The Symmetry Groupoid and Weighted Signature of a Geometric Object

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Abstract. We refine the concept of the symmetry group of a geometric object through its symmetry groupoid, which incorporates both global and local symmetries in a common framework. The symmetry groupoid is related to the weighted differential invariant signature of a submanifold, that is introduced to capture its fine grain equivalence and symmetry properties. Applications to the recognition and symmetry properties of digital images are indicated.

1. Introduction.

Roughly speaking, the "symmetry group" of a geometric object is the set of transformations that leave the object unchanged. The fact that symmetries form a group follows from the fact that the composition of any two symmetries is again a symmetry, as is the inverse of any symmetry.

However, to be precise, one must, a priori, specify the overall class of allowed transformations G to which the symmetries are required to belong. In Klein geometries, [**15**], G represents a finite-dimensional Lie transformation group, or, slightly more generally, a local Lie group action, [**28**], the most familiar example (of the former) being the Euclidean group of rigid motions of Euclidean space. Other well-studied examples are the geometrically-based (local) Lie transformation groups consisting of affine transformations, of projective transformations, or of conformal transformations relative to some, possibly indefinite, metric, the last example requiring that the dimension of the space be ≥ 3.

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More generally, the class G could be a pseudo-group of transformations, [**8**, **10**, **11**, **21**, **35**], for example the pseudo-group of all local homeomorphisms, of local diffeomorphisms, of volume-preserving local diffeomorphisms, the pseudo-group of canonical transformations of a symplectic or Poisson manifold, or the pseudo-group of conformal transformations of the Euclidean or Minkowski plane. Since finite-dimensional local and global Lie group actions can be viewed as Lie pseudo-groups of finite type, it is tempting to develop our ideas in the latter more general, but more technically demanding, framework. However, to keep the exposition here at a reasonable level, we shall restrict ourselves to the Lie group context, leaving extensions of these results to infinite-dimensional pseudo-group actions as projects for the interested reader.

The specification of the underlying transformation group G is important, because the symmetry group associated to an object depends upon it. For example, suppose $S = \partial Q \subset \mathbb{R}^2$ is the boundary of the square $Q = \{-1 \leq x, y \leq 1\}$. If $G = \mathrm{SE}(2)$ is the special Euclidean group consisting of orientation-preserving rigid motions — translations and rotations — then the symmetry group of S is the finite group $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z} \subset \mathrm{SE}(2)$ consisting of rotations by multiples of 90° around the origin. Expanding to $G = \mathrm{E}(2)$, the full Euclidean group containing translations, rotations, and reflections, produces an 8 element symmetry group $\mathbb{Z}_2 \ltimes \mathbb{Z}_4 \subset \mathrm{E}(2)$ containing the above rotations along with the four reflections across the two coordinate axes and the two diagonals. On the other hand, if $G = \mathcal{H}(\mathbb{R}^2)$ is the pseudo-group of all local homeomorphisms, then the induced symmetry group of S is considerably larger. For example, one can easily construct a homeomorphism $\psi: \mathbb{R}^2 \to \mathbb{R}^2$ that maps S to the unit circle $C = \{x^2 + y^2 = 1\}$. Then the composition $\psi^{-1} \circ R \circ \psi$, in which R is any rotation around the origin, defines a homeomorphism symmetry of S. Clearly, in this much broader context, the square now possesses an infinite number of symmetries. One can clearly produce a similar example in the smooth category, that is, the pseudo-group of local diffeomorphisms of the plane, by symmetrically smoothing out the corners of the square.

Let us, from here on, fix the underlying Lie transformation group G acting on the space M, and require symmetries of objects $S \subset M$ to be elements thereof. As emphasized by Weinstein, [**39**], the usual characterization of the symmetry group fails to capture all the symmetry properties of common geometric objects, and its full set of symmetries forms a more general object, known as a *groupoid*, [**25**]. A good example is a square tiling of the plane. If the tiling is of infinite extent, then its Euclidean symmetry group is an infinite discrete group generated by translations in directions of the two tiling axes, combined with the rotation and reflection symmetries of each underlying square tile. However, a bounded portion of the square tiling, e.g. a bathroom floor as in Figure 1, has no translation symmetries and possibly, depending on the shape of the outline of the portion, no rotation or reflection symmetries either, even though to the "untrained" eye a tiled floor remains highly symmetric. One productive way to mathematically retain our aesthetic notion of symmetry is to introduce the concept of a "local symmetry" of a geometric object. Local symmetries no longer form a group, but, rather, have the structure of a groupoid, which we call the *symmetry groupoid*. Indeed, in his survey article, Brown, [**5**], emphasizes that groupoids form the appropriate framework for studying objects with *variable symmetry*; a relevant example is a surface, part of which is spherical, of constant curvature and hence

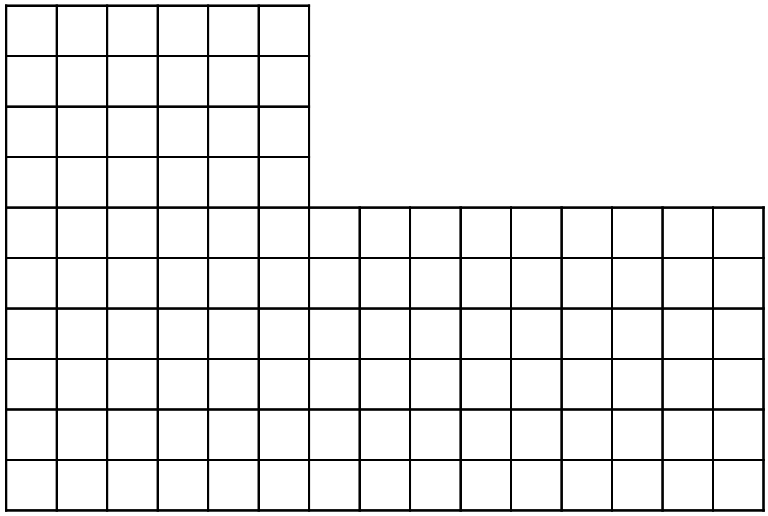

Figure 1. A Bounded Tiling.

a high degree of local symmetry, another part is contained in a surface of revolution, and hence admit a one-parameter family of local symmetries, while other parts have variable principal curvatures, and hence at most a discrete set of local symmetries. While the overall surface may have trivial global symmetry group, its symmetry groupoid retains the inherent local symmetries of its component parts. In Section 2, we present our version of this basic construction, illustrated by several examples.

The key distinction between groups and groupoids is that one is only allowed to multiply elements of the latter under certain conditions. Historically, groupoids were first introduced in the 1920's by Brandt, [**3**], to study quadratic forms. More directly relevant to our concerns is Ehresmann's fundamental paper, [**10**], in which he introduced the powerful tools of Lie groupoids and jet bundles order to study the geometric properties of partial differential equations and, in particular, Lie pseudo-groups. Here, the fundamental examples are the groupoids consisting of Taylor polynomials or series, a.k.a. jets of diffeomorphisms belonging to the pseudo-group, [**11, 35**]. Observe that one can only algebraically compose a pair of Taylor polynomials, $p_2 \circ p_1$, if the source of p_2, meaning its base point, matches the target of p_1, meaning its image point.

While the symmetry groupoid construction developed in Section 2 applies to very general geometric objects, our primarily focus is on (sufficiently smooth) submanifolds. In the equivalence method of Élie Cartan, [**9, 14, 29**], the functional relationships or *syzygies* among its differential invariants serve to prescribe the local equivalence and symmetry properties of a sufficiently regular submanifold. Motivated by a range of applications in image processing, one employs a suitable collection of differential invariants to parametrize the differential invariant signature of a submanifold, [**6**]. The second topic of this paper is to understand how the differential invariant signature is related to the symmetry groupoid of the underlying submanifold. In particular, the codimension and index of the signature directly correspond to the dimension and cardinality of the symmetry groupoid at the corresponding point. In particular, the cogwheel curves of [**27**], which appear to violate the signature symmetry conditions, are explained correctly in the context of their local symmetry groupoids. Motivated by other examples appearing in [**27**], and in response, the extended curve signatures proposed in [**17**] — which was then applied to develop remarkably effective algorithms for the automatic assembly of apictorial jigsaw puzzles, [**18**] — we shall also investigate submanifolds of variable signature rank. The goal is to determine to what extent the equivalence method of Cartan can be extended to this more general context. To this end, we introduce a weighted version of the differential invariant signature, based on uniformly sampling the submanifold relative to some group-invariant measure. We then show how the weighted signature might be employed to analyze the equivalence and symmetry groupoid properties for a broad range of submanifolds. In our analysis of the weighted signature, we are led to a group-invariant version of the celebrated coarea formula of geometric measure theory, [**12, 26**], that is of independent interest; see Section 4 for details. Applications in image processing and elsewhere will be the subject of future investigations.

2. The Symmetry Groupoid of a Geometric Object.

In this section, we shall formalize the notion of the symmetry groupoid of a geometric object, and explain how it better captures the inherent symmetry properties than the more traditional, coarser symmetry group. For simplicity, we shall assume that G is a finite-dimensional Lie group acting smoothly and globally (but not necessarily transitively) on a manifold M. As noted above, with some care, one can readily extend the following constructions to local Lie group actions as well as to infinite-dimensional Lie pseudo-groups.

Let $S \subset M$ be a subset. For us, the most important case is when S is a submanifold, but for the time being we allow S to be arbitrary. The most common definition of a *symmetry* of the subset S is a group transformation $g \in G$ that preserves S, meaning that

$$g \cdot S = \{\, g \cdot z \mid z \in S \,\} = S. \tag{2.1}$$

The set of such symmetries is easily shown to form a subgroup

$$G_S = \{\, g \in G \mid g \cdot S = S \,\} \subset G, \tag{2.2}$$

which we call the *global symmetry group* of $S \subset M$. Its elements will be referred to as *global symmetries* from here on.

As noted in the introduction, global symmetries may fail to fully capture the more subtle symmetry properties of objects, and thus we now refine the concept by suitably localizing the symmetry requirement.

Definition 2.1. A group transformation $g \in G$ is a *local symmetry of S based at the point $z \in S$* if there is an open neighborhood $z \in U \subset M$ such that

$$g \cdot (S \cap U) = S \cap (g \cdot U). \tag{2.3}$$

We denote the set of local symmetries based at z by $G_z \subset G$. Note that, at the very least, the identity $e \in G_z$.

Any global symmetry of S is clearly a local symmetry at each $z \in S$, and so $G_S \subset G_z$. Moreover, the set of global symmetries can be identified as the intersection of all local symmetry sets:

$$G_S = \bigcap_{z \in S} G_z.$$

Although the set of all local symmetries does not, in general, form a group — see below for examples — it does form a groupoid, in accordance with the following definition, [**25**].

Definition 2.2. A *groupoid* over a base S is a set $\mathcal{G}$ along with a pair of surjective maps $\boldsymbol{\sigma}, \boldsymbol{\tau}: \mathcal{G} \to S$, called the *source* and *target* maps, a binary operation $(\alpha, \beta) \longmapsto \alpha \cdot \beta$, called *multiplication*, that is defined on the set

$$\mathcal{G} \star \mathcal{G} = \{\, (\alpha, \beta) \mid \boldsymbol{\sigma}(\alpha) = \boldsymbol{\tau}(\beta) \,\} \subset \mathcal{G} \times \mathcal{G},$$

and an injective *identity map* $\varepsilon: S \to \mathcal{G}$, satisfying the following conditions:

- *Source and target of products*: $\boldsymbol{\sigma}(\alpha \cdot \beta) = \boldsymbol{\sigma}(\beta)$, $\boldsymbol{\tau}(\alpha \cdot \beta) = \boldsymbol{\tau}(\alpha)$, for $(\alpha, \beta) \in \mathcal{G} \star \mathcal{G}$.
- *Associativity*: $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$ when defined, that is for $(\alpha, \beta),\ (\beta, \gamma) \in \mathcal{G} \star \mathcal{G}$, which implies that $(\alpha \cdot \beta, \gamma),\ (\alpha, \beta \cdot \gamma) \in \mathcal{G} \star \mathcal{G}$.
- *Identity*: $\boldsymbol{\sigma}(\varepsilon(x)) = x = \boldsymbol{\tau}(\varepsilon(x))$, while $\alpha \cdot \varepsilon(x) = \alpha$ when $x = \boldsymbol{\sigma}(\alpha)$, and $\varepsilon(y) \cdot \alpha = \alpha$ when $y = \boldsymbol{\tau}(\alpha)$.
- *Inverses*: each $\alpha \in \mathcal{G}$ has a two-sided inverse $\alpha^{-1} \in \mathcal{G}$ such that $\boldsymbol{\sigma}(\alpha) = \boldsymbol{\tau}(\alpha^{-1})$, $\boldsymbol{\tau}(\alpha) = \boldsymbol{\sigma}(\alpha^{-1})$, and $\alpha^{-1} \cdot \alpha = \varepsilon(\boldsymbol{\sigma}(\alpha))$, $\ \alpha \cdot \alpha^{-1} = \varepsilon(\boldsymbol{\tau}(\alpha))$.

Remark: For the more abstract-minded reader, a streamlined version of this definition is to say that a groupoid is "a small category such that every morphism has an inverse". See the survey paper [**5**] and text [**25**] for further details, references, and applications in algebra, geometry, topology, crystallography, and elsewhere.

Example 2.3. Let G be a Lie group acting on a manifold M. The *action groupoid* is the principal bundle $\mathcal{G} = G \times M$, with source and target maps

$$\boldsymbol{\sigma}(g, z) = z, \qquad \boldsymbol{\tau}(g, z) = g \cdot z \qquad \text{for} \qquad (g, z) \in G \times M. \tag{2.4}$$

The groupoid multiplication, inversion, and identity are explicitly given by

$$(h, g \cdot z) \cdot (g, z) = (h \cdot g, z), \qquad (g, z)^{-1} = (g^{-1}, g \cdot z), \qquad \varepsilon(z) = (e, z), \tag{2.5}$$

for $g, h \in G$, $z \in S$, and e the identity element of G. The verification of the groupoid axioms is straightforward.

Definition 2.4. Given a Lie group G acting on a manifold M, the *symmetry groupoid* of a subset $S \subset M$ is the following subgroup of the action groupoid:

$$\mathcal{G}_S = \{\, \alpha = (g, z) \mid z \in S,\ g \in G_z \,\} \subset G \times S, \tag{2.6}$$

Thus, the induced source and target maps, multiplication, inversion, and identity on $\mathcal{G}_S$ are given by the same formulas (2.4), (2.5). Clearly, if $g \in G_z$ then $g^{-1} \in G_{g \cdot z}$; further, if $h \in G_{g \cdot z}$, then $h \cdot g \in G_z$, which thus confirms the groupoid structure of $\mathcal{G}_S$. Set

$$\mathcal{G}_z = \boldsymbol{\sigma}^{-1}\{\, z \,\} = G_z \times \{\, z \,\} \tag{2.7}$$

to be the *source fiber* of $z \in S$.

Remark: A groupoid is a *Lie groupoid* if $\mathcal{G}$ and M are smooth, meaning C^∞, manifolds, the source and target maps are smooth surjective submersions, and the identity and multiplication maps are smooth. The action groupoid is trivially a Lie groupoid. However, as we will see in later examples, symmetry groupoids, even of smooth submanifolds $S \subset M$, are not necessarily Lie groupoids.

Remark: Some authors relax the global symmetry condition (2.1) by only requiring that $g \cdot S \subset S$. Under this definition, the set of global symmetries only forms a semigroup in general, since the inverse of g may fail to satisfy the relaxed symmetry condition. For example, consider the three-parameter semi-direct product group $G = \mathbb{R}^+ \ltimes \mathbb{R}^2$ that acts on $M = \mathbb{R}^2$ by translations and dilations:

$$(x, y) \longmapsto (\lambda\, x + a, \lambda\, y + b) \qquad \text{for} \qquad \lambda \in \mathbb{R}^+, \quad (a, b) \in \mathbb{R}^2.$$

Then, for the line segment

$$S = \{\, (x, 0) \mid\ -1 < x < 1 \,\}\,, \tag{2.8}$$

say, the set of group transformations $g \in G$ satisfying $g \cdot S \subset S$ is the scaling semigroup

$$\{\, (\lambda, 0, 0) \mid\ 0 < \lambda \leq 1 \,\} \subsetneq \mathbb{R}^+.$$

Similarly, relaxing the local symmetry condition (2.3) to $g \cdot (S \cap U) \subset S$ will produce a "semi-groupoid" of local symmetries. We will not pursue this further extension of the theory here.

The groupoid elements that fix a point $z \in S$ form a bona fide group, $G_z^* \subset G_z$, satisfying

$$G_z^* \times \{z\} = \{\, \alpha \in \mathcal{G} \mid\ \boldsymbol{\sigma}(\alpha) = z = \boldsymbol{\tau}(\alpha) \,\} \subset \mathcal{G}_z, \tag{2.9}$$

and known in the groupoid literature as the *vertex group* of z, [**25**]. However, I would prefer to call it the *isotropy group* or, when seeking to emphasize that these are not global symmetries, the *local isotropy group* of the point z, in keeping with standard transformation group terminology, and also in view of our later use of the term "vertex" in the context of curve geometry, [**15**]. Note that if $\alpha, \beta \in \mathcal{G}$ have the same source, $z = \boldsymbol{\sigma}(\alpha) = \boldsymbol{\sigma}(\beta)$, and target, $\boldsymbol{\tau}(\alpha) = \boldsymbol{\tau}(\beta)$ then $\gamma = \beta^{-1} \cdot \alpha \in G_z^*$. Moreover, the local isotropy groups form a "normal system" within $\mathcal{G}$ in the sense that if $g \in G_z$ and $h \in G_z^*$, then $g \cdot h \cdot g^{-1} \in G_{g \cdot z}$. Or, to state this in another way, adapting the standard notation for the adjoint map of a group, [**28**],

$$\operatorname{Ad} \alpha \cdot G^*_{\boldsymbol{\sigma}(\alpha)} = G^*_{\boldsymbol{\tau}(\alpha)} \quad \text{for any} \quad \alpha \in \mathcal{G}. \tag{2.10}$$

Example 2.5. Let $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ be the *special Euclidean group* consisting of all orientation-preserving rigid planar motions — translations and rotations[†] — acting

[†] Extending this discussion to the full *Euclidean group* $G = \mathrm{E}(2) = \mathrm{O}(2) \ltimes \mathbb{R}^2$, that also includes reflections, is a worthwhile exercise for the reader.

on the plane $M = \mathbb{R}^2$. We parametrize G by $g = (\theta, a, b)$, for $\theta \in \mathrm{SO}(2) \simeq S^1$, $(a, b) \in \mathbb{R}^2$, representing the rigid motion

$$(x, y) \longmapsto (x\cos\theta - y\sin\theta + a,\ x\sin\theta + y\cos\theta + b). \tag{2.11}$$

Let us determine the Euclidean symmetry groupoid for a few basic subsets $S \subset \mathbb{R}^2$.

If S is a circle, then its symmetry groupoid is simply the Cartesian product (principal bundle) $\mathcal{G}_S = G_S \times S$ where $G_S \simeq \mathrm{SO}(2) \subset \mathrm{SE}(2)$ is its global symmetry group — the one-parameter subgroup of rotations around the center of the circle. For example, the unit circle

$$S_1 = \{x^2 + y^2 = 1\},$$

has (special) Euclidean symmetry groupoid

$$\mathcal{G}_{S_1} = \left\{ (\theta, 0, 0; x, y) \;\middle|\; 0 \le \theta < 2\pi,\ x^2 + y^2 = 1 \right\} \simeq \mathrm{SO}(2) \times S_1 \subset \mathrm{SE}(2) \times S_1.$$

The source and target maps on $\mathcal{G}_{S_1}$ are

$$\boldsymbol{\sigma}(\theta, 0, 0; x, y) = (x, y), \qquad \boldsymbol{\tau}(\theta, 0, 0; x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta),$$

with consequent groupoid multiplication

$$(\varphi, 0, 0; x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \cdot (\theta, 0, 0; x, y) = (\varphi + \theta \mod 2\pi, 0, 0; x, y), \tag{2.12}$$

which is only defined when the target of the right hand element matches the source of the left hand element:

$$\begin{aligned} \boldsymbol{\tau}(\theta, 0, 0; x, y) &= \boldsymbol{\sigma}(\varphi, 0, 0; x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \\ &= (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta). \end{aligned}$$

If S is only a circular arc, then it has no global special Euclidean symmetries other than the identity. On the other hand, its symmetry groupoid is not trivial. For example, the semicircle

$$S_+ = \left\{ (x, y) \;\middle|\; x^2 + y^2 = 1,\ x > 0 \right\},$$

has symmetry groupoid

$$\mathcal{G}_{S_+} = \left\{ (\theta, 0, 0; x, y) \;\middle|\; x^2 + y^2 = 1,\ x > 0,\ -\frac{\pi}{2} < \theta + \tan^{-1}\frac{y}{x} < \frac{\pi}{2} \right\} \subsetneq \mathrm{SO}(2) \times S,$$

with the same groupoid multiplication rule (2.12). The symmetry groupoid of a closed circular arc is of the same form at the interior points, but the endpoints only have the trivial identity map as a local symmetry. (Keep in mind that we are only considering orientation-preserving Euclidean symmetries here.)

If $S \subset \mathbb{R}^2$ is a straight line of infinite extent, its symmetry groupoid is $\mathcal{G}_S = G_S \times S$, where $G_S \simeq \mathbb{Z}_2 \ltimes \mathbb{R} \subset \mathrm{SE}(2)$ is its global symmetry group consisting of translations in the direction of the line and 180° rotations centered at any point thereon. A bounded open line segment S — for example (2.8) — has only two global special Euclidean symmetries: the identity and the 180° rotation around its center. On the other hand, the local Euclidean symmetry set G_z of a point $z \in S$ is generated by all translations in the direction of S that map z to another point on S, combined with the $180°$ rotation centered at z; the latter forms (along with the identity) the isotropy symmetry group $G_z^* \simeq \mathbb{Z}_2$ of a point $z \in S$.

The case when S is a square is more interesting. The local symmetry set of one of the corner points is the four element subgroup, $G_z \simeq \mathbb{Z}_4 \subset \mathrm{SE}(2)$, containing all rotations by multiples of $\frac{1}{2}\pi$ around the center of the square. Indeed, as noted in the introduction, this is the global special Euclidean symmetry group of the square: $G_S = \mathbb{Z}_4$. On the other hand, if $z \in S$ is not a corner, its local symmetry set G_z consists of those translations that map z to another point on the same side of the square, as in the case of the line segment, and, in addition, the transformations obtained by following such a translation by a rotation in the global symmetry group $\mathbb{Z}_4$ and/or by a $180°$ rotation centered at the image point.

Definition 2.6. The *symmetry orbit* of $z \in S$ is the image of source fiber over z under the target map:

$$\mathcal{O}_z = \boldsymbol{\tau}(\mathcal{G}_z) = \boldsymbol{\tau} \circ \boldsymbol{\sigma}^{-1}\{z\} = \{\, g \cdot z \mid\ g \in G_z \,\}. \tag{2.13}$$

By the preceding remarks, there is an evident one-to-one correspondence between the orbit through z and the quotient of its local symmetry set by its local isotropy group: $\mathcal{O}_z \simeq G_z/G_z^*$. We use the orbits to define an equivalence relation on S, with $z \sim \widehat{z}$ if and only if they belong to the same orbit, or, equivalently, $\widehat{z} = g \cdot z$ for some $g \in G_z$. Note that this is well-defined because if $\widehat{z} = g \cdot z$ for $g \in G_z$, then $z = g^{-1} \cdot \widehat{z}$ with $g^{-1} \in G_{\hat{z}}$; equivalently, $\widehat{z} \in \mathcal{O}_z$ if and only if $z \in \mathcal{O}_{\hat{z}}$. Moreover, if $\widetilde{z} = h \cdot \widehat{z}$ with $h \in G_{\hat{z}}$, then $\widetilde{z} = (h \cdot g) \cdot z$ and $h \cdot g \in G_z$. The set of equivalence classes is the *symmetry orbit space* or *symmetry moduli space* of S, and denoted $S^{\mathcal{G}} = S/\sim$. We let $\pi^{\mathcal{G}} \colon S \to S^{\mathcal{G}}$ denote the projection that maps a point $z \in M$ to the equivalence class defined by its orbit $\mathcal{O}_z$.

In Example 2.5, the circle, open circular arc, line, and open line segment each consist of a single orbit. The square has two orbits, one containing the four corners and the other consisting of all the remaining points. The tiling in Figure 1, where S consists of all the line segments and vertices in the figure, i.e., the tile grout†, has four orbits: the points on the interior of any edge; the interior vertices along with the concave corner vertex where four edges meet; the side vertices where three edges meet; and the remaining 5 convex corner vertices where two orthogonal edges meet. The reader is encouraged to determine the local symmetry sets in each case.

Finally, we make the trivial observation that two globally equivalent subsets have isomorphic symmetry groupoids.

Proposition 2.7. *If $\widetilde{S} = h \cdot S$ are congruent under a group element $h \in G$, then so are their symmetry groupoids:*

$$\mathcal{G}_{\widetilde{S}} = \operatorname{Ad} h \cdot \mathcal{G}_S \equiv \left\{\, \beta = (h \cdot g \cdot h^{-1}, h \cdot z) \;\middle|\; \alpha = (g, z) \in \mathcal{G}_S \,\right\}.$$

† Weinstein, [**39**], discusses a similar example in detail, but also includes the interior points of the tiles; however this requires treating the tiling as something other than a single subset of the plane.

3. Signature.

We now direct our attention to the important case when $S \subset M$ is a smooth[†] — meaning C^∞ — embedded, connected submanifold. We set $m = \dim M$, and $1 \le p = \dim S < m$. The following definition[‡] will prove useful throughout.

Definition 3.1. A *piece* of the submanifold S is a connected subset $\widehat{S} \subset S$ whose interior, under the induced topology of S, is a non-empty submanifold of the same dimension $p = \dim \widehat{S} = \dim S$, and whose boundary $\partial \widehat{S}$ is a piecewise smooth submanifold of dimension $p-1$. A piece can contain some or all of its boundary points.

For example, a piece of a curve is a connected sub-curve, possibly containing one or both of its endpoints.

Cartan's solution to the local equivalence problem for submanifolds under Lie group and Lie pseudo-group actions relies on the functional dependencies or *syzygies* among their differential invariants, [**9**, **14**, **29**]. Following [**6**, **13**, **29**], we employ a suitable finite collection of differential invariants $I_1, \ldots, I_l$ to parametrize the *differential invariant signature*[§] $\Sigma = \Sigma_S$ of the submanifold S. Having prescribed the signature invariants, we define the associated *signature map*

$$\chi : S \longrightarrow \Sigma = \chi(S) \subset \mathbb{R}^n, \qquad \chi(z) = (I_1(z), \ldots, I_l(z)), \tag{3.1}$$

whose image is the *signature set* of S. The determination of an appropriate set of signature invariants follows from either the exterior differential systems approach to the equivalence method, [**14**, **29**], or, alternatively, the recurrence formulae for the normalized differential invariants resulting from the calculus of equivariant moving frames, [**13**]. The key requirement is that the syzygies among the signature invariants serve to uniquely prescribe the syzygies among *all* the differential invariants; examples appear below and in the indicated references. The selection and overall number of signature invariants required for this purpose depends upon the group G as well as the dimension and, possibly, other intrinsic properties of the submanifold S.

Example 3.2. Consider the preceding action (2.11) of the special Euclidean group SE(2) on plane curves $S \subset M = \mathbb{R}^2$. According to [**6**, **13**], the Euclidean signature of a plane curve is the set $\Sigma \subset \mathbb{R}^2$ parametrized by the Euclidean curvature invariant κ and its derivative with respect to the Euclidean arc length element ds:

$$\chi : S \longrightarrow \Sigma \subset \mathbb{R}^2, \qquad \chi(z) = (\kappa(z), \kappa_s(z)). \tag{3.2}$$

The Euclidean signature degenerates to a single point if and only if $\kappa = c$ is constant, and so $\kappa_s = 0$, which is equivalent to the curve being an arc of a circle of radius $1/c$, or, when

[†] The smoothness assumption can evidently be weakened in much of what follows.

[‡] I could not find a suitable term in the standard literature. The word "piece" is inspired by "puzzle piece", which is an important application, [**18**].

[§] In [**13**, **29**], the signature is called the *classifying set* of S. The term *signature* was adopted later in light of significant applications in image processing, [**6**], and is now consistently used in the literature.

$c = 0$, a straight line segment. Otherwise, at least away from singularities and degeneracies, the signature map traces out an immersed plane curve, typically with self-intersections.

In this case, a complete system of differential invariants is provided by the successive derivatives of the curvature invariant with respect to arc length: $\kappa, \kappa_s, \kappa_{ss}, \ldots$. Assuming $\kappa_s \neq 0$, the signature curve (3.2) locally determines the syzygy

$$\kappa_s = H(\kappa) \tag{3.3}$$

relating the two signature invariants. Successive differentiation of this syzygy produces the corresponding syzygies among all the higher order differential invariants; for example, by the chain rule,

$$\kappa_{ss} = \frac{d}{ds}\,\kappa_s = \frac{d}{ds}\,H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa),$$

and so on. This justifies our choice of κ, κ_s as the signature invariants.

In Euclidean curve geometry, [**15**], a *vertex* is defined as a point at which the curvature κ has an extremal — maximum or minimum — and hence where $\kappa_s = 0$; inflection points of curvature are usually excluded. In [**17**], the notion of a *generalized vertex* was introduced, and characterized as a point or connected piece of the curve on which $\kappa_s \equiv 0$, the latter being either a circular arc or straight line segment. Each generalized vertex maps to a single point $\chi_0 = (\kappa_0, 0)$ of Σ, lying on the κ axis, where κ_0 equals the reciprocal of the (signed) radius of the circular arc, or, in the case of a line segment, 0. This implies that one cannot use the curve traced by the signature map to determine the overall length of such vertices. This observation was exploited by Musso and Nicolodi, [**27**], in their construction of inequivalent closed curves possessing a common Euclidean signature; see also [**16**]. In [**17**], an extension of the standard Euclidean signature that would overcome these difficulties was proposed, and then exploited in the design of a rather successful algorithm for automated jigsaw puzzle assembly, [**18**].

More generally, consider the action of a Lie group G on $M = \mathbb{R}^2$. We assume that the action is "ordinary", meaning that G acts transitively and its prolonged actions on the curve jet space do not "pseudo-stabilize". Almost all transitive planar group actions are ordinary; see [**29**] for a complete list, based on Lie's classification tables, [**23**]. In this situation, for the induced action of G on plane curves, there is a unique, up to functions thereof, differential invariant κ of lowest order, known as the *G-invariant curvature*, along with a unique, up to scalar multiple, invariant differential form $\omega = ds$ of lowest order, known as the *G-invariant arc length element*†. The corresponding *G-invariant signature* of a curve $C \subset M$ takes an identical form, (3.2), parametrized by the curvature invariant κ and its derivative κ_s with respect to the G-invariant arc length. Here, a *generalized vertex* $V \subset S$ is either a stationary point of the curvature invariant, or a maximal connected subset upon which $\kappa \equiv \kappa_0$ is constant, and hence $\kappa_s \equiv 0$. As a consequence of the discussion following Theorem 3.6 below, this implies that the latter is, in fact, a piece of an orbit of a suitable one-parameter subgroup $H \subset G$; in other words $V \subset H \cdot z$ for any $z \in V$. (See

† More correctly, ds is the contact-invariant horizontal component of a fully G-invariant one-form on the jet space, [**13**]. See Section 4 below for further comments.

[**33**] for a general method for determining the value of the curvature invariant κ_0 along such an orbit.)

Remark: A one-parameter subgroup $H \subset G$ is "suitable" when its orbits have a one-dimensional global symmetry group, whose connected component containing the identity is H itself. Orbits that have global symmetry groups of dimension > 1, which hence act non-freely thereon, are called *totally singular*, and do not admit well-defined signatures. A well-known example is when $G = \mathrm{SA}(2)$ is the equi-affine group consisting of area-preserving affine transformations of $M = \mathbb{R}^2$, [**7, 15**]. The orbits of one-parameter subgroups $H \subset \mathrm{SA}(2)$ are conic sections, which, when nonsingular, are the curves of constant equi-affine curvature $\kappa = \kappa_0$. However, straight lines admit a three-parameter equi-affine symmetry group, and hence are totally singular. Indeed, the equi-affine curvature is not defined at inflection points or on straight line segments. See [**30**] for details, including a Lie algebraic characterization of the totally singular orbits.

Typically, as in the case of equi-affine plane curves, the differential invariants are not defined on all submanifolds of the given dimension, and their common domain is a dense open subset of the submanifold jet bundle. For simplicity, from here on we will assume that the signature map (3.1) is well-defined at all points of the submanifolds S under consideration. There are extensions of the theory, involving higher order differential invariants and the like, that can be employed to extend the range of validity of the signature method. Alternatively, one may be able to avoid singularities where the differential invariants blow up by assuming that the signature map takes values in a suitably projectivized version of Euclidean space. However, this approach has yet to be explored in any depth.

Example 3.3. Consider the standard action of the special Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ on $M = \mathbb{R}^3$ by orientation-preserving rigid motions: translations and rotations. The basic differential invariants of a smooth curve $S \subset M$ are its curvature, which must be nonzero† $\kappa \neq 0$, and torsion τ and their successive derivatives with respect to the Euclidean-invariant arc length element ds, namely $\kappa_s, \tau_s, \kappa_{ss}, \tau_{ss}, \ldots$. A signature map can be constructed from the first three of these, so that

$$\chi \colon S \longrightarrow \Sigma \subset \mathbb{R}^3, \qquad \chi(z) = (\kappa(z), \kappa_s(z), \tau(z)). \tag{3.4}$$

Observe that we do not need to include τ_s since, assuming $\kappa_s \neq 0$, i.e., we are not at a vertex, we can locally express $\kappa_s = F(\kappa)$, and $\tau = H(\kappa)$, which, by the chain rule, uniquely determines the syzygy

$$\tau_s = \frac{d}{ds} H(\kappa) = H'(\kappa)\, \kappa_s = H'(\kappa)\, F(\kappa).$$

As in the case of plane curves, the higher order syzygies can all be obtained by repeated differentiation, justifying the above choice of signature invariants.

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† This assumption is needed in order that the indicated complete set of differential invariants be well defined. In contrast, straight lines, with $\kappa \equiv 0$, are totally singular orbits since they admit a two-parameter Euclidean symmetry group.

Similarly, to analyze the equivalence of two-dimensional surfaces $S \subset \mathbb{R}^3$ under the same Euclidean group action, there are two familiar second order differential invariants: the Gauss curvature K and the mean curvature H, being, respectively, the product and average of the two principal curvatures κ_1, κ_2. Again, one can produce an infinite collection of higher order differential invariants by invariantly differentiating the Gauss and mean curvature. Specifically, at a non-umbilic point where $\kappa_1 \neq \kappa_2$, there exist two non-commuting invariant differential operators† $\mathcal{D}_1, \mathcal{D}_2$, that effectively differentiate in the direction of the diagonalizing Darboux frame. A complete system of differential invariants is provided by

$$K,\ H,\ \mathcal{D}_1 K,\ \mathcal{D}_2 K,\ \mathcal{D}_1 H,\ \mathcal{D}_2 H,\ \mathcal{D}_1^2 K,\ \mathcal{D}_1\mathcal{D}_2 K,\ \mathcal{D}_2\mathcal{D}_1 K,\ \mathcal{D}_2^2 K,\ \mathcal{D}_1^2 H,\ \ldots\ .$$

For a generic surface, one can use those of order ≤ 1 to parametrize a signature:

$$\chi\colon S \longrightarrow \Sigma \subset \mathbb{R}^6, \qquad \chi(z) = (K(z), H(z), \mathcal{D}_1 K(z), \mathcal{D}_2 K(z), \mathcal{D}_1 H(z), \mathcal{D}_2 H(z)). \tag{3.5}$$

The exceptional surfaces are those for which the mean and Gauss curvatures are functionally dependent and yet there is a second functionally independent differential invariant among their first order invariant derivatives; a complete signature in these cases requires second order differential invariants. See [**29**] for further details.

Alternatively, a non-umbilic surface is called *mean curvature degenerate* if, for any $z_0 \in S$, there exist scalar functions $F_1(t), F_2(t)$, such that

$$\mathcal{D}_1 H = F_1(H), \qquad \mathcal{D}_2 H = F_2(H), \tag{3.6}$$

at all points $z \in S$ in a small neighborhood of z_0. According to a slight refinement of the result in [**32**] — see [**36**] for details — if S is mean curvature *non-degenerate*, then one can, in fact, express its Gauss curvature as a universal rational function of the mean curvature and its invariant derivatives of order ≤ 4. In other words, for such submanifolds, the mean curvature H serves to generate the entire algebra of Euclidean surface invariants, and one can employ it and its derivatives‡

$$\widetilde{\chi}\colon S \longrightarrow \Sigma \subset \mathbb{R}^5, \qquad \widetilde{\chi}(z) = (H(z), \mathcal{D}_1 H(z), \mathcal{D}_2 H(z), \mathcal{D}_1^2 H(z), \mathcal{D}_2^2 H(z)). \tag{3.7}$$

to parametrize an alternative Euclidean differential invariant signature.

Returning to the general situation, since differential invariants are, by definition, invariant under the (prolonged) action of G, the signature map (3.1) is not affected by the local symmetries of S. In other words, if $g \in G_z$ is a local symmetry based at $z \in S$, then

$$\chi(g \cdot z) = \chi(z), \qquad \text{whenever} \qquad \alpha = (g, z) \in \mathcal{G}_S. \tag{3.8}$$

† These are *not* the same as the operators of covariant differentiation, but are closely related, [**15**, **29**].

‡ One can show that the mixed derivatives $\mathcal{D}_1\mathcal{D}_2 H$, $\mathcal{D}_2\mathcal{D}_1 H$ are not needed here.

This implies that the signature map is constant on the symmetry groupoid orbits, and hence factors through the symmetry moduli space:

$$\begin{tikzcd} S \arrow[rr, "\chi"] \arrow[dr, "\pi^{\mathcal{G}}"'] & & \Sigma \\ & S^{\mathcal{G}} \arrow[ur, "\widetilde{\chi}"'] & \end{tikzcd} \qquad \text{with} \qquad \widetilde{\chi}(\pi^{\mathcal{G}}(z)) = \widetilde{\chi}(\mathcal{O}_z) = \chi(z). \tag{3.9}$$

Let us define the *signature rank*, or *rank* for short, of a point $z \in S$ to be the rank of the signature map at z, i.e., the rank of its Jacobian matrix or, equivalently, its differential:

$$0 \leq r_z = \operatorname{rank} d\chi|_z \leq p = \dim S. \tag{3.10}$$

For example, in the case of the Euclidean signature of a plane curve (3.2), the signature rank is 0 if both $\kappa_s = \kappa_{ss} = 0$; otherwise it is 1. We note that r_z is a bounded lower semicontinuous function of $z \in S$, meaning that $r_z \leq r_{\widehat{z}}$ for all $\widehat{z} \in S$ in a sufficiently small neighborhood of z. Equation (3.8) implies that the signature rank is constant on the symmetry orbits of S. A point $z \in S$ is called *regular* if the signature rank is constant in a neighborhood of z, i.e., on $S \cap U$ where $U \subset M$ is an open subset containing $z \in S \cap U$. Clearly, if $z \in S$ is regular, then any point in its symmetry orbit $\widehat{z} \in \mathcal{O}_z$ is also regular.

Proposition 3.4. *If $z \in S$ is regular of rank k, then the image of the signature map restricted to a suitable neighborhood $z \in U$ forms a k-dimensional submanifold $\chi(S \cap U) \subset \Sigma$.*

Remark: In general, [**29**], the *differential invariant rank* of a point $z \in S$ is defined as the maximum, over all $n \geq 0$, of the rank of the map defined by all its differential invariants of order $\leq n$. Clearly the signature rank is bounded from above by the differential invariant rank although generically, away from singularities, they agree. If we knew in advance the order at which the differential invariant rank is achieved, then we could extend the signature map by including all the differential invariants up to that order and thus replace signature rank by differential invariant rank. However, (a) the maximum may not be achieved until high order, and (b) for most submanifolds, the resulting extended signature is highly redundant and of scant practical value. For these reasons, we will restrict our attention to the signature rank from here on.

With the proper choice of signature invariants, the Cartan Equivalence Theorem, [**13, 29**], states that the resulting signature completely determines the symmetry orbit of a regular point. More specifically, two points lie on the same symmetry orbit if and only if their signatures locally coincide.

Theorem 3.5. *If $z \in S$ is regular, then $\widehat{z} = g \cdot z \in \mathcal{O}_z$ for $g \in G_z$ if and only if there exist neighborhoods $z \in U \subset M$ and $\widehat{z} \in \widehat{U} \subset M$ such that $\chi(S \cap U) = \chi(S \cap \widehat{U})$.*

Note that it is not sufficient to require $\chi(z) = \chi(\widehat{z})$ in order that z and $\widehat{z}$ lie in the same symmetry orbit. For example, the Euclidean signature of a plane curve (3.2) is typically an immersed plane curve with self-intersections. If $\chi(z) = \chi(\widehat{z}) \in \Sigma$ is an intersection point, then the condition in the Equivalence Theorem 3.5 will not hold when the parts of S near z and $\widehat{z}$ map to different branches of the self-intersecting signature curve.

Let $S_{\mathrm{reg}} \subset S$ be the open dense subset containing all regular points. We decompose

$$S_{\mathrm{reg}} = \bigcup_{k=0}^{p} S_k, \tag{3.11}$$

where $S_k \subset S_{\mathrm{reg}}$ is the set of regular points of rank k. Note that points on its boundary, $z \in \partial S_k$, are not regular, and, by lower semicontinuity, have rank $0 \le r_z \le k$; further, if $r_z = k$, then $z \in \partial S_l$ for some $l > k$. In particular, all points on ∂S_p have rank strictly less than $p = \dim S$. Let $\Sigma_{\mathrm{reg}} = \chi(S_{\mathrm{reg}})$ be the corresponding regular part of the signature set, which we also decompose by setting $\Sigma_k = \chi(S_k)$. Since $\operatorname{rank}\chi = k$ on S_k, we conclude that $\Sigma_k \subset \mathbb{R}^n$ is an immersed k-dimensional submanifold. We further decompose

$$S_k = \bigcup_{i} S_{k,i}$$

into a disjoint union of connected pieces $S_{k,i} \subset S$, whose images $\Sigma_{k,i} = \chi(S_{k,i}) \subset \Sigma$ are connected subsets of the signature.

Another consequence of the Cartan Equivalence Theorem is that, at regular points, the signature rank also prescribes the size of the local symmetry set, [**13**, **29**].

Theorem 3.6. *A point $z \in S_k$ is regular of rank $0 \le k \le p$ if and only if its local symmetry set G_z is a local Lie group of dimension $p-k$, or, more precisely, its connected component containing the identity, $e \in \widetilde{G}_z \subset G_z$, is a relatively open subset of a $(p-k)$-dimensional Lie subgroup, $\widetilde{G}_z \subset \widehat{G}_z \subset G$. We call this subgroup $\widehat{G}_z$ the* completion *of the local symmetry set G_z.*

In particular, the *maximally symmetric component* $S_0 \subset S$ consists of regular points of signature rank 0 (if any). Each of its pieces $S_{0,i}$ maps to a single point of the signature Σ and, in fact, is a piece of an orbit of a suitable p-dimensional subgroup $\widehat{G}_{0,i} \subset G$. Indeed, $\widehat{G}_{0,i} = \widehat{G}_z$ is the completion of the local symmetry set of any point $z \in S_{0,i}$. More generally, [**29**]:

Theorem 3.7. *If S is connected and of constant rank k, then its local symmetry sets all have a common $(p-k)$-dimensional connected Lie subgroup $\widehat{G}_S \subset G$ as their completion: $\widehat{G}_z = \widehat{G}_S$ for all $z \in S$. Moreover, S is the disjoint union of a k-parameter family of pieces of orbits of $\widehat{G}_S$. The connected component of its symmetry groupoid,*

$$\mathcal{G}_S^* = \left\{ (g,z) \,\middle|\, z \in S,\ g \in \widetilde{G}_z \right\} \subset S \times \widehat{G}_S, \tag{3.12}$$

is an open subbundle of the principal bundle $S \times \widehat{G}_S$ containing the identity section, and hence a Lie groupoid.

For example, a surface $S \subset \mathbb{R}^3$ contained in a non-cylindrical surface of revolution has constant Euclidean rank 1, and is the disjoint union of circular arcs centered on a common axis, each contained in a circular orbit of the one-parameter subgroup $\widehat{G}_S \subset \mathrm{SE}(3)$ of rotations around the axis.

More generally, if S is of variable rank, containing more than one nonempty component $S_k \subset S$ consisting of regular points of rank k, then $\mathcal{G}_S$ is *not* a Lie groupoid since, according to Theorem 3.6, its source fiber dimension is not constant on all of S. An example would be a Euclidean plane curve of the type considered in [**27**], which contains several pieces of rank 1 along with some circular arcs and/or straight line segments of rank 0. According to Theorem 3.7, each connected component $S_{k,i} \subset S_k$ is the disjoint union of pieces of a k-parameter family of orbits of a common $(p-k)$-dimensional Lie subgroup $\widehat{G}_{k,i} \subset G$, which can be identified as the completion $\widehat{G}_{k,i} = \widehat{G}_z$ of the local symmetry set of any point $z \in S_{k,i}$ therein. Of course, the subgroup may well vary from component to component.

In particular, points $z \in S_p$ belonging to the component of maximal rank (if such exists) have a purely discrete local symmetry set $G_z \subset G$. In [**6**, **29**], the number of local symmetries, $\# G_z$, was characterized by the *signature index*, which, roughly, counts the number of points on the original submanifold mapping to the same point in the signature. More rigorously, let us call a point $z \in S_p$ *completely regular* if Σ is, locally, an embedded p-dimensional submanifold in a neighborhood of the image point $\chi(z) \in \Sigma$. Thus, complete regularity excludes points of self-intersection and other singularities of the signature. Let $S_p^* \subset S_p$ denote the open dense subset of all completely regular points of rank p. The *index* of $z \in S_p^*$ is then defined as the number of points in S_p^* that are mapped to the same signature point $\zeta = \chi(z)$:

$$\operatorname{ind} z = \# \left\{ \widehat{z} \in S_p^* \;\middle|\; \chi(\widehat{z}) = \chi(z) = \zeta \right\} = \# \, \chi^{-1}\{\zeta\} = \operatorname{ind} \zeta. \tag{3.13}$$

Complete regularity implies that the signatures near any point $z \in \chi^{-1}\{\zeta\}$ are locally identical.

The index determines the number of discrete local symmetries of z that do *not* fix it, in the following sense. Recall that $G_z^* \subset G_z$ denotes the local isotropy group of the point z, that is, all the local symmetries that fix z. The quotient set G_z/G_z^* must be discrete, since it is in one-to-one correspondence with the pieces in S that are equivalent to a suitably small piece containing z. Theorem 3.5 then implies the following:

Proposition 3.8. *The index of a completely regular point $z \in S_p^*$ is equal to the cardinality of its quotient local symmetry set*: $\operatorname{ind} z = \# \, (G_z/G_z^*)$.

Remark: The index is not necessarily constant on S_p, even when it is connected, since the number of local symmetries can vary as one traverses the submanifold. For example, S_p could contain j rigidly equivalent pieces $\widehat{S}_1, \ldots, \widehat{S}_j$, whose points would have index j, as well as $k \neq j$ different rigidly equivalent pieces $\widetilde{S}_1, \ldots, \widetilde{S}_k$, whose points are of index k.

Example 3.9. In [**27**], Musso and Nicolodi construct examples of "cogwheels", which are closed plane curves that have identical Euclidean signatures, are everywhere of rank 1 and index n, and yet possess different discrete global symmetry groups $G_S \subset \mathrm{SO}(2)$. Roughly, the construction starts with a closed curve obtained by smoothly joining n suitable rigidly equivalent curve pieces, each of which is individually of index 1, implying

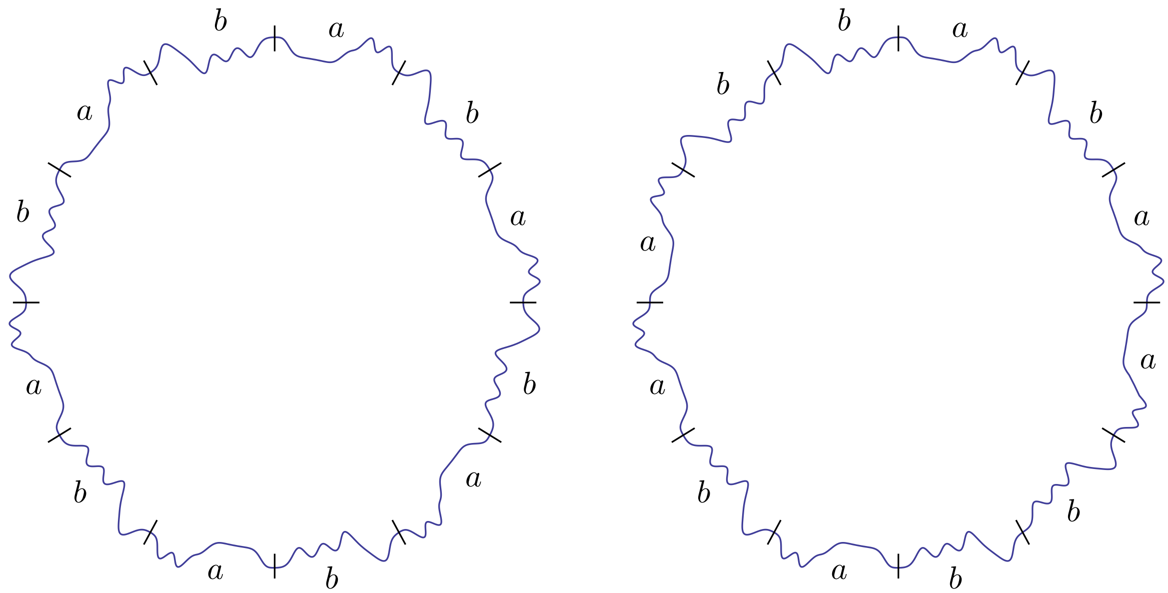

Figure 2. Cogwheels.

that there are no rigid motions, other than the identity, in its symmetry groupoid. Think of a regular n–gon whose sides have been symmetrically deformed so that its corners have also been smoothed out. The resulting curve is of rank 1 and signature index n, and, moreover, has global symmetry group $\mathbb{Z}_n \simeq \{\, \theta = 2k\pi/n \,|\, 0 \leq k < n \,\} \subset \mathrm{SO}(2)$ consisting of all rotations through multiples of $2\pi/n$ around its center.

One then splits, in the same manner, each constituent piece into a pair of rigidly inequivalent subpieces. By suitably rearranging the resulting $2n$ curve pieces, one can produce a closed curve, that they call a *cogwheel*, that has the same signature and index as the original, but whose symmetry group is only $\mathbb{Z}_m$ where m is a given divisor of n. Figure 2 sketches two examples. The left hand cogwheel has $\mathbb{Z}_6$ symmetry, consisting of 12 pieces of two inequivalent types, labelled a and b, respectively; the short transverse lines mark the ends of the individual pieces. The right hand cogwheel has the same 12 pieces, rearranged so that it has only $\mathbb{Z}_2$ as its global symmetry group. The more subtle aspects of the construction ensure that the cogwheels remain closed and sufficiently smooth in order that their signatures be well-defined.

Observe that, while the global symmetry group of such rearranged cogwheels can vary, their local symmetry set at each point (except possibly at the endpoints where the pieces have been rejoined) remains of cardinality n and equal to the index. In other words, the cogwheels all have (essentially) the same index, the same number of local symmetries, away from the joins, but non-isomorphic symmetry groupoids, which thus serves to characterize their global inequivalence, cf. Proposition 2.7.

One final remark: the open curve obtained by deleting one of the original pieces of the cogwheel has no global symmetry (save the identity of course) but this only eliminates one of the local symmetries at each point, leaving a residual local symmetry set of cardinality $n - 1$, and a consequential reduction of its symmetry groupoid.

Let us now extend the notion of index to include points which have non-discrete symmetry, as well as points in S_p that are not completely regular.

Definition 3.10. The *index* of a regular point $z \in S_k$ is defined as the maximal number of connected components of $\chi^{-1}[\chi(S_k \cap U)]$ where $U \subset M$ is a sufficiently small open neighborhood of z such that $S_k \cap U$ is connected.

This definition allows us to generalize Proposition 3.8 as follows.

Theorem 3.11. *If $z \in S_{\mathrm{reg}}$, the number of connected components of the quotient G_z/G_z^* is equal to its index, denoted* $\operatorname{ind} z$.

Theorem 3.12. *If $\Sigma_k = \chi(S_k)$ is an embedded submanifold, of dimension $k =$* $\operatorname{rank} S_k$, *then there is a one-to-one correspondence between it and the corresponding component of the symmetry moduli space*: $\Sigma_k \longleftrightarrow S_k^{\mathcal{G}} = S_k/\!\sim\, \subset S^{\mathcal{G}}$.

4. Weighted Submanifolds and Weighted Signatures.

In practical applications to image processing and elsewhere, one typically approximates the signature by discretizing (sampling) the original submanifold, and then employing suitable† numerical approximations to compute the signature differential invariants, the net effect being an approximate discretization of the corresponding signature, [**6**]. To maintain accuracy along the entire curve, the sample points should be more or less uniformly distributed, which, in the Euclidean curve case, means that the sample points are fairly evenly spaced with respect to the arc length. Alternatively, one can employ a random discretization of the curve, as in [**4**], based on the probability measure determined by rescaling arc length by the total length of the curve. In the ensuing discussion, we will ignore numerical inaccuracies, and focus our analysis solely on the effect that uniformly sampling the submanifold has on its signature.

Remark: We do not discuss the practicalities of constructing such uniform samplings of submanifolds. Even the "elementary" case of a sphere under surface area measure leads to the *generalized Fekete problem*, which appears as one of Smale's celebrated 18 problems for the twenty-first century, [**38**]. Powerful new algorithms for distributing points on general curved surfaces based on graph Laplacians have been recently developed in [**24**].

Let us focus our attention initially on the case of curves under the Euclidean group. After uniform sampling, the number of points in any constituent curve piece is approximately (or probabilistically) proportional to its length. On the other hand, the corresponding points in the signature curve will be distributed according to the measure induced by pushing forward the arc length measure of the original curve via the signature map. This remains valid even on a generalized vertex — circular arc or straight line segment —

† In [**6**, **2**], the use of symmetry-preserving numerical algorithms for this purpose is advocated and used. See [**31**, **19**, **37**, **1**] for further developments of this approach, based on the calculus of equivariant moving frames, [**13**], leading to important new algorithms for symmetry-preserving numerical schemes for integration of ordinary and partial differential equations.

since the number of points clustering at the corresponding signature point $(\kappa_0, 0)$ will be proportional to its overall length. In the limit, as the discretization becomes more and more dense, the sample point distribution converges, in the usual sense, to the arc length measure $d\mu = ds$ on the original curve, and to the push-forward of arc length measure $d\nu = \chi^{\#} d\mu$ on the signature, defined so that if $\Gamma \subset \Sigma$, then

$$\nu(\Gamma) = \mu(\chi^{-1}(\Gamma)) = \int_{\chi^{-1}(\Gamma)} ds. \tag{4.1}$$

If we parametrize the signature Σ by κ, which can be done locally provided $\kappa_s \neq 0$, and hence away from the problematic rank 0 points corresponding to generalized vertices, then the measure on Σ is

$$d\nu = \chi^{\#}(ds) = \operatorname{ind}\zeta \, \frac{d\kappa}{|\,\kappa_s\,|} = \operatorname{ind}\zeta \, \frac{d\kappa}{|\,H(\kappa)\,|}\,, \qquad \zeta = (\kappa, \kappa_s) \in \Sigma, \tag{4.2}$$

where $\operatorname{ind}\zeta$ denotes the index of the signature point ζ, and reflects the fact that $\chi^{-1}(\Gamma)$ may contain multiple disjoint yet equivalent pieces of the original curve C, while $\kappa_s = H(\kappa)$ is the local characterization of the signature curve as a graph, cf. (3.3). The absolute value comes from the fact that, as s increases, in accordance with the chosen orientation on C, the signature curve is traversed in the left-to-right orientation when $\kappa_s > 0$, but in the opposite orientation when $\kappa_s < 0$. We conclude that the weight on the signature curve can be used to determine the index, and hence the number of local symmetries, through formula (4.2).

On the other hand, at the vertices, where $\kappa_s = 0$, the push-forward measure ν acquires an atomistic term concentrated at the signature point $\zeta_0 = (\kappa_0, 0)$, whose weight is proportional to the total Euclidean length of the circular arc(s) or line segment(s) that map to the point ζ_0. Thus, the induced weighted measure on Σ is given by

$$\nu(\Gamma) = \int_{\Gamma} \operatorname{ind}\zeta \, \frac{d\kappa}{|\,\kappa_s\,|} \; + \sum_{\zeta_0 \in \Gamma \,\cap\, \{\kappa_s = 0\}} \ell(\zeta_0), \tag{4.3}$$

whenever $\Gamma \subset \Sigma$. Here, the atomistic weight

$$\ell(\zeta_0) = \int_{\chi^{-1}\{\zeta_0\}} ds \tag{4.4}$$

equals the sum of the lengths of all the circular arc/line segments mapping to $\zeta_0 = (\kappa_0, 0)$, whose cardinality equals their common index. However, this has the implication that *the weighted signature does not, in general, uniquely determine the original curve up to equivalence*, since the weight (4.4) at any point $\zeta_0 = (\kappa_0, 0) \in \Sigma$ only measures the total length of all the circular arcs of radius $1/\kappa_0$ (or straight line segments when $\kappa_0 = 0$), and not the number thereof nor how their individual lengths are apportioned.

Turning to the general scenario, let us first introduce some notation for measure concentrated on weighted submanifolds. The simplest is the *atomic* or *delta measure* concentrated at a point $a \in \mathbb{R}^n$, denoted by

$$\delta_a(D) = \begin{cases} 1, & a \in D, \\ 0, & a \notin D, \end{cases} \qquad \text{for} \qquad D \subset \mathbb{R}^n. \tag{4.5}$$

Definition 4.1. A *weighted submanifold* of dimension p is a pair $\mathbf{S} = (S, |\,\omega\,|)$ in which S is a smooth p-dimensional manifold and $|\,\omega\,| \geq 0$ a non-negative p-density on S.

Remark: We use p-densities, [**22**], rather than p-forms in our characterization for two reasons: (a) so that we do not have to worry about orientability of S, and, more importantly, (b) when we integrate the density on its underlying submanifold, we obtain a non-negative number — assuming the integral converges. One can, in many places, weaken the underlying smoothness assumptions, but this is not necessary for the basic applications we present in the sequel.

Given a weighted submanifold $\mathbf{S} = (S, |\,\omega\,|)$ with $S \subset \mathbb{R}^n$, we define the corresponding measure $\delta_{\mathbf{S}}$ concentrated on $\mathbf{S}$ by

$$\delta_{\mathbf{S}}(D) = \int_{S \cap D} |\,\omega\,| \qquad \text{for} \qquad D \subset \mathbb{R}^n. \tag{4.6}$$

In particular, if $S = (a, w)$ is a single point $a \in \mathbb{R}^n$ with weight $w \geq 0$, then $\delta_{\mathbf{S}} = w\,\delta_a$. More generally, if $h(z) \geq 0$ is a measurable function of $z \in \mathbb{R}^n$, then

$$h\,\delta_{\mathbf{S}} = \delta_{\widetilde{\mathbf{S}}}, \qquad \text{where} \qquad \widetilde{\mathbf{S}} = (S, h\,|\,\omega\,|\,) \tag{4.7}$$

is a reweighted version of $\mathbf{S}$.

Now, let $S \subset M$ be a p-dimensional submanifold. Let Ω be a G-contact-invariant† horizontal p-form on the submanifold jet space $\mathrm{J}^n(M, p)$, for suitable $n \geq 0$. For example, one might set

$$\Omega = \omega^1 \wedge \cdots \wedge \omega^p, \tag{4.8}$$

where $\omega^1, \ldots, \omega^p$ form the contact-invariant horizontal coframe constructed by applying the equivariant moving frame invariantization process to the basic horizontal coframe: $\omega^i = \iota(dx^i)$, cf. [**13**]. Another option is to set

$$\Omega = dI_1 \wedge \cdots \wedge dI_p, \tag{4.9}$$

where $I_1, \ldots, I_p$ are functionally independent differential invariants. The most general contact-invariant p-form has the form

$$\Omega = J\,\omega^1 \wedge \cdots \wedge \omega^p, \tag{4.10}$$

where J is an arbitrary differential invariant, [**29**]. This added flexibility in the choice of p-form can play an important role in practical applications. For instance, in the case of Euclidean plane curves, we could employ an invariantly re-weighted version of arc length,

† The term *contact-invariant* refers to the fact that, under the prolonged group action on the submanifold jet space, Ω is invariant modulo contact forms. This holds even in the case of Euclidean plane curves, where the arc length form $\Omega = ds$ is, in fact, only a contact-invariant one-form on jet space under prolonged Euclidean transformations. Any contact-invariant form can be made fully invariant by the addition of a suitable contact correction. In the present situation, the contact forms play no role — indeed, their defining property is that they vanish on jets of submanifolds — and so can be safely ignored. See [**13**, **20**, **29**] for complete details.

$\Omega = J\,ds$, where J is a suitable differential invariant, that is, a function of the basic curvature invariants $\kappa, \kappa_s, \kappa_{ss}, \ldots$. For example, to emphasize (or de-emphasize) parts of high curvature, one might choose $\Omega = \phi(\kappa)\,ds$ where $\phi(\cdot)$ is a suitably chosen scalar function.

Specification of the contact-invariant p-form Ω induces an invariant measure on p-dimensional submanifolds $S \subset M$. (For simplicity, we will avoid submanifolds whose jet does not belong to the domain of the p-form in jet space.) In order to preserve non-negativity, we replace the p-form Ω by the corresponding *p-density* $|\,\Omega\,|$, [**22**]. This induces the G-invariant measure

$$\mu(S) = \int_S |\,\Omega\,| \tag{4.11}$$

on compact p-dimensional submanifolds, both orientable and non-orientable. Mimicking (4.1), we introduce the push-forward measure $\nu = \chi^{\#}(\mu)$ on the signature:

$$\nu(\Gamma) = \mu(\chi^{-1}(\Gamma)) = \int_{\chi^{-1}(\Gamma)} |\,\Omega\,| \qquad \text{for} \qquad \Gamma \subset \Sigma. \tag{4.12}$$

The component $\Sigma_p = \chi(S_p)$ of maximal rank, corresponding to those points in S with only a discrete local symmetry set, can be locally parametrized by $p = \dim \Sigma_p$ independent differential invariants, say $I_1, \ldots, I_p$. Thus, using (4.10), we can write

$$d\nu = \operatorname{ind} \zeta \left| \frac{J\,dI_1 \wedge \cdots \wedge dI_p}{\dfrac{\partial(I_1, \ldots, I_p)}{\partial(\omega^1, \ldots, \omega^p)}} \right|, \qquad \zeta = (I_1, \ldots, I_l) \in \Sigma, \tag{4.13}$$

where the denominator is the determinant of the $p \times p$ Jacobian matrix whose entries are $\partial I_j / \partial \omega^k = I_{j,k}$, i.e., the corresponding first order derived invariants, which may well be included in the signature invariants, or, more generally, can be written in terms of them using the moving frame recurrence formulae, [**13**]. In particular, if Ω is given in terms of the invariants by (4.9), then the denominator in (4.13) is equal to 1.

The key considerations already arise in the case of two-dimensional surfaces in three-dimensional space, and so we will discuss this situation in some detail.

Example 4.2. As an educational, elementary example, consider the two-dimensional abelian group $G = \mathbb{R}^2$ acting intransitively on $M = \mathbb{R}^3$ by translation in the two "horizontal" directions:

$$(x, y, u) \longmapsto (x + a, y + b, u), \qquad (a, b) \in G. \tag{4.14}$$

We consider the induced action of G on surfaces $S \subset \mathbb{R}^3$ which, for simplicity, we assume to be given as the graphs of smooth functions $u = f(x, y)$. In this case, the differential invariants are simply

$$u, u_x, u_y, u_{xx}, , u_{xy}, u_{yy}, \ \ldots \ ,$$

and the invariant differential operators the ordinary total derivatives D_x, D_y. The differential invariants of order ≤ 2 can be used to form the signature map

$$\chi(x, y, u(x, y)) = \big(\, u(x, y), u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yy}(x, y) \,\big)\,, \tag{4.15}$$

albeit with some redundancy as discussed below. The associated *signature rank* of the surface is the rank of its Jacobian matrix, and so

$$\operatorname{rank} S|_{z=(x,y,u(x,y))} = \operatorname{rank} \begin{pmatrix} u_x & u_{xx} & u_{xy} & u_{xxx} & u_{xxy} & u_{xyy} \\ u_y & u_{xy} & u_{yy} & u_{xxy} & u_{xyy} & u_{yyy} \end{pmatrix}. \tag{4.16}$$

The basic invariant[†] horizontal coframe consists of the one-forms $\omega^1 = dx,\ \ \omega^2 = dy$, which produce the invariant two-form

$$\Omega = \omega^1 \wedge \omega^2 = dx \wedge dy.$$

The corresponding G-invariant measure (4.11) is

$$\mu(S) = \int_S |\, dx \wedge dy \,|. \tag{4.17}$$

Let $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be projection to the xy–plane, so $\pi(x,y,u) = (x,y)$. Since we are assuming that the surface S is the graph of a function $u(x,y)$ for $(x,y) \in D \subset \mathbb{R}^2$, then $\pi\colon S \to D$ is one-to-one and $\mu(S) = A(D)$ equals the area of its projection onto the plane.

Uniformly sampling S with respect to the measure (4.17) is equivalent to uniformly sampling its projection D with respect to planar Lebesgue measure, and then mapping each sample point $(x_k, y_k) \in D$ back to the corresponding point $(x_k, y_k, u(x_k, y_k)) \in S$. This induces a sampling of the signature obtained by evaluating the signature map prescribed by the differential invariants (4.15) at the sample points. As the sample points become denser and denser, the result converges to the push forward of the flat horizontal measure (4.17) to the signature.

Let us look at the three basic cases of surfaces of constant rank $0 \leq k \leq 2$.

Rank 0: According to Theorem 3.6, a connected surface has rank 0 if and only if its signature is a single point, $\Sigma = \{\,\zeta_0\,\}$, if and only if all its differential invariants are constant, so $u = c,\ \ u_x = u_y = \ \cdots\ = 0$, if and only if S is a piece of a horizontal plane: $S \subset \{\,u = c\,\}$ for some constant c, if and only if its local symmetry sets G_z for $z \in S$ are two-dimensional, and hence open neighborhoods of the identity in the full group: $0 \in G_z \subset \mathbb{R}^2$.

In this case, the weight of the signature point equals the area of the projection $D = \pi(S)$, as above, which, because S is horizontal, equals the area of S. In other words, the signature measure ν of a surface of rank 0 is a delta measure concentrated at the signature point ζ_0 weighted by the area of the surface:

$$\nu = A(S)\,\delta_{\zeta_0} = A(D)\,\delta_{\zeta_0}. \tag{4.18}$$

Thus, the weighted signature of such a surface will only prescribe its area, and not its overall shape.

[†] In this case, because the group acts projectably, meaning that the transformed independent variables do not depend upon the dependent variable, the horizontal one-forms are fully invariant — no contact corrections are needed.

Rank 1: Again, according to Theorem 3.6, a connected surface is everywhere of rank 1 if and only if it has only precisely one functionally independent differential invariant, namely u — since if u is constant the manifold is of rank 0 — if and only if its local symmetry sets G_z are one-dimensional, and whose common completion is a one-parameter subgroup $\widehat{G} \subset G$, if and only if S is the union of pieces of orbits of $\widehat{G}$. Since $\widehat{G}$ must be a one-parameter group of translations, it has the form

$$(x, y, u) \longmapsto (x + b\,t, y - a\,t, u)$$

for some constants $(a, b) \neq (0, 0)$. Thus, the surface S is a piece of a non-horizontal ruled surface, i.e., a union of line segments of the form $\{(x + b\,t, y - a\,t, u)\}$, not all lying in a common horizontal plane $\{u = c\}$. If the projection $D = \pi(S)$ is convex, or, more generally, its intersection $D \cap L$ with any line of the form

$$L_c = \{a\,x + b\,y = c\} \tag{4.19}$$

is connected, then

$$S \subset \{ (x, y, u) \mid u = h(a\,x + b\,y) \}, \tag{4.20}$$

where the scalar function h is not constant, as otherwise the surface would have rank 0. Thus, the surface can be viewed as the graph of a traveling wave, [**34**; §2.1].

The signature Σ of such a rank 1 surface is a curve. Under the above assumption on D, the line (4.19) forms what we will call a *normal cross-section* to the projected group orbits. When restricted to the curve

$$C_c = \{ (x, y, u(x, y)) \mid (x, y) \in L_c \} \subset S,$$

lying over L_c, the signature map $\chi : C_c \to \Sigma$ is locally one-to-one. Globally, the number of points in C_c mapping to the same point in the signature curve equals its index, as in Definition 3.10, which also counts the number of connected components in the local symmetry set of each $z \in \chi^{-1}(\zeta)$, that is the number of discrete local symmetries at the point z beyond the given translational symmetry.

Since C_c is a normal cross-section, the area of the projection $D = \pi(S)$ is equal to

$$A(D) = \int_{L_c} \ell(K_{x,y})\, ds, \tag{4.21}$$

where the integrand equals the length of the line segment

$$K_{x,y} = \{(x + b\,t, y - a\,t)\} \cap D$$

contained in D that passes through $(x, y) \in L_c$. Thus the weighted signature will have the probability measure given by the push-forward of the weighted arc-length measure $d\mu = \ell(K_{x,y})\, ds$ on the signature curve:

$$\nu = \chi^{\#}(dA) = \delta_{\boldsymbol{\Sigma}} \qquad \text{where} \qquad \boldsymbol{\Sigma} = (\Sigma, |\,\omega\,|) = (\Sigma, \chi^{\#}(\ell\, ds)). \tag{4.22}$$

Rank 2: A surface has rank 2 everywhere if and only if, nearby any point, it admits two functionally independent differential invariants, and hence at most a discrete translational local symmetry set. As above, one of these independent invariants must be u. Assume that the other is u_x, where functional independence means that their Jacobian determinant does not vanish:

$$\frac{\partial(u,u_x)}{\partial(x,y)} = u_x u_{xy} - u_y u_{xx} \neq 0. \tag{4.23}$$

This implies that we can (locally) write

$$u_y = H(u,u_x) \tag{4.24}$$

for some function H. Differentiation of this syzygy implies

$$\begin{aligned} u_{xy} &= H_u u_x + H_{u_x} u_{xx}, \\ u_{yy} &= H_u u_y + H_{u_x} u_{xy} = H\,H_u + H_u H_{u_x} u_x + H_{u_x}^2 u_{xx}. \end{aligned} \tag{4.25}$$

Thus, once we also prescribe the second order syzygy

$$u_{xx} = K(u,u_x), \tag{4.26}$$

we can determine all the other second order syzygies, and hence, by repeated differentiation, all the higher order syzygies too. Thus, parametrization of a reduced differential invariant signature for rank 2 surfaces of this type requires only the 4 signature invariants u, u_x, u_y, u_{xx}. The only rank 2 surfaces not covered are those for which $u_x = h(u)$ but u_y is functionally independent of u. These can be analyzed separately, and the details left to the reader. To cover both cases simultaneously requires using the signature invariants

$$\widetilde{\chi}(x,y,u(x,y)) = \big(\, u(x,y), u_x(x,y), u_y(x,y), u_{xx}(x,y), u_{yy}(x,y) \,\big) \tag{4.27}$$

to parametrize the reduced *signature map* $\widetilde{\chi}: S \to \Sigma \subset \mathbb{R}^5$.

Remark: The signature syzygy functions (4.24), (4.26) are not arbitrary. Cross differentiation produces the single integrability condition

$$\mathfrak{d}_x^2 H = \mathfrak{d}_y K, \tag{4.28}$$

where

$$\begin{aligned} \mathfrak{d}_x &= u_x \frac{\partial}{\partial u} + K(u,u_x)\,\frac{\partial}{\partial u_x}\,, \\ \mathfrak{d}_y &= H(u,u_x)\,\frac{\partial}{\partial u} + \left(u_x\,\frac{\partial H}{\partial u} + K\,\frac{\partial H}{\partial u_x} \right) \frac{\partial}{\partial u_x}\,, \end{aligned} \tag{4.29}$$

are the derivations (vector fields) representing implicit differentiation with respect to x and y, respectively. Note that the integrability condition (4.28) is exactly the requirement that the derivations (4.29) commute:

$$[\, \mathfrak{d}_x, \mathfrak{d}_y \,] = 0. \tag{4.30}$$

Moreover, the differential invariants u, u_x satisfy the functional independence condition (4.23) if and only if the derivations $\mathfrak{d}_x, \mathfrak{d}_y$ are linearly independent.

If we uniformly discretize S, as above, and let the number of points go to ∞, then this endows the signature Σ with the weighted measure

$$d\nu = \operatorname{ind}\zeta \left| \frac{du \wedge du_x}{u_x u_{xy} - u_y u_{xx}} \right| = \operatorname{ind}\zeta \left| \frac{du \wedge du_x}{u_x^2 H_u + u_x H_{u_x} K - H\,K} \right|, \tag{4.31}$$

where, as usual, $\operatorname{ind}\zeta$ denotes the index of the signature point ζ, which equals the number of non-isotropy local symmetries of any point $z \in S$ mapping to $\zeta = \chi(z)$.

In general, a surface $S \subset \mathbb{R}^3$ may have variable rank. Suppose, for example, that $S_2 \subsetneq S$ is a nonempty open subset of rank 2. Let's assume, for simplicity, that the remainder $S_1 = S \setminus S_2$ is of rank 1. Let $\Sigma_1 = \widetilde{\chi}(S_1)$, $\Sigma_2 = \widetilde{\chi}(S_2)$, be, respectively, the *regular* and *singular* parts of the signature. Thus, $\Sigma_2 \subset \mathbb{R}^5$ is a two-dimensional immersed surface, typically with self-intersections, which is a relatively open, dense subset of Σ, while $\Sigma_1 = \partial\Sigma_2$ represents some sort of one-dimensional "singular boundary" of Σ_2. The weighted signature measure induced by uniformly discretizing S, as above, will merely be the sum of the corresponding terms (4.22), (4.31) on the appropriate components of Σ.

Example 4.3. Let us next look at the case of surfaces $S \subset \mathbb{R}^3$ under the Euclidean group $\mathrm{SE}(3)$, as in Example 3.3. We assume that S is non-umbilic in order that it admits a well-defined Euclidean signature.

A rank 0 connected surface is contained in the orbit of a suitable two-dimensional subgroup $G^* \subset \mathrm{SE}(3)$. Being non-umbilic implies that S must be a piece of a cylinder, and so $G^* \simeq \mathrm{SO}(2) \ltimes \mathbb{R}$. (Spheres and planes are totally umbilic, and, moreover, in the language of [**30**], are totally singular submanifolds since their respective symmetry groups, $\mathrm{SO}(3)$ and $\mathrm{SE}(2)$, have dimension $3 > 2$ and hence act non-freely thereon.) Since the mean and Gauss curvatures are both constant on S, its signature consists of a single point ζ_0. Indeed, the Gauss curvature is $K = 0$ while the mean curvature $H = 1/(2\,R)$ is one half the reciprocal of the radius of the cylinder, while all higher order differentiated invariants vanish. Uniform sampling of S with respect to the Euclidean-invariant surface area measure implies, in the limit as the number of sample points goes to ∞, that the weight of ζ_0 is given by the surface area $A(S)$. Consequently, the distribution representing the weighted signature of the connected rank zero surface S is an atomic measure concentrated at the point ζ_0 that is weighted by the area of the surface:

$$\nu = A(S)\, \delta_{\zeta_0}. \tag{4.32}$$

Observe that the weighted signature only determines the area and radius of the cylindrical piece S, and not its overall shape. Consequently, the weighted signature does not uniquely determine the global geometry of a connected rank 0 surface.

At the other extreme, a surface S of rank 2 has a two-dimensional signature $\Sigma = \chi(S)$. Generically, the mean and Gauss curvatures are functionally independent, and hence can be used to parametrize Σ. Thus, according to (4.13), the corresponding weight on Σ represents the push-forward of the surface area element on S, giving

$$d\nu = \operatorname{ind}\zeta \left| \frac{dH \wedge dK}{\mathcal{D}_1 H\, \mathcal{D}_2 K - \mathcal{D}_2 H\, \mathcal{D}_1 K} \right|, \tag{4.33}$$

where $\mathcal{D}_1, \mathcal{D}_2$ are the invariant differential operators corresponding to the orthonormal Darboux frame on S, and $\operatorname{ind}\zeta$ is the index of the signature point $\zeta \in \Sigma$, which counts the number of distinct, locally equivalent points on the original surface that map to ζ. Alternatively, assuming the surface is mean curvature non-degenerate — see the discussion surrounding (3.6) — one can use the mean curvature H and one of its derivatives, say $H_1 = \mathcal{D}_1 H$, to parametrize Σ, as in (3.7). The corresponding weight on Σ takes the form

$$d\nu = \operatorname{ind}\zeta \left| \frac{dH \wedge dH_1}{H_1\, H_{12} - H_2\, H_{11}} \right|, \tag{4.34}$$

where $H_2 = \mathcal{D}_2 H$, $H_{11} = \mathcal{D}_1^2 H$, $H_{12} = \mathcal{D}_2 \mathcal{D}_1 H$, etc.

The most interesting case is a rank 1 surface, that possesses a symmetry groupoid with one-dimensional fibers at each point $z \in S$. Assuming S is connected, let $\widehat{G} \subset \mathrm{SE}(3)$ denote the completion of the local symmetry set G_z, which is independent of $z \in S$, as implied by Theorem 3.7. Thus $\widehat{G}$ is a one-dimensional subgroup consisting of either translations, rotations, or screw motions, whose orbits are straight lines, circles, or helices, respectively. The surface S is thus ruled by the orbits, and can be considered as a piece of the surface obtained by applying $\widehat{G}$ to a curve $C \subset \mathbb{R}^3$ that intersects the orbits of $\widehat{G}$ transversally. More specifically, let $\widehat{S} = \widehat{G} \cdot C$, which is a surface of translation (a traveling wave), of rotation (a surface of revolution), or a helical surface. Then $S \subset \widehat{S}$ is a piece thereof, and $C \cap S$ can be viewed as a cross-section to the local action of $\widehat{G}$ on S.

Our goal is to compute the surface area of S and thereby deduce the weighted measure on its one-dimensional signature curve $\Sigma = \chi(S)$. To this end, let us assume that there exists such a cross-section $C_0 \subset S$ that is *orthogonal* to the group orbits. Let $\widehat{\mathbf{v}} \in \widehat{\mathfrak{g}}$ be the infinitesimal generator of the action of $\widehat{G}$, which spans the tangent spaces to the orbits. Orthogonality requires that the tangent vector $\mathbf{t}$ to C_0 be everywhere orthogonal to the infinitesimal generator: $\mathbf{t}|_z \cdot \widehat{\mathbf{v}}|_z = 0$ for all $z \in C_0$. We will refer to C_0 as a *normal cross-section*. Normal cross-section curves always exist locally in a neighborhood of any $z \in S$. Globally, S can be decomposed into a union of pieces each possessing a normal cross-section curve. Under this assumption, the area of S can then be computed using the following interesting formula.

Theorem 4.4. *Let $S \subset \widehat{G} \cdot C_0$ be a surface of rank 1, such that $C_0 \subset S$ is a normal cross-section to the orbits of the one-parameter subgroup $\widehat{G} \subset \mathrm{SE}(3)$. Let*

$$\ell(z) = L(\mathcal{O}_z \cap S) = \int_{\mathcal{O}_z \cap S} ds$$

denote the length of the piece of the orbit $\mathcal{O}_z$ through z that is contained in S. Then

$$A(S) = \int_{C_0} \ell(z(s))\, ds. \tag{4.35}$$

Remark: Equation (4.35) is reminiscent of the coarea formula of geometric measure theory, [**12**, **26**]. In anticipation of our subsequent generalization to arbitrary transformation groups, we will refer to it as the *Euclidean coarea formula*.

Proof: We parametrize S by $w(s,t) = \exp(t\,\mathbf{v})\, z(s)$, where $z(s)$, for $s_0 \le s \le s_1$, is the arc length parametrization of C_0, whereby $\|\, dz/ds \,\| = 1$, and where $t_0(z(s)) < t < t_1(z(s))$ parametrizes the piece of the group orbit $\mathcal{O}_{z(s)}$ passing through $z(s) \in C_0$. Then

$$\frac{\partial w}{\partial s} = \exp(t\,\mathbf{v})_* \left(\frac{dz}{ds}\right), \qquad \frac{\partial w}{\partial t} = \mathbf{v}|_{\exp(t\,\mathbf{v})\, z(s)} = \exp(t\,\mathbf{v})_* \big[\, \mathbf{v}|_{z(s)} \,\big]. \tag{4.36}$$

Thus,

$$\begin{aligned} A(S) &= \int_{s_0}^{s_1} \int_{t_0(z(s))}^{t_1(z(s))} \left\| \frac{\partial w}{\partial s} \times \frac{\partial w}{\partial t} \right\| dt\, ds = \int_{s_0}^{s_1} \int_{t_0(z(s))}^{t_1(z(s))} \left\| \frac{dz}{ds} \times \mathbf{v}|_{z(s)} \right\| dt\, ds \\ &= \int_{s_0}^{s_1} \int_{t_0(z(s))}^{t_1(z(s))} \left\| \frac{dz}{ds} \right\| \left\| \mathbf{v}|_{z(s)} \right\| dt\, ds = \int_{s_0}^{s_1} \left(\int_{t_0(z(s))}^{t_1(z(s))} \left\| \frac{\partial w}{\partial t} \right\| dt \right) ds \\ &= \int_{C_0} \ell(z(s))\, ds. \end{aligned}$$

The second and next-to-last equalities follow from (4.36) and the norm-preserving properties of Euclidean transformations, while the third equality is a consequence of the normality condition on C_0. *Q.E.D.*

Orthogonality of the orbits to C_0 is essential for the validity of (4.35). Indeed, the simplest case is when C_0 is a line segment on the x axis, $\widehat{G}$ is the translation group in the direction of the u axis, so that S is a flat planar region of the form $\{\, (x,0,u) \mid 0 < u < h(x) \,\}$ for some scalar positive function h. The orbits $\mathcal{O}_z$ for $z = (x,0,0) \in C_0$ are the line segments from $(x,0,0)$ to $(x,0,h(x))$, of length $h(x)$ and hence (4.35) reduces to the triviality

$$A(S) = \int_{C_0} h(x)\, dx.$$

Clearly this formula requires that C_0 be orthogonal to the orbits; it is not valid for the skew translation $(x,y,u) \longmapsto (x + c\,t, y, u + t)$, for $c \ne 0$, whose orbits are not perpendicular to the x axis. More generally, the Euclidean coarea formula (4.35) gives a noteworthy formula for the areas of surfaces of rotation and helicoidal surfaces.

Now consider the weighted signature Σ of the surface S. Since S is assumed to have rank 1, $\Sigma \subset \mathbb{R}^l$ is an immersed curve, where l equals the number of Euclidean signature invariants, which depends upon which version of the Euclidean signature one employs, e.g., (3.5) or (3.7) or some alternative. The signature map is constant on the orbits of the underlying one-parameter subgroup $\widehat{G} \subset \mathrm{SE}(3)$, and hence reduces to a locally one-to-one map that identifies the normal cross-section C_0 as a covering of the signature curve: $\chi\colon C_0 \stackrel{\sim}{\longrightarrow} \Sigma$. The cardinality of the inverse image $\chi^{-1}\{\,\zeta\,\} \cap C_0$ of a point $\zeta \in \Sigma$ equals the index, and counts the number of discrete local symmetries not in the one-parameter subgroup $\widehat{G}$. Since the invariant measure on S is just the surface area element, its push forward to the signature curve will produce a measure concentrated on Σ whose weight is obtained by pushing forward the weight on C_0 given by the integrand in (4.35). Thus,

$$\nu = \chi^{\#}(dA) = \delta_{\boldsymbol{\Sigma}} \qquad \text{where} \qquad \boldsymbol{\Sigma} = \big(\Sigma, \chi^{\#}(\ell(z)\, ds) \big) \tag{4.37}$$

Keep in mind that, while the normal cross-section to the group orbits is not unique, the weighted signature is independent of the choice of cross-section.

Thus, the information supplied by the weighted signature curve can be viewed as representing the normal cross-section weighted by the lengths of the intersecting orbits, which thereby determine the overall area of the surface via the Euclidean coarea formula (4.35). This is clearly insufficient to uniquely reconstruct the surface up to rigid motion, since one can rearrange the orbit pieces through the cross-section while preserving their individual lengths, and hence construct a new, inequivalent surface that still has the same area and the same signature. For example, the Euclidean-inequivalent parabolic surfaces of translation

$$S = \{u = x^2,\ -1 \le x, y \le 1\}, \qquad \widehat{S} = \{u = x^2,\ -1 \le x \le 1,\ x - 1 \le y \le x + 1\},$$

have identical weighted signature curves since they have a common normal cross-section $C_0 = \{\, (x, 0, 0) \mid -1 \le x \le 1 \,\}$, and the line segments belonging to S and $\widehat{S}$ passing through a common point in C_0 have the same overall length. Thus, while the weighted signature of a rank 1 surface again does not uniquely determine the surface, it does, as always, prescribe its local geometry.

More generally, suppose we have a surface $S \subset \mathbb{R}^3$ of variable rank. We decompose

$$S = S_0 \cup S_1 \cup S_2 \cup S_{\text{sing}}$$

where each S_k has rank k and S_{sing} is the set of irregular points, which forms the boundary of the open dense subset formed by the union of the regular subsets S_0, S_1, S_2. We decompose each S_k into connected components, and then form the corresponding signature weight according to (4.32), (4.37) and (4.33) or (4.34). The full weighted signature measure is obtained by combining the various components, making sure to multiply locally equivalent parts, which have identical signatures, by the corresponding index.

Our final goal is to generalize the preceding coarea formula and resulting weighted signature measure to group actions on submanifolds of arbitrary dimension. However, for simplicity, we will continue to restrict our attention to a Lie group G acting on surfaces $S \subset M = \mathbb{R}^3$, deferring the completely general case to later contributions. Suppose that Ω is a G-contact invariant 2-form defined on the surface jet bundle $\mathrm{J}^n(M, 2)$, while ω is a G-contact invariant one-form on the curve jet bundle $\mathrm{J}^n(M, 1)$. Both can be systematically constructed through the method of equivariant moving frames, [**13**, **20**]. We interpret the corresponding densities, $|\,\Omega\,|$ and $|\,\omega\,|$ as, respectively, the G-invariant surface area and arc length elements. Of course, these are not uniquely defined, because either can be premultiplied by an arbitrary differential invariant (of the appropriate kind).

The case of rank 0 and rank 2 surfaces proceeds in an evidently analogous fashion to the above Euclidean case, and so the only case of genuine novelty is the case of a rank 1 surface. Let $\widehat{G}$ be the one-parameter group generated by the local symmetries of the surface, as per Theorem 3.7. The orbits $\mathcal{O}_z = \widehat{G} \cdot z$ of the (connected component) of $\widehat{G}$ are prescribed by the flow of its infinitesimal generator $0 \ne \widehat{\mathbf{v}} \in \widehat{\mathfrak{g}}$.

Proposition 4.5. *Let $\widehat{\mathbf{v}}$ and ω be as above. Then the arc length of the orbit piece*

$$\widehat{\mathcal{O}}_z = \{\, \exp(t\,\widehat{\mathbf{v}})\, z \mid\ t_0 \le t \le t_1 \,\} \subset \mathcal{O}_z$$

is given by

$$\ell(\widehat{\mathcal{O}}_z) = \int_{\widehat{\mathcal{O}}_z} |\,\omega\,| = \Big|\,\langle\, \omega|_z\,;\widehat{\mathbf{v}}|_z\,\rangle\,\Big|\,(t_1 - t_0), \tag{4.38}$$

where $\langle\,\cdot\,;\cdot\,\rangle$ *denotes the natural pairing between the tangent bundle* $T\widehat{\mathcal{O}}_z$ *and cotangent bundle* $T^*\widehat{\mathcal{O}}_z$.

Proof: Keep in mind that $\widehat{\mathbf{v}}$ is everywhere tangent to its orbits. We first note that the scalar quantity $\langle\,\omega\,;\widehat{\mathbf{v}}\,\rangle$ is, in fact, constant along the orbit. Indeed, the derivative of

$$\Big|\,\langle\,\omega|_{\exp(t\,\widehat{\mathbf{v}})\,z}\,;\widehat{\mathbf{v}}|_{\exp(t\,\widehat{\mathbf{v}})\,z}\,\rangle\,\Big|$$

with respect to t is a sum of two terms, the first involving the Lie derivative of $\widehat{\mathbf{v}}$ with respect to $\widehat{\mathbf{v}}$, which trivially vanishes, and the second Lie derivative of ω with respect to $\widehat{\mathbf{v}}$, which vanishes owing to the assumed G-invariance of the arc length.

Thus, we can easily compute the arc length integral using the given orbit parametrization:

$$\begin{aligned}\ell(\widehat{\mathcal{O}}_z) &= \int_{\widehat{\mathcal{O}}_z} \Big|\,\exp(t\,\widehat{\mathbf{v}})^*\omega|_{\exp(t\,\widehat{\mathbf{v}})\,z}\,\Big| \\ &= \int_{t_0}^{t_1} \Big|\,\langle\,\omega|_{\exp(t\,\widehat{\mathbf{v}})\,z}\,;\widehat{\mathbf{v}}|_{\exp(t\,\widehat{\mathbf{v}})\,z}\,\rangle\,\Big|\,dt = \Big|\,\langle\,\omega|_z\,;\widehat{\mathbf{v}}|_z\,\rangle\,\Big|\,(t_1 - t_0),\end{aligned}$$

as claimed. *Q.E.D.*

Example 4.6. Consider the case of a space curve, parametrized by $(x, y(x), z(x))$, under the Euclidean group SE(3). The Euclidean arc length element is

$$\omega = ds = \sqrt{1 + y_x^2 + z_x^2}\ dx. \tag{4.39}$$

In particular, if the curve coincides with a piece of a Euclidean orbit $z(t) = \exp(t\,\widehat{\mathbf{v}})z_0$ — a straight line, circle, or helix, whose character depends upon the infinitesimal generator

$$0 \neq \widehat{\mathbf{v}} = \xi\,\partial_x + \eta\,\partial_y + \zeta\,\partial_z \in \mathfrak{se}(3). \tag{4.40}$$

Note that, on $\mathcal{O}_z$,

$$y_x = \frac{y_t}{x_t} = \frac{\eta}{\xi}\,, \qquad\qquad z_x = \frac{z_t}{x_t} = \frac{\zeta}{\xi}\,,$$

and hence

$$|\,\langle\,\omega\,;\widehat{\mathbf{v}}\,\rangle\,| = \sqrt{1 + \frac{\eta^2}{\xi^2} + \frac{\zeta^2}{\xi^2}}\ |\,\xi\,| = \sqrt{\xi^2 + \eta^2 + \zeta^2} = \|\,\widehat{\mathbf{v}}\,\| \tag{4.41}$$

which is, indeed, constant along the orbit.

The first issue is to determine what replaces the Euclidean condition that the curve intersect the group orbits in a normal direction, given that there is no G-invariant notion of inner product in general. Here is the proposed generalization of this condition.

Definition 4.7. Under the given choice of arc length and surface area forms, a curve $C \subset S \subset \mathbb{R}^3$ will be called a *normal cross-section* provided it forms a cross-section to the orbits of the one-parameter symmetry group of the rank 1 surface S that is generated by $0 \neq \widehat{\mathbf{v}} \in \mathfrak{g}$ and, moreover, at each point $z \in C$, satisfies

$$\langle\, \Omega \,;\widehat{\mathbf{v}} \wedge \mathbf{w} \,\rangle = \langle\, \omega \,;\widehat{\mathbf{v}} \,\rangle \,\langle\, \omega \,;\mathbf{w} \,\rangle \qquad \text{for all} \qquad \mathbf{w} \in TC|_z. \tag{4.42}$$

Remark: The normality condition (4.42) defines an underdetermined system of ordinary differential equations whose solutions prescribe the curve C; their local existence can be established by standard techniques.

Example 4.8. Let us investigate (4.42) in the case of the Euclidean group SE(3). We use the standard Euclidean arc length element (4.39) and surface area element, which, assuming S is the graph of $u(x,y)$, is

$$\Omega = \sqrt{1 + u_x^2 + u_y^2}\; dx \wedge dy. \tag{4.43}$$

Let the curve be parametrized by $z(\tau) = (x(\tau), y(\tau), u(\tau))$. Since $C \subset S$, we must have

$$u(\tau) = u(x(\tau), y(\tau)) \qquad \text{hence} \qquad u_\tau = u_x x_\tau + u_y y_\tau.$$

Similarly, given that the orbits with infinitesimal generator (4.40) must lie in S, the same computation implies that

$$\zeta = u_x \xi + u_y \eta.$$

Solving the last two equations for u_x, u_y produces

$$u_x = \frac{\eta\, u_\tau - \zeta\, y_\tau}{\eta\, x_\tau - \xi\, y_\tau}\,, \qquad\qquad u_y = \frac{\zeta\, x_\tau - \xi\, u_\tau}{\eta\, x_\tau - \xi\, y_\tau}\,.$$

Thus, on C

$$|\,\widehat{\mathbf{v}} \mathbin{\lrcorner} \Omega\,| = \sqrt{(\eta\, x_\tau - \xi\, y_\tau)^2 + (\zeta\, x_\tau - \xi\, u_\tau)^2 + (\eta\, u_\tau - \zeta\, y_\tau)^2}\;\left|\, \frac{\xi\, dy - \eta\, dx}{\eta\, x_\tau - \xi\, y_\tau} \,\right| = \|\,\widehat{\mathbf{v}} \times z_\tau \,\|\, d\tau.$$

On the other hand, in view of formula (4.41), this is equal to the right hand side of (4.42) if and only if

$$\|\,\widehat{\mathbf{v}} \times z_\tau \,\| = \|\,\widehat{\mathbf{v}}\,\|\; \|\, z_\tau \,\|$$

which demonstrates orthogonality of the tangent to the curve and the tangent to the orbit under the Euclidean inner product.

With this in hand, we can state a general *G-invariant coarea formula.*

Theorem 4.9. *Let $C \subset S$ be a normal cross-section in a rank 1 surface $S \subset M$. Then*

$$\iint_S |\,\Omega\,| = \int_C \ell(\widehat{\mathcal{O}}_z \cap S)\, |\,\omega\,|\,. \tag{4.44}$$

Here, for each $z \in C$, the G-invariant arc length $\ell(\widehat{\mathcal{O}}_z \cap S)$ of the piece of the orbit of a one-parameter subgroup generated by $\widehat{\mathbf{v}} \in \mathfrak{g}$ lying in S is given by formula (4.38).

Proof: Let $z(\tau)$ for $\tau_0 \leq \tau \leq \tau_1$ parametrize C, and hence $w(t,\tau) = \exp(t\,\widehat{\mathbf{v}})\,z(\tau)$, for $t_0(\tau) \leq t \leq t_1(\tau)$, parametrizes our rank 1 surface S. Then,

$$\iint_S |\,\Omega\,| = \int_{\tau_0}^{\tau_1}\int_{t_0(\tau)}^{t_1(\tau)} \langle\, \Omega\,;\widehat{\mathbf{v}}\wedge w_\tau\,\rangle\,dt\,d\tau = \int_{\tau_0}^{\tau_1}\int_{t_0(\tau)}^{t_1(\tau)} \langle\,\omega\,;\widehat{\mathbf{v}}\,\rangle\,\langle\,\omega\,;w_\tau\,\rangle\,dt\,d\tau$$
$$= \int_{\tau_0}^{\tau_1} \langle\,\omega\,;\widehat{\mathbf{v}}\,\rangle\big[\,t_0(\tau)-t_1(\tau)\,\big]\,\langle\,\omega\,;w_\tau\,\rangle\,d\tau = \int_{\tau_0}^{\tau_1} \ell(\widehat{\mathcal{O}}_z)\,\langle\,\omega\,;w_\tau\,\rangle\,d\tau = \int_C \ell(\widehat{\mathcal{O}}_z)\,|\,\omega\,|,$$

where we used (4.42), the constancy of $\langle\,\omega\,;\widehat{\mathbf{v}}\,\rangle$ along the orbit, and formula (4.38). *Q.E.D.*

We conclude that the weighted signature of such a rank 1 surface is the distribution concentrated on the signature curve which, as in the Euclidean case, is in one-to-one correspondence with the normal cross-section curve C. The weight is given by the push-forward, under the signature map, of the measure on C given by the integrand in the G coarea formula (4.44). Similar results extend to surfaces of variable rank.

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