)$ of S will be identified with their graphs; which are certain p-dimensional submanifolds in the cartesian product space $X \times U$. A symmetry group of S will be a local group of transformations G acting on $X \times U$ in such a way that "G transforms solutions of S to other solutions of S." Note that we are allowing arbitrary, nonlinear transformations of both the independent and dependent variables in G.

To proceed rigorously, we must first explain exactly how the group G transforms functions. Given a function $u = f(x)$, defined in a neighborhood N of a point $x_0 \in X$, let $\Gamma = \{(x, f(x)) : x \in N\}$ be the graph of f. If Γ is relatively compact in $X \times U$, then, for g sufficiently close to the identity, the set $g\Gamma = \{g(x,u) : (x,u) \in \Gamma\}$ is defined. The set $g\Gamma$ is not necessarily the graph of some other function. However, since G acts continuously and e leaves Γ unchanged, by possibly shrinking N we can find a neighborhood of e in G such that for every g in this neighborhood, $g\Gamma$ is defined and is the graph of some new function $\tilde{u} = g \cdot f(\tilde{x})$, called the transform of f by g.



The explicit construction of the transformed function gf follows. Suppose the transformation g is given by

$$g(x, u) = (\Xi_g(x,u), \Psi_g(x, u)) = (\tilde{x}, \tilde{u}).$$

The graph $g\Gamma$ is given by the parametric equations

$$\tilde{x} = \Xi_g(x, f(x)) = \Xi_g \circ (I \times f)(x) ,$$

$$\tilde{u} = \Psi_g(x, f(x)) = \Psi_g \circ (I \times f)(x) .$$

(Here I is the identity function on X.) To find $g \cdot f$, we must eliminate x from these equations. For g sufficiently close to e, using the inverse function theorem we can locally solve for x:

$$x = [\Xi_g \circ (I \times f)]^{-1}(\tilde{x}) .$$

Substitution into the second equation yields

$$g \cdot f = [\Psi_g \circ (I \times f)] \circ [\Xi_g \circ (I \times f)]^{-1} , \tag{7.3}$$

whenever the second factor is invertible.

Example 7.5. Let $p = q = 1$, so $X = U = \mathbb{R}$. Let $G = S_1$ be the rotation group acting on $X \times U \simeq \mathbb{R}^2$, so the transformations in G are given by

$$(\tilde{x}, \tilde{u}) = (x \cos\theta - u \sin\theta,\ x \sin\theta + u \cos\theta)$$

for $0 \leq \theta < 2\pi$. Therefore

$$\Xi_\theta(x, u) = x\cos\theta - u\sin\theta ,$$

$$\Psi_\theta(x, u) = x\sin\theta + u\cos\theta .$$

Consider the linear function $f(x) = ax+b$. Note that if θ is sufficiently large, the graph of f will be rotated so that it is vertical and is no longer the graph of a function. Now

$$\tilde{x} = \Xi_\theta(x, f(x)) = x(\cos\theta - a\sin\theta) - b\sin\theta ,$$

$$\tilde{u} = \Psi_\theta(x, f(x)) = x(a\cos\theta + \sin\theta) + b\cos\theta .$$

Whenever $\cos\theta - a\sin\theta \neq 0$, in particular for θ sufficiently close to 0, the first equation is solvable for x, and

$$x = \frac{\tilde{x} + b\sin\theta}{\cos\theta - a\sin\theta} .$$

Therefore the transform of f by θ according to formula (7.3) is the linear function

$$\theta f(\tilde{x}) = \frac{\sin\theta + a\cos\theta}{\cos\theta - a\sin\theta} \cdot \tilde{x} + \frac{b}{\cos\theta - a\sin\theta} .$$

Definition 7.6. Given a system of partial differential equations, in p independent and q dependent variables, a symmetry group is a local group of transformations G acting on $X \times U$ such that whenever $u = f(x)$ is a local solution of this system, and for each g such that $g \cdot f$ is defined, then $\tilde{u} = gf(\tilde{x})$ is also a solution of the system.

For example, in the case of the heat equation $u_t = u_{xx}$, the group of translations in the spatial variable $(x, t, u) \mapsto (x + \lambda, t, u)$ is a symmetry group since $f(x + \lambda, t)$ is a solution of the heat equation whenever $f(x, t)$ is. Another example is provided by the group of Chapter 4 leaving the Navier-Stokes equations invariant. The ultimate goal of this chapter is to provide a readily verifiable criterion which will enable us to check whether or not a given group is a symmetry group of a given system of equations, and also find the symmetry group, meaning "the largest local group of symmetries," of the equations. The criterion will be infinitesimal, in direct analogy with the criterion of invariance of algebaraic equations of Theorem 7.3. In fact, when we finish constructing an appropriate machinery, we will be able to directly invoke Theorem 7.3 to prove invariance.

The key step is to construct spaces representing the various derivatives present in our system of partial differential equations and then concretely "realize" the system of equations as a subvariety of such a space. (This construction is a greatly simplified version of the construction of "jet bundles" in the differential-geometric theory of partial differential equations, cf [4] or [7].) Now for a given function of p independent variables, there are

$p_k = \binom{p+k-1}{k}$ different k-th order partial derivatives, which we denote by

$$\partial_J = \frac{\partial^{\Sigma J}}{\partial x_1^{j_1} \cdots \partial x_p^{j_p}} ,$$

where $J = (j_1, \ldots, j_p)$, each j_i is a nonnegative integer, and $\Sigma J = j_1 + \ldots + j_p = k$. Then given a function $f: X \to U$, so $u^\ell = f^\ell(x)$, there are $q \cdot p_k$ different numbers $u_J^\ell = \partial_J f^\ell(x)$ for $\ell = 1, \ldots, q$, $\Sigma J = k$ which give all the k-th order derivatives of f. Let $U_k = \mathbb{R}^{qp_k}$, with coordinates u_J^ℓ as above, be the space representing all these k-th order derivatives. Let $U^{(k)} = U \times U_1 \times \ldots \times U_k$ be the space representing all partial derivatives of functions $f: X \to U$ of order $\leq k$. Thus, given a function $f: X \to U$, there is a corresponding function $pr^{(k)} f: X \to U^{(k)}$, called the k-th prolongation of f, whose graph is given by the equations

$$u_J^\ell = \partial_J f^\ell (x) .$$

In other words, the value of $pr^{(k)} f(x)$ is a vector whose $q + qp_1 + \ldots + qp_k$ entries are the values of f and its partial derivatives of order $\leq k$ at the point x. Another way of looking at $pr^{(k)} f(x)$ is that it represents the Taylor polynomial of degree k of f at x, since the derivatives determine the Taylor polynomial and vice versa. (The total space $X \times U^{(k)}$ is called the k-jet space of $X \times U$, and the k-th prolongation of f is also called the k-jet of f in differential geometry.) We will use the symbol $u^{(k)}$ to denote points in $U^{(k)}$, so the entries of $u^{(k)}$ are the u_J^ℓ's.

A system of partial differential equations in p independent and q dependent variables is given by m equations of the form

$$\Delta^i(x, u^{(k)}) = 0 \ , \quad i = 1, \ldots, m.$$

Therefore, we may identify the system of equations with a subvariety $S_\Delta \subset X \times U^{(k)}$ given by the vanishing of a smooth function $\Delta: X \times U^{(k)} \to \mathbb{R}^m$, i.e.,

$$S_\Delta = \{(x, u^{(k)}) : \Delta(x, u^{(k)}) = 0\} \ .$$

Then a solution of these equations is just a smooth function $f: X \to U$, such that $\Delta(x, pr^{(k)}f(x)) = 0$. In other words, the graph of the k-th prolongation of f lies entirely within S_Δ.

$$\{(x, pr^{(k)}f(x))\} \subset S_\Delta \ .$$

Now suppose G is a local group of transformations acting on the space $X \times U$. There is an induced local action of G on the space $X \times U^{(k)}$, called the k-th prolongation of G, and denoted by $pr^{(k)}G$. This prolonged action is such that the transform of the derivatives of a function is the derivatives of the transformed function, the latter being defined by (7.3). More rigorously, given $(x, u^{(k)}) \in X \times U^{(k)}$, choose a smooth function f defined in a neighborhood of x such that $\partial_J f^\ell(x) = u^\ell_J$. (Such a choice is always possible, for example f might be the finite Taylor polynomial at x corresponding to $u^{(k)}$:

$$f^\ell(\xi) = \sum_J \frac{u^\ell_J}{J!} (\xi - x)^J \ .)$$

Then for $g \in G$ sufficiently close to the identity the function $g \cdot f$ is defined in a neighborhood of $(\tilde{x}, \tilde{u}) = g(x, u)$ by formula (7.3). We then define

$$pr^{(k)} g \cdot (x, u^{(k)}) = (\tilde{x}, \tilde{u}^{(k)}) \ .$$

where the coordinates of $\tilde{u}^{(k)}$ are

$$\tilde{u}^\ell_J = \partial_J (g \cdot f)^\ell (\tilde{x}) \ . \tag{7.4}$$

It is a straightforward matter to check that this definition is independent of the choice of function f to represent the point $(x, u^{(k)})$. The formula (7.4), when expanded using (7.3), will define the prolonged group action.

Example 7.7. Let U, X, G be as in example 7.5. Note first that $U_k \simeq \mathbb{R}$ with coordinate u_k representing the k-th derivative of a function $f(x)$. Thus

$$pr^{(k)}f(x) = (f(x), f'(x), \dots, f^{(k)}(x)) .$$

We proceed to construct the first prolongation of the rotation group S_1. Let $(x^*, u^*, u_x^*) \in X \times U \times U_1 = X \times U^{(1)}$. Choose the linear Taylor polynomial

$$f(x) = u^* + (x - x^*)u_x^* = u^* - x^*u_x^* + xu^*$$

as a representative of (x^*, u^*, u_x^*), so that $f(x^*) = u^*$, $f'(x^*) = u_x^*$. According to the calculations of example 7.5, the transform of f by a rotation through an angle θ is the function

$$\theta \cdot f(\tilde{x}) = \frac{\sin\theta + u_x^* \cos\theta}{\cos\theta - u_x^* \sin\theta} \tilde{x} + \frac{u^* - x^* u_x^*}{\cos\theta - u_x^* \sin\theta} .$$

Now $\tilde{x}^* = x^* \cos\theta - u^* \sin\theta$, so

$$\tilde{u}^* = \theta \cdot f(\tilde{x}^*) = x^* \sin\theta + u^* \cos\theta ,$$

as we already knew. Moreover,

$$\tilde{u}_x^* = (\theta f)'(\tilde{x}^*) = \frac{\sin\theta + u_x^* \cos\theta}{\cos\theta - u_x^* \sin\theta} .$$

Therefore the prolonged action $pr^{(1)}S_1$ on $X \times U^{(1)}$ is given by

$$pr^{(1)}\theta \cdot (x, u, u_x) = \left(x\cos\theta - u\sin\theta, x\sin\theta + u\cos\theta, \frac{\sin\theta + u_x\cos\theta}{\cos\theta - u_x \sin\theta}\right),$$

which is defined whenever

$$|\theta| < |\text{arc cot } u_x| .$$

Note that even though S_1 is a global transformation group, its prolongation $pr^{(1)}S_1$ is only a local transformation group. The infinitesimal generator of the prolonged group action is found by differentiating the last equation with respect to θ and setting $\theta = 0$. This yields

$$pr^{(1)}\alpha = -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x} .$$

(Here and in the sequel we will occasionally use ∂_x to denote $\partial/\partial x$, etc.)

Theorem 7.8. Suppose $\Delta = 0$ is a system of p.d.e.'s with corresponding subvariety $S_\Delta \subset X \times U^{(k)}$. Suppose S_Δ is invariant under $pr^{(k)}G$ for some group G acting on $X \times U$. Then G is a symmetry group of the system as in Definition 7.6.

Proof. Suppose $u = f(x)$ is a local solution. This means that the graph $\Gamma_f^{(k)}$, of $pr^{(k)}f$ lies entirely inside S_Δ. Now for $g \in G$ such that $g \cdot f$ is defined, the graph $\Gamma_{gf}^{(k)}$ of $pr^{(k)}(g \cdot f$ is just the transform of the graph of $pr^{(k)}f$ under $pr^{(k)}g$, i.e.,

$$\Gamma_{gf}^{(k)} = pr^{(k)}g \cdot \Gamma_f^{(k)} .$$

This is just a restatement of formula (7.4) defining $pr^{(k)}g$. Now since S_Δ is invariant under $pr^{(k)}g$, the graph of $pr^{(k)}(g \cdot f)$ lies entirely within S_Δ. But this is just another way of saying that $g \cdot f$ is a solution of $\Delta = 0$. This completes the proof.

Given an infinitesimal generator α of a one-parameter subgroup $\exp(t\alpha)$ of G, define the <u>k-th prolongation</u> of α to be the infinitesimal generator of the prolonged one-parameter subgroup $pr^{(k)}\exp(t\alpha)$; i. e.,

$$pr^{(k)}\alpha = \frac{d}{dt}\bigg|_{t=0} pr^{(k)}[\exp(t\alpha)]. \tag{7.5}$$

Combining Theorem 7.8 and Theorem 7.3, we get the following infinitesimal criterion for G to be the symmetry group of a system of p. d. e.'s.

<u>Corollary 7.9.</u> <u>Suppose</u> $\Delta(x, u^{(k)}) = 0$ <u>is a system of p. d. e.'s (such that the Jacobian matrix of</u> Δ <u>has maximal rank everywhere). Suppose</u> G <u>is a connected local transformation group acting on</u> $X \times U$ <u>such that for every infinitesimal generator</u> α <u>of</u> G

$$pr^{(k)}\alpha[\Delta(x, u^{(k)})] = 0 \tag{7.6}$$

<u>whenever</u> $\Delta(x, u^{(k)}) = 0$. <u>Then</u> G <u>is a symmetry group of the equations</u> $\Delta = 0$.

<u>Example 7.10.</u> Let X, U, G be as in examples 7.5, 7.7. Consider the first order ordinary differential equation

$$\Delta = (u - x)u_x + (u + x) = 0.$$

Applying the infinitesimal generator of $pr^{(1)}S_1$ to this equation yields

$$pr^{(1)}\alpha \cdot \Delta = (-u\partial_x + x\partial_u + (1+u_x^2)\partial_{u_x})[(u-x)u_x + (u+x)]$$

$$= u_x[(u-x)u_x + (u+x)]$$

$$= u_x \Delta .$$

Therefore $pr^{(1)}\alpha \cdot \Delta = 0$ whenever $\Delta = 0$, and condition (7.6) is verified. Then Corollary 7.10 shows that if $u = f(x)$ is any solution of $\Delta = 0$, then so is the rotated function $\tilde{u} = \theta \cdot f(\tilde{x})$. Indeed, in polar coordinates $x = r\cos\theta$, $u = r\sin\theta$, the equation $\Delta = 0$ becomes

$$dr/d\theta = r ,$$

whose solutions are the spirals

$$r = ce^{\theta} .$$

Obviously, any one of these spirals, when rotated, is another spiral of the same type. (For a discussion of the use of symmetry groups of ordinary differential equations for finding solutions by quadratures, the reader should consult reference [2].)

Theorem 7.8 and Corollary 7.9 admit converses if we further assume that the system of p. d. e.'s is "solvable for arbitrary initial dats." Then (7.6) becomes a necessary and sufficient condition for symmetry.

Theorem 7.11 Suppose $\Delta(x, u^{(k)}) = 0$ is a system of partial differential equations in p independent variables $x_1, \ldots, x_p$ and q dependent variables $u^1, \ldots, u^q$, such that the Jacobian matrix of Δ has maximal rank everywhere. Suppose further that for any point

$$(x_0, u_0^{(k)}) \in S_\Delta = \{(x, u^{(k)}) : \Delta(x, u^{(k)}) = 0\} \subset X \times U^{(k)}$$

there is a solution $u = f(x)$ defined in a neighborhood of x_0 such that $u_0^{(k)} = \mathrm{pr}^{(k)} f(x_0)$. Suppose G is a connected local transformation group acting on $X \times U = \mathbb{R}^p \times \mathbb{R}^q$, the space of independent and dependent variables. Then G is a symmetry group of the system if and only if for every infinitesimal generator α of G,

$$\mathrm{pr}^{(k)}\alpha[\Delta(x, u^{(k)})] = 0 \qquad (7.7)$$

whenever $\Delta(x, u^{(k)}) = 0$.

Proof We need only show the necessity of (7.7). In view of Theorem 7.3, we must show that S_Δ is invariant under $\mathrm{pr}^{(k)}G$, since this will imply (7.7). Given $(x_0, u_0^{(k)}) \in S_\Delta$, let $u = f(x)$ be a local solution with $u_0^{(k)} = \mathrm{pr}^{(k)} f(x_0)$. For $g \in G$ such that $g \cdot f$ is defined,

$$\mathrm{pr}^{(k)} g \cdot (x_0, u_0^{(k)}) = (\tilde{x}_0, \mathrm{pr}^{(k)}(gf)(\tilde{x}_0)) \in S_\Delta ,$$

since gf is also a solution. This proves the theorem.

3. The Prolongation Formula

In light of Theorem 7.11, the primary task remaining is to find a formula for the prolongation of a vector field. Even though the prolonged group action, as determined by (7.4), is exceedingly complicated, we will find that the prolonged infinitesimal generators are expressed relatively simply. First we need the concept of a t total derivative.

Definition 7.12. Given a differentiable function $\Delta : X \times U^{(k)} \to \mathbb{R}$, the total derivative D_i $(1 \leq i \leq p)$ is the function $D_i \Delta : X \times U^{(k+1)} \to \mathbb{R}$ such that, for any smooth function $f: X \to U$,

$$D_i \Delta(x, pr^{(k+1)} f(x)) = \frac{\partial}{\partial x_i} \Delta(x, pr^{(k)} f(x)) .$$

In other words, $D_i \Delta$ is just the derivative of $\Delta(x, u^{(k)})$, treating u as a function of x.

It is easy to check that

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\ell=1}^{q} \sum_{J} u^{\ell}_{J_i} \frac{\partial}{\partial u^{\ell}_{J}} , \tag{7.8}$$

where $J_i = (j_1, \ldots, j_{i-1}, j_i + 1, j_{i+1}, \ldots, j_p)$, and the sum is over all J's with $j_1 + \ldots + j_p \leq k$. For instance, if $X = U = \mathbb{R}$, then there is just one total derivative

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xxx} \frac{\partial}{\partial u_{xx}} + \cdots .$$

Given a multi-index $J = (j_1, \ldots, j_p)$, we abbreviate

$$D^J = D_1^{j_1} D_2^{j_2} \cdots D_p^{j_p} .$$

Theorem 7.13. Suppose α is a smooth vector field on $X \times U$, given by

$$\alpha = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x_i} + \sum_{\ell=1}^{q} \varphi_\ell(x, u) \frac{\partial}{\partial u^\ell} .$$

The k-th prolongation of α, as defined by (7.5), is the vector field

$$\mathrm{pr}^{(k)}\alpha = \alpha + \sum_{\ell=1}^{q} \sum_{J} \varphi_\ell^J(x, u^{(k)}) \frac{\partial}{\partial u_J^\ell} \tag{7.9}$$

on $X \times U^{(k)}$, where the sum is over all J's with $0 < j_1 + \ldots + j_p \leq k$. The coefficient functions φ_ℓ^J are given by the following formula:

$$\varphi_\ell^J = D^J(\varphi^\ell - \sum_{i=1}^{p} u_i^\ell \xi_i) + \sum_{i=1}^{p} u_{J_i}^\ell \xi^i , \tag{7.10}$$

where $u_i^\ell = \partial u^\ell / \partial x_i$ and J_i is as defined above.

Proof. First the formula will be proved for the case $k = 1$. Let

$$(\tilde{x}_t, \tilde{u}_t) = \exp(t\alpha)(x, u) = (\Xi_t(x, u), \Phi_t(x, u)) ,$$

so that

$$\left.\frac{d}{dt}\right|_{t=0} \Xi_t^i(x,u) = \xi^i(x,u) \qquad i = 1,\ldots,p ,$$

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_t^\ell(x,u) = \varphi_\ell(x,u) \qquad \ell = 1,\ldots,q .$$

Now given $(x, u^{(1)}) \in X \times U^{(1)}$, let $u = f(x)$ be any representative, so that $u_i^\ell = \partial f(x)/\partial x_i$. According to (7.3), for t sufficiently small, the transform of f by the group element $\exp(t\alpha)$ is well-defined and is given by

$$f_t(\tilde{x}_t) = [\Phi_t \circ (I \times f)] \circ [\Xi_t \circ (I \times f)]^{-1}(x_t) .$$

Using the chain rule, the Jacobian matrix of f_t at x_t is therefore

$$Jf_t(\tilde{x}_t) = J[\Phi_t \circ (I \times f)](x) \cdot [J[\Xi_t \circ (I \times f)](x)]^{-1} . \qquad (7.11)$$

This serves to define the prolonged group action $pr^{(1)}\exp(t\alpha)$. Thus to find the infinitesimal generator $pr^{(1)}\alpha$, we must differentiate (7.11) with respect to t and set $t = 0$. Recall that for any matrix valued function $A(t)$,

$$\frac{d}{dt}[A^{-1}(t)] = -A^{-1}(t)\frac{dA(t)}{dt} A^{-1}(t) .$$

Also note that since $t = 0$ corresponds to the identity group element,

$$\Xi_0 \circ (I \times f) = I , \qquad \Phi_0 \circ (I \times f) = f .$$

Therefore, by Leibnitz' rule,

$$\frac{d}{dt}\bigg|_{t=0} Jf_t(\tilde{x}_t) = \frac{d}{dt}\bigg|_{t=0} J[\Phi_t \circ (I \times f)](x) -$$

$$- J f(x) \cdot \frac{d}{dt}\bigg|_{t=0} J[\Xi_t \circ (I \times f)](x)$$

$$= J[\varphi \circ (I \times f)](x) - Jf(x) \cdot J[\xi \circ (I \times f)](x) .$$

Now the matrix entries of this are just the coordinate functions of the first prolongation of α ; namely

$$\varphi_\ell^j = \frac{\partial}{\partial x_j}[\varphi_\ell(x, f(x))] - \sum_{i=1}^{p} \frac{\partial f^\ell}{\partial x_i} \cdot \frac{\partial}{\partial x_j}[\xi^i(x, f(x))]$$

$$= D_j \varphi_\ell(x, u^{(1)}) - \sum_{i=1}^{p} u_i^\ell D_j \xi^i(x, u^{(1)})$$

$$= D_j[\varphi^\ell - \sum_{i=1}^{p} u_i^\ell \xi^i] \quad \sum_{i=1}^{p} u_{ij}^\ell \xi^i ,$$

where we have used the definition of the total derivative, and $u_{ij}^\ell = \partial^2 u^\ell / \partial x_i \partial x_j$. This proves the theorem when $k = 1$.

To prove the theorem in general, we proceed by induction. Notice that $X \times U^{(k+1)}$ can be viewed as a subspace of $X \times [U^{(k)}]^{(1)}$. Therefore given a multi-index J, by what we have already proven,

$$\varphi_\ell^{J_j} = D_j \varphi_\ell^J - \sum_{i=1}^{p} u_{J_i}^\ell D_j \xi^i . \tag{7.12}$$

(Equation (7.10) is a useful recursion relation for the φ_ℓ^J's, and may be also found in [3; page 106].) It is a simple matter to check that (7.10) satisfies the recursion relation (7.12). Indeed,

$$D_j\varphi_\ell^J - \sum_i u^\ell_{J_i} D_j\xi^i = D^{J_j}(\varphi^\ell - \sum_i u_i^\ell \xi^i) +$$

$$+ \sum_i (u^\ell_{J_{ij}} \xi^i + u^\ell_{J_i} D_j\xi^i) - \sum_i u^\ell_{J_i} D_j\xi^i$$

$$= D^{J_j}(\varphi^\ell - \sum_i u_i^\ell \xi^i) + \sum_i u^\ell_{J_{ij}} \xi^i .$$

(Here $J_{ij} = (J_i)_j$.) This completes the proof of Theorem 7.13.

<u>Example 7.14.</u> Let X, U, G be as in examples 7.5, 7.7 and 7.10. The infinitesimal generator of the rotation group is $\alpha = -u\partial_x + x\partial_u$. The first prolongation, according to (7.9, 10), is the vector field $pr^{(1)}\alpha = \alpha + \varphi^x \partial_{u_x}$, where

$$\varphi^x = D_x(x + uu_x) - uu_{xx}$$
$$= 1 + u_x^2 ,$$

as we have already discovered. Similarly, the coefficient function φ^{xx} of $\partial/\partial u_{xx}$ in the second prolongation of α is

$$\varphi^{xx} = D_x^2(x + u_x^2 + uu_{xx}) - uu_{xxx}$$

$$= D_x(1 + u_x^2 + uu_{xx}) - uu_{xxx}$$

$$= 3u_x u_{xx} \ .$$

Therefore the infinitesimal generator of the second prolongation of the rotation group is the vector field

$$\mathrm{pr}^{(2)}\alpha = -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x} + 3u_x u_{xx}\partial_{u_{xx}} \ .$$

(The reader is invited to attempt to deduce this formula directly from the prolonged group action!) Using the infinitesimal criterion of Theorem 7.11, we see that the differential equation $u_{xx} = 0$ is invariant under S_1, since $\mathrm{pr}^{(2)}\alpha(u_{xx}) = 3u_x u_{xx} = 0$ whenever $u_{xx} = 0$. This is just a restatement of the fact that rotations preserve straight lines. Similarly the function $F(u_x, u_{xx}) = u_{xx}(1+u_x^2)^{-3/2}$ is invariant under $\mathrm{pr}^{(2)}S_1$ since $\mathrm{pr}^{(2)}\alpha \cdot F = 0$. This just says that the curvature of a curve is invariant under rotations.

Example 7.15. Consider the case $p = 2$, $q = 1$, so we are looking at partial differential equations for functions $u = f(x, t)$. A vector field on $X \times U$ is of the form

$$\alpha = \xi\partial_x + \tau\partial_t + \varphi\partial_u \ ,$$

where ξ, τ, φ are functions of x, t, u. The first prolongation of α is the vector field

$$\mathrm{pr}^{(1)}\alpha = \alpha + \varphi^x \partial_{u_x} + \varphi^t \partial_{u_t} ,$$

where

$$\begin{aligned}
\varphi^x &= D_x(\varphi - u_x\xi - u_t\tau) + u_{xx}\xi + u_{xt}\tau \\
&= D_x\varphi - u_x D_x\xi - u_t D_x\tau \\
&= \varphi_x + (\varphi_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t
\end{aligned}$$

$$(7.13)$$

$$\begin{aligned}
\varphi^t &= D_t(\varphi - u_x\xi - u_t\tau) + u_{xt}\xi + u_{tt}\tau \\
&= D_t\varphi - u_x D_t\xi - u_t D_t\tau \\
&= \varphi_t - \xi_t u_x + (\varphi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_x^2 .
\end{aligned}$$

Similarly, the second prolongation of α is

$$\mathrm{pr}^{(2)}\alpha = \mathrm{pr}^{(1)}\alpha + \varphi^{xx}\partial_{u_{xx}} + \varphi^{xt}\partial_{u_{xt}} + \varphi^{tt}\partial_{u_{tt}}$$

where, for example,

$$\varphi^{xx} = D_x^2(\varphi - u_x\xi - u_t\tau) + u_{xxx}\xi + u_{xxt}\tau$$

$$= D_x^2\varphi - u_x D_x^2\xi - u_t D_x^2\tau - 2u_{xx}D_x\xi - 2u_{xt}D_x\tau$$

$$= \varphi_x + u_x(2\varphi_{xu} - \xi_{xx}) - u_t\tau_{xx} + \qquad (7.14)$$

$$+ u_x^2(\varphi_{uu} - 2\xi_{xu}) - u_x u_t \tau_{xu} - u_x^3\xi_{uu} - u_x^2 u_t \tau_{uu} +$$

$$+ u_{xx}(\varphi_u - 2\xi_x) - 2u_{xt}\tau_x - 3u_{xx}u_x\xi_u - u_{xx}u_t\tau_u - 2u_{xt}u_x\tau_u .$$

These expressions will be used in the following section to compute the symmetry group of the heat equation.

Theorem 7.16. Suppose α and β are smooth vector fields on $X \times U$. Then

$$pr^{(k)}[\alpha, \beta] = [pr^{(k)}\alpha, pr^{(k)}\beta] .$$

Corollary 7.17. If $\Delta(x, u^{(k)}) = 0$ is a system of p. d. e. 's satisfying the conditions of Theorem 7.12, then the set of all infinitesimal symmetries of $\Delta = 0$, meaning the set of all vector fields α on $X \times U$ generating one-parameter symmetry groups, is a Lie algebra.

The most straightforward proof of 7.16 is computational using the prolongation formula of Theorem 7.13. The details are left to the reader. (See also [4].)

4. Applications of the Theory.

The basic method for finding the (connected component) of the symmetry group of a given system of p. d. e.'s is to substitute the prolongation formula of Theorem 7.13 for a vector field α on $X \times U$ into the infinitesimal criterion of invariance (7.6). The coefficients of the various partial derivatives of the dependent variables in the resulting equations are equated, which gives a large system of elementary p. d. e.'s for the coefficient functions of α, called the symmetry equations. The general solution of the symmetry equations is then the most general infinitesimal symmetry of the given system. The symmetry group itself may be found via exponentiation.

As a first example, consider the one-dimensional heat equation

$$u_t = u_{xx} . \tag{7.15}$$

Note that $p = 2$, $q = 1$ and $k = 2$; the heat equation being the linear subvariety of $X \times U^{(2)}$ given by the vanishing of the function

$\Delta = u_t - u_{xx}$. Given a vector field $\alpha = \xi\partial_x + \tau\partial_t + \varphi\partial_u$ on $X \times U$, the second prolongation $pr^{(2)}\alpha$ is given in example 7.15. The infinitesimal criterion (7.6) is just

$$\varphi^t = \varphi^{xx} , \tag{7.16}$$

which must be satisfied whenever $u_t = u_{xx}$. Substituting (7.13, 14) into (7.16), replacing u_t by u_{xx}, and equating the coefficients of the various partial derivatives of u, yields the following system of equations:

$u_{xx}u_{xt}$:	$0 = -\tau_u$	(a)
u_{xt} :	$0 = -2\tau_x$	(b)
u_{xx}^2 :	$-\tau_u = -\tau_u$	(c)
$u_x^2 u_{xx}$:	$0 = -\tau_{uu}$	(d)
$u_x u_{xx}$:	$-\xi_u = -2\tau_{xu} - 3\xi_u$	(e)
u_{xx} :	$\varphi_u - \tau_t = -\tau_{xx} + \varphi_u - 2\xi_x$	(f)
u_x^3:	$0 = -\xi_{uu}$	(g)
u_x^2:	$0 = \varphi_{uu} - 2\xi_{xu}$	(h)
u_x:	$-\xi_t = 2\varphi_{xu} - \xi_{xx}$	(i)
1:	$\varphi_t = \varphi_{xx}$	(j)

These are the symmetry equations. Now (a) and (b) show that τ is just a function of t. Then (e) shows that ξ depends only on x, t, and (f) shows $\tau_t = 2\xi_x$, hence $\xi_{xx} = 0$. Then (h) shows that $\varphi = g(x,t)u + f(x,t)$, and by (i), $\xi_t = -2g_x$. Finally (j) implies $g_t = g_{xx}$ and $f_t = f_{xx}$, hence $g_{xxx} = 0 = g_{xt}$, so $\xi_{tt} = 0$. Therefore the most general solution of the symmetry equations is

$$\begin{aligned} \xi &= c_1 + c_4 x + 2c_5 t + 4c_6 xt \,, \\ \tau &= c_2 + 2c_4 t + 4c_6 t^2 \,, \\ \varphi &= (c_3 - c_5 x - 2c_6 t - c_6 x^2)u + f(x,t) \,, \end{aligned} \tag{7.17}$$

where $c_1, \ldots, c_6$ are arbitrary constants and $f(x,t)$ an arbitrary solution of the heat equation. Thus the Lie algebra of infinitesimal symmetries of the heat equation is spanned by the six vector fields

$$\begin{aligned} \alpha_1 &= \partial_x \\ \alpha_2 &= \partial_t \\ \alpha_3 &= u\partial_u \\ \alpha_4 &= x\partial_x + 2t\partial_t \\ \alpha_5 &= 2t\partial_x - xu\partial_u \\ \alpha_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u \end{aligned} \tag{7.18}$$

and the infinite-dimensional subalgebra

$$\alpha_f = f(x,t)\partial_u$$

where f is an arbitrary solution of the heat equation. The one-parameter groups $G_i = \exp(\lambda\alpha_i)$ generated by α_i are given by the expressions

$$
\begin{aligned}
&G_1: && (x+\lambda, t, u) \\
&G_2: && (x, t+\lambda, u) \\
&G_3: && (x, t, e^{\lambda}u) \qquad \lambda \in \mathbb{R} \\
&G_4: && (e^{\lambda}x, e^{2\lambda}t, u) \\
&G_5: && (x-2\lambda t, t, u\exp(x\lambda-\lambda^2 t)) \\
&G_6: && \left(\frac{x}{4\lambda t+1}, \frac{t}{4\lambda t+1}, u\sqrt{4\lambda t+1}\,\exp\left[\frac{-\lambda x^2}{4\lambda t+1}\right]\right) \\
&G_f: && (x, t, u+\lambda f(x,t)).
\end{aligned}
\tag{7.19}
$$

Note that the symmetries G_3, G_f are consequences of the fact that the heat equation is linear. G_1 and G_2 reflect the fact that the heat equation has constant coefficients. G_4 is the well-known scale symmetry, G_5 a kind of Gallilean boost. This result is not new, see for instance [1] and [2] for a more complete discussion of these symmetries. Note also that G_6 is a local group.

For our second example, the symmetry group of the Navier-Stokes equations

$$
\begin{aligned}
p_x + uu_x + vu_y + wu_z &= \Delta u \\
p_y + uv_x + vv_y + wv_z &= \Delta v \\
p_z + uw_x + vw_y + ww_z &= \Delta w \\
u_x + v_y + w_z &= 0
\end{aligned}
\tag{7.20}
$$

will be computed. In this case $p = 3$, with coordinates (x, y, z), and $q = 4$, with coordinates (u, v, w, p). A typical vector field on $X \times U \simeq \mathbb{R}^3 \times \mathbb{R}^4$ is given by

$$\alpha = \xi\partial_x + \eta\partial_y + \zeta\partial_z + \varphi\partial_u + \psi\partial_v + \chi\partial_w + \pi\partial_p ,$$

where the coefficient functions depend on (x, y, z, u, v, w, p). The infinitesimal symmetry criterion (7.6) for (7.20) is

$$\begin{aligned}
\pi^x + u\varphi^x + v\varphi^y + w\varphi^z + u_x\varphi + u_y\psi + u_z\chi &= \varphi^{xx} + \varphi^{yy} + \varphi^{zz} \\
\pi^y + u\psi^x + v\psi^y + w\psi^z + v_x\varphi + v_y\psi + v_z\chi &= \psi^{xx} + \psi^{yy} + \psi^{zz} , \qquad (7.21) \\
\pi^z + u\chi^x + v\chi^y + w\chi^z + w_x\varphi + w_y\psi + w_z\chi &= \chi^{xx} + \chi^{yy} + \chi^{zz} , \\
\varphi^x + \psi^y + \chi^z &= 0 ,
\end{aligned}$$

which must be satisfied whenever (7.20) is. Here φ^x, etc. are the coefficient functions of the second prolongation of α, and are given by the prolongation formula (7.10). (See also (7.13, 14) for prototypical examples.) In the first equation of (7.21), the coefficient of p_{xx} is

$$0 = \varphi_p - u_x\xi_p - u_y\eta_p - u_z\zeta_p ,$$

hence, $\xi, \eta, \zeta, \varphi$, and, by similar arguments, ψ and χ do not depend on p. Next the coefficient of u_{xy} and other mixed second-order derivatives of u, v and w in the first three equations of (7.21) shows that

$$\xi_y + \eta_x = 0, \qquad \xi_z + \zeta_x = 0, \qquad \eta_z + \zeta_y = 0 . \tag{7.22}$$

If, in the first equation of (7.21), we replace p_x by its value as given in (7.20), then the resulting coefficient of v_{yy} shows that $\varphi_v + \eta_x = 0$. Similarly, we find

$$\begin{array}{lll} \varphi_v + \eta_x = 0 & \psi_u + \xi_y = 0 & \chi_u + \xi_z = 0 , \\ \varphi_w + \zeta_x = 0 & \psi_w + \zeta_y = 0 & \chi_v + \eta_z = 0 . \end{array} \tag{7.23}$$

Next the coefficient of u_{xx} (again after replacing p_x) in the first equation of (7.21) shows that ξ (and, similarly, η and ζ) depends only on x, y, z. Moreover, we find

$$\begin{aligned} \varphi_u &= \pi_p + \xi_x , \\ \psi_v &= \pi_p + \eta_y , \\ \chi_w &= \pi_p + \zeta_z \end{aligned} \tag{7.24}$$

Note that this implies that π depends linearly on p. The coefficient of u_y in the first equation of (7.21), using (7.24), yields

$$\psi = \eta_x u + (\eta_y - 2\xi_x)v + \eta_z w - \Delta\eta + 2\varphi_{yu} .$$

Comparison with (7.24) requires that $\pi_p = -2\xi_x$. Therefore we get the following representations of φ, ψ, and χ:

$$\varphi = -\xi_x u + \xi_y v + \xi_z w - 3\xi_{xx} - \xi_{yy} - \xi_{xx} ,$$

$$\psi = \eta_x u - \eta_y v + \eta_z w - \eta_{xx} - 3\eta_{yy} - \eta_{zz} , \qquad (7.25)$$

$$\chi = \zeta_x u + \zeta_y v - \zeta_z w - \zeta_{xx} - \zeta_{yy} - 3\zeta_{zz} .$$

Moreover

$$\xi_x = \eta_y = \zeta_z = 0 . \qquad (7.26)$$

Consideration of other first derivatives of u, v, w in the first three equations of (7.20) shows that π is linear in u, v, w and p. Also the mixed partial derivatives of ξ, η, ζ, for instance, ξ_{xz}, are all 0. Finally the term in the first equation of (7.20) not involving any derivatives of u, v, w, p is

$$\pi_x + u\varphi_x + v\varphi_t + 2\varphi_z = \Delta\varphi .$$

The quadratic terms u^2, v^2, and w^2 of this last equation imply that $\xi_{xx} = \xi_{yy} = \xi_{zz} = 0$, so that ξ, and also η and ζ, must be linear in x, y, z. The general solution of (7.21) is then found to be given by

$$\xi = c_1 + c_7 x + c_4 y + c_5 z$$

$$\eta = c_2 - c_4 x + c_7 y + c_6 z$$

$$\zeta = c_3 - c_5 x - c_6 y + c_7 z$$

$$\varphi = -c_7 u + c_4 v + c_3 w$$

$$\psi = -c_4 u - c_7 v + c_6 w$$

$$\chi = -c_5 u - c_6 v - c_7 w$$

$$\pi = c_8 - 2c_7 p .$$

Therefore the Lie algebra of infinitesimal symmetries of the Navier-Stokes equations is spanned by the vector fields

$$\alpha_1 = \partial_x$$

$$\alpha_2 = \partial_y$$

$$\alpha_3 = \partial_z$$

$$\alpha_4 = y\partial_x - x\partial_y + v\partial_u - u\partial_v$$

$$\alpha_5 = z\partial_x - x\partial_z + w\partial_u - u\partial_w$$

$$\alpha_6 = z\partial_y - y\partial_z + w\partial_v - v\partial_w$$

$$\alpha_7 = x\partial_x + y\partial_y + z\partial_z - u\partial_u - v\partial_v - w\partial_w - 2p\partial_p$$

$$\alpha_8 = \partial_p \ .$$

It is easily verified that the first six of these vector fields are just the infinitesimal generators of the action of the group of rigid motions given previously in Chapter 4, section 5. Hence $\alpha_1, \alpha_2, \alpha_3$ generate the translations, and $\alpha_4, \alpha_5, \alpha_6$ the rotations. The vector field α_7 generates a group of scale transformations:

$$G_7: \quad (e^{\lambda}x, \ e^{\lambda}y, \ e^{\lambda}z, \ e^{-\lambda}u, \ e^{-\lambda}v, \ e^{-\lambda}w, \ e^{-2\lambda}p).$$

This means that if

$$(u, v, w, p) = (f(x, y, z), \ g(x, y, z), \ h(x, y, z), j(x, y, z))$$

is a solution of the Navier-Stokes equations, so is

$$(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}) = (e^{-\lambda}f(e^{-\lambda}x, e^{-\lambda}y, e^{-\lambda}z), e^{-\lambda}g, e^{-\lambda}h, e^{-2\lambda}j),$$

where g, h, j are also evaluated at $e^{-\lambda}(x, y, z)$. Finally, α_φ comes from the fact that the Navier-Stokes equations are invariant under pressure translations.

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