

# Differential Invariants for Lie Pseudo-groups

Peter Olver and Juha Pohjanpelto

**Key words.** Pseudo-group, moving frame, differential invariant, Gröbner basis

**AMS classification.** 58H05, 53A55, 58J70

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## 1 Introduction

Lie pseudo-groups, roughly speaking, are the infinite-dimensional counterparts of local Lie groups of transformations. Pseudo-groups were first studied systematically at the end of the 19th century by Sophus Lie, whose great insight in the subject was to postulate the additional condition that pseudo-group transformations form the general solution to a system of partial differential equations, the determining equations for the pseudo-group. In contrast to finite dimensional Lie groups, which since Lie's day have been rigorously formalized and have become a widely used mathematical tool, the foundations of infinite-dimensional pseudo-groups remain to date in a relatively undeveloped stage. Infinite dimensional Lie pseudo-groups can for the most part only be studied through their concrete action on space, which makes the classification problems and analytical foundations of the subject thorny, particularly in the intransitive situation. We refer the reader to the original papers of Lie, Medolaghi, and Vessiot [37, 47, 69, 71] for the classical theory of pseudo-groups, to Cartan [13] for their reformulation in terms of exterior differential systems, and [20, 29, 30, 35, 36, 39, 62, 63, 67, 68] for a variety of modern approaches.

Lie pseudo-groups appear in many fundamental physical and geometrical contexts, including gauge symmetries [6], Hamiltonian mechanics and symplectic and Poisson

geometry [51], conformal geometry of surfaces and conformal field theory [19, 21], the geometry of real hypersurfaces [16], symmetry groups of both linear and nonlinear partial differential equations, such as the Navier-Stokes and Kadomtsev–Petviashvili (KP) equations appearing in fluid and plasma mechanics [5, 18, 51], geometric hydrodynamics [2], Vessiot’s method of group splitting for producing explicit solutions to nonlinear partial differential equations [46, 50, 61, 71], mathematical morphology and computer vision [66, 72], and geometric numerical integration [45]. Pseudo-groups also appear as foliation-preserving groups of transformations, with the associated characteristic classes defined by certain invariant forms [24]. Also sufficiently regular local Lie group actions can be regarded as Lie pseudo-groups.

In a series of collaborative papers, starting with [22, 23], the first author has successfully reformulated the classical theory of moving frames in a general, algorithmic, and equivariant framework that can be readily applied to a wide range of finite-dimensional Lie group actions. Applications have included complete classifications of differential invariants and their syzygies, equivalence and symmetry properties of submanifolds, rigidity theorems, invariant signatures in computer vision [3, 7, 10, 54], joint invariants and joint differential invariants [8, 54], rational and algebraic invariants of algebraic group actions [27, 28], invariant numerical algorithms [31, 55, 72], classical invariant theory [4, 53], Poisson geometry and solitons [42, 43, 44], and the calculus of variations [32]. New applications of these methods to computation of symmetry groups and classification of partial differential equations can be found in [41, 48]. Furthermore, MAPLE software implementing the moving frame algorithms, written by E. Hubert, can be found at [26].

Our main goal in this contribution is to survey the extension of the moving frame theory to general Lie pseudo-groups recently put forth by the authors in [57, 58, 59, 60], and in [14, 15] in collaboration with J. Cheh. Following [32], we develop the theory in the framework of the variational bicomplexes over the bundles of (infinite) jets of mappings  $J^\infty(M, M)$  of  $M$  into  $M$  and of  $p$ -dimensional submanifolds  $J^\infty(M, p)$  of  $M$ , cf. [1, 32, 70]. The interactions between the two bicomplexes provide the key to understanding the moving frame constructions. Importantly, the invariant contact forms on the diffeomorphism jet bundle  $\mathcal{D}^{(\infty)} \subset J^\infty(M, M)$  will play the role of Maurer–Cartan forms for the diffeomorphism pseudo-group which enables us to formulate explicitly the structure equations for  $\mathcal{D}^{(\infty)}$ . Restricting the diffeomorphism-invariant forms to the pseudo-group subbundle  $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$  yields a complete system of Maurer–Cartan forms for the pseudo-group. Remarkably, the restricted Maurer–Cartan forms satisfy an “invariantized” version of the linear infinitesimal determining equations for the pseudo-group, which can be used to produce an explicit form of the pseudo-group structure equations. Application of these results to directly determining the structure of symmetry (pseudo-)groups of partial differential equations can be found in [5, 14, 15, 49].

Assuming freeness of the action, the explicit construction of a moving frame for a pseudo-group is based on a choice of local cross-section to the pseudo-group orbits in  $J^\infty(M, p)$ , [23]. The moving frame induces an invariantization process that projects general differential functions and differential forms on  $J^\infty(M, p)$  to their invariant counterparts. In particular, invariantization of the standard jet coordinates results in a complete system of normalized differential invariants, while invariantization of the

horizontal and contact one-forms yields an invariant coframe. The corresponding dual invariant total derivative operators will map invariants to invariants of higher order. The structure of the algebra of differential invariants, including the specification of a finite generating set of differential invariants, and the syzygies or differential relations among the generators, will then follow from the recurrence formulas that relate the differentiated and normalized differential invariants. Remarkably, besides the choice of a cross section, this final step requires only linear algebra and differentiation based on the infinitesimal determining equations for the pseudo-group, and not the explicit formulas for either the differential invariants, the invariant differential operators, or the moving frame. Except possibly for some low order details, the underlying structure of the differential invariant algebra is then entirely governed by two commutative algebraic modules: the symbol module of the infinitesimal determining system of the pseudo-group and a new module, named the prolonged symbol module, that quantifies the symbols of the prolonged action of the pseudo-group on  $J^\infty(M, p)$ .

## 2 The Diffeomorphism Pseudo-Group

Let  $M$  be a smooth  $m$ -dimensional manifold and let  $\mathcal{D} = \mathcal{D}(M)$  denote the pseudo-group of all local diffeomorphisms  $\varphi: M \rightarrow M$ . For each  $0 \leq n \leq \infty$ , let  $J^n(M, M)$  denote the bundle of  $n^{\text{th}}$  order jets of smooth mappings  $\phi: M \rightarrow M$  and  $\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset J^n(M, M)$  the groupoid of  $n^{\text{th}}$  order jets of local diffeomorphisms [40]. The source map  $\sigma^n: \mathcal{D}^{(n)} \rightarrow M$  and target map  $\tau^n: \mathcal{D}^{(n)} \rightarrow M$  are given by

$$\sigma^n(j_z^n \varphi) = z, \quad \tau^n(j_z^n \varphi) = \varphi(z), \quad (2.1)$$

respectively, and groupoid multiplication is induced by composition of mappings,

$$j_{\varphi(z)}^n \psi \cdot j_z^n \varphi = j_z^n (\psi \circ \varphi). \quad (2.2)$$

Let  $\hat{\pi}_k^n: \mathcal{D}^{(n)} \rightarrow \mathcal{D}^{(k)}$ ,  $0 \leq k \leq n$ , denote the natural projections.

Given local coordinates  $(z, Z) = (z^1, \dots, z^m, Z^1, \dots, Z^m)$  on an open subset of  $M \times M$ , the induced local coordinates of  $g^{(n)} = j_z^n \varphi \in \mathcal{D}^{(n)}$  are denoted  $(z, Z^{(n)})$ , where the components

$$Z_B^a = \frac{\partial^{|B|} \varphi^a}{\partial z^B}(z), \quad \text{for } 1 \leq a \leq m, \quad 0 \leq \#B \leq n,$$

of  $Z^{(n)}$ , represent the partial derivatives of the coordinate expression of  $\varphi$  at the source point  $z = \sigma^n(g^{(n)})$ . We will consistently use lower case letters,  $z, x, u, \dots$  for the source coordinates and the corresponding upper case letters  $Z^{(n)}, X^{(n)}, U^{(n)}, \dots$  for the derivative target coordinates of our diffeomorphisms  $\varphi$ .

The groupoid  $\mathcal{D}^{(\infty)} \subset J^\infty(M, M)$  of infinite order jets inherits the structure of a variational bicomplex from  $J^\infty(M, M)$ , [1, 70]. This provides a natural splitting of the cotangent bundle  $T^*\mathcal{D}^{(\infty)}$  into horizontal and vertical (or contact) components [1, 52],

and we use  $d = d_M + d_G$  to denote the induced splitting of the exterior derivative on  $\mathcal{D}^{(\infty)}$ . In terms of local coordinates  $(z, Z^{(\infty)})$ , the horizontal subbundle of  $T^*\mathcal{D}^{(\infty)}$  is spanned by the one-forms  $dz^a = d_M z^a$ ,  $a = 1, \dots, m$ , while the vertical subbundle is spanned by the basic *contact forms*

$$\Upsilon_B^a = d_G Z_B^a = dZ_B^a - \sum_{c=1}^m Z_{Bc}^a dz^c, \quad a = 1, \dots, m, \quad \#B \geq 0. \quad (2.3)$$

Composition by a local diffeomorphism  $\psi \in \mathcal{D}$  induces an action by right multiplication on diffeomorphism jets,

$$\mathbf{R}_\psi(j_z^n \varphi) = j_{\psi(z)}^n(\varphi \circ \psi^{-1}). \quad (2.4)$$

A differential form  $\mu$  on  $\mathcal{D}^{(n)}$  is *right-invariant* if  $\mathbf{R}_\psi \mu = \mu$ , where defined, for every  $\psi \in \mathcal{D}$ . Since the splitting of forms on  $\mathcal{D}^{(\infty)}$  is invariant under this action, the differentials  $d_M \mu$  and  $d_G \mu$  of a right-invariant form  $\mu$  are again invariant. The target coordinate functions  $Z^a$  are obviously right-invariant, and hence their horizontal differentials

$$\sigma^a = d_M Z^a = \sum_{b=1}^m Z_b^a dz^b \quad (2.5)$$

form an invariant horizontal coframe, while their vertical differentials

$$\mu^a = d_G Z^a = \Upsilon^a = dZ^a - \sum_{b=1}^m Z_b^a dz^b, \quad a = 1, \dots, m, \quad (2.6)$$

are the invariant contact forms of order zero. Let

$$\mathbb{D}_{Z^a} = W_a^b \mathbb{D}_{z^b} \quad (2.7)$$

denote the total total derivative operators on  $\mathcal{D}^{(\infty)}$  dual to the horizontal forms (2.5), where

$$\mathbb{D}_{z^b} = \frac{\partial}{\partial z^b} + \sum_{\#B \geq 0} Z_{Bb}^c \frac{\partial}{\partial Z_B^c} = \frac{\partial}{\partial z^b} + Z_b^c \frac{\partial}{\partial Z^c} + Z_{b_1 b}^c \frac{\partial}{\partial Z_{b_1}^c} + Z_{b_1 b_2 b}^c \frac{\partial}{\partial Z_{b_1 b_2}^c} + \dots, \quad (2.8)$$

$b = 1, \dots, m$ , are the standard total derivative operators on  $\mathcal{D}^{(\infty)}$  and where  $W_b^a = (Z_b^a)^{-1}$  is the inverse Jacobian matrix. Then the higher-order invariant contact forms are obtained by successively Lie differentiating the invariant contact forms (2.6),

$$\mu_B^a = \mathbb{D}_{Z^B} \mu^a = \mathbb{D}_{Z^B} \Upsilon^a, \quad (2.9)$$

where  $\mathbb{D}_{Z^B} = \mathbb{D}_{z^{b_1}} \cdots \mathbb{D}_{z^{b_k}}$ ,  $a = 1, \dots, m$ ,  $k = \#B \geq 0$ .

The next step in our program is to establish the structure equations for the diffeomorphism groupoid  $\mathcal{D}^{(\infty)}$ , which can be derived efficiently by employing Taylor series. Let

$$Z^a \llbracket h \rrbracket = \sum_{\#B \geq 0} \frac{1}{B!} Z_B^a h^B, \quad 1 \leq a \leq m, \quad (2.10)$$

be the individual components of the column vector-valued Taylor series  $Z^a[[h]]$ , depending on  $h = (h^1, \dots, h^m)$ , obtained by expanding a local diffeomorphism  $Z = \varphi(z + h)$  at  $h = 0$ . Further, let  $\Upsilon[[h]]$ ,  $\mu[[H]]$  denote the column vectors of contact form-valued and invariant contact form-valued power series with individual components

$$\Upsilon^a[[h]] = \sum_{\#B \geq 0} \frac{1}{B!} \Upsilon_B^a h^B, \quad \mu^a[[H]] = \sum_{\#B \geq 0} \frac{1}{B!} \mu_B^a H^B, \quad a = 1, \dots, m, \quad (2.11)$$

respectively. Equations (2.9) imply that

$$\mu[[H]] = \Upsilon[[h]] \quad \text{when } H = Z[[H]] - Z[[0]], \quad (2.12)$$

which, after an application of the exterior derivative, can be used to derive the diffeomorphisms pseudo-group structure equations.

**Theorem 2.1** *The complete structure equations for the diffeomorphism pseudo-group are obtained by equating coefficients in the power series identity*

$$d\mu[[H]] = \nabla_H \mu[[H]] \wedge (\mu[[H]] - dZ[[0]]), \quad d\sigma = -d\mu[[0]] = \nabla_H \mu[[0]] \wedge \sigma. \quad (2.13)$$

Here  $\nabla_H \mu[[H]] = \left( \frac{\partial Z^a}{\partial H^b} [[H]] \right)$  denotes the  $m \times m$  power series Jacobian matrix obtained by differentiating  $\mu[[H]]$  with respect to  $H = (H^1, \dots, H^m)$ .

Let  $\mathcal{X} = \mathcal{X}(M)$  denote the space of locally defined vector fields on  $M$ , which we write in local coordinates as

$$\mathbf{v} = \sum_{a=1}^m \zeta^a(z) \frac{\partial}{\partial z^a}. \quad (2.14)$$

We regard  $\mathcal{X}$  as the space of infinitesimal generators of the diffeomorphism pseudo-group. Let  $\mathcal{X}^{(n)} = J^n TM$ ,  $0 \leq n \leq \infty$ , denote the tangent  $n$ -jet bundle. The  $n$ -jet  $j_z^n \mathbf{v} \in \mathcal{X}^{(n)}$  of the vector field (2.14) at a point  $z$  is prescribed by all the partial derivatives of its coefficients up to order  $n$ , which we denote by

$$\zeta^{(n)} = (\dots, \zeta_B^a, \dots), \quad a = 1, \dots, m, \quad 0 \leq \#B \leq n. \quad (2.15)$$

### 3 Lie Pseudo-Groups

Several variants of the precise technical definition of a Lie pseudo-group appear in the literature. Ours is:

**Definition 3.1** A sub-pseudo-group  $\mathcal{G} \subset \mathcal{D}$  will be called a *Lie pseudo-group* if there exists  $n_o \geq 1$  such that for all  $n \geq n_o$ :

1. the corresponding sub-groupoid  $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$  forms a smooth, embedded subbundle,
2. every smooth local solution  $Z = \varphi(z)$  to the determining system  $\mathcal{G}^{(n)}$  belongs to  $\mathcal{G}$ ,
3.  $\mathcal{G}^{(n)} = \text{pr}^{(n-n_o)}\mathcal{G}^{(n_o)}$  is obtained by prolongation.

The minimal value of  $n_o$  is called the *order* of the pseudo-group.

Thus on account of conditions (1) and (3), for  $n \geq n_o$ , the pseudo-group subbundle  $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$  is defined in local coordinates by a formally integrable system of  $n^{\text{th}}$  order partial differential equations

$$F^{(n)}(z, Z^{(n)}) = 0, \quad (3.1)$$

the *determining equations* for the pseudo-group, whose local solutions  $Z = \varphi(z)$ , by condition (2), are precisely the pseudo-group transformations. Our assumptions, moreover, imply that the isotropy jets  $\mathcal{G}_z^{(n)} = \{g^{(n)} \in \mathcal{G}^{(n)} \mid \sigma^n(g^{(n)}) = \tau^n(g^{(n)}) = z\}$  form a finite dimensional Lie algebra for all  $z \in M$ .

Given a Lie pseudo-group  $\mathcal{G}$ , let  $\mathfrak{g} \subset \mathcal{X}$  denote the local Lie algebra of infinitesimal generators, i.e., the set of locally defined vector fields whose flows belong to  $\mathcal{G}$ . Let  $\mathfrak{g}^{(n)} \subset \mathcal{X}^{(n)}$  denote their jets. In local coordinates, the subspace  $\mathfrak{g}^{(n)} \subset \mathcal{X}^{(n)}$  is defined by a linear system of partial differential equations

$$L^{(n)}(z, \zeta^{(n)}) = 0 \quad (3.2)$$

for the vector field coefficients (2.13), called the *linearized* or *infinitesimal determining equations* for the pseudo-group. Conversely, any vector field  $\mathbf{v}$  satisfying infinitesimal determining equations (3.2) is an infinitesimal generator for  $\mathcal{G}$ , [57]. In practice, the linearized determining equations are constructed by linearizing the  $n^{\text{th}}$  order determining equations (3.1) at the identity transformation. If  $\mathcal{G}$  is the symmetry group of a system of differential equations, then the linearized determining equations (3.2) are (the completion of) the usual determining equations for its infinitesimal generators obtained via Lie's algorithm [51].

Let us explain how the underlying structure of the pseudo-group is explicitly prescribed by its infinitesimal determining equations. As with finite-dimensional Lie groups, the structure of a pseudo-group is described by its Maurer–Cartan forms. A complete system of right-invariant one-forms on  $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$  is obtained by restricting (or pulling back) the Maurer–Cartan forms (2.5), (2.9). For simplicity, we continue to denote these forms by  $\sigma^a, \mu_B^a$ . The restricted Maurer–Cartan forms are, of course, no longer linearly independent, but are subject to certain constraints dictated by the pseudo-group. Remarkably, these constraints can be explicitly characterized by an invariant version of the linearized determining equations (3.2), obtained by replacing the source coordinates  $z^a$  by the corresponding target coordinates  $Z^a$  and the vector field jet coordinates  $\zeta_B^a$  by the corresponding Maurer–Cartan form  $\mu_B^a$ .

**Theorem 3.2** *The linear system*

$$L^{(n)}(Z, \mu^{(n)}) = 0 \quad (3.3)$$

*serves to define the complete set of linear dependencies among the right-invariant Maurer–Cartan forms  $\mu^{(n)}$  on  $\mathcal{G}^{(n)}$ .*

In this way, we effectively and efficiently bypass Cartan's more complicated prolongation procedure [9, 13] for accessing the pseudo-group structure equations.

**Example 3.3** In this example we derive the structure equations for the pseudo-group  $\mathcal{G}$  consisting of transformations  $\varphi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  with

$$\begin{aligned} X &= f(x), & Y &= e(x, y) \equiv f'(x)y + g(x), \\ U &= u + \frac{e_x(x, y)}{f'(x)} = u + \frac{f''(x)y + g'(x)}{f'(x)}, \end{aligned} \quad (3.4)$$

where  $f(x) \in \mathcal{D}(\mathbb{R})$  and  $g(x)$  is an arbitrary smooth function. The transformations (3.4) form the general solution to the first order system of determining equations

$$X_y = X_u = 0, \quad Y_y = X_x \neq 0, \quad Y_u = 0, \quad Y_x = (U - u)X_x, \quad U_u = 1 \quad (3.5)$$

for  $\mathcal{G}$ . The infinitesimal generators are given by

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \varphi \frac{\partial}{\partial u} = a(x) \frac{\partial}{\partial x} + [a'(x)y + b(x)] \frac{\partial}{\partial y} + [a''(x)y + b'(x)] \frac{\partial}{\partial u}, \quad (3.6)$$

where  $a(x), b(x)$  are arbitrary functions, forming the general solution to the first order infinitesimal determining system

$$\xi_x = \eta_y, \quad \xi_y = \xi_u = \eta_u = \varphi_u = 0, \quad \eta_x = \varphi, \quad (3.7)$$

which is obtained by linearizing the determining system (3.5) at the identity jet.

In accordance with Theorem 3.2, the pull-backs of the Maurer–Cartan forms (2.9) satisfy the invariantized version

$$\mu_X^x = \mu_Y^y, \quad \mu_Y^x = \mu_U^x = \mu_U^y = \mu_U^u = 0, \quad \mu_X^y = \mu^u, \quad (3.8)$$

of the linearized determining equations. By a repeated application of the invariant total derivative operators  $\mathbb{D}_X, \mathbb{D}_Y, \mathbb{D}_U$  (cf. (2.7)) we find that

$$\mu_{X^n Y}^y = \mu_{X^{n+1}}^x, \quad \mu_{X^n}^u = \mu_{X^{n+1}}^y, \quad \mu_{X^n Y}^u = \mu_{X^{n+2}}^x, \quad n \geq 0, \quad (3.9)$$

while all the other pulled-back basis Maurer–Cartan forms vanish. As a result, the one-forms

$$\sigma^x, \quad \sigma^y, \quad \sigma^u, \quad \mu_n^x = \mu_{X^n}^x, \quad \mu_n^y = \mu_{X^n}^y, \quad n \geq 0, \quad (3.10)$$

form a  $\mathcal{G}$ -invariant coframe on  $\mathcal{G}^{(\infty)}$ .

The structure equations are obtained by substituting the expansions

$$\begin{aligned} \mu^x \llbracket H \rrbracket &= \sum_{n=0}^{\infty} \frac{1}{n!} \mu_n^x H^n, \\ \mu^y \llbracket H, K \rrbracket &= \sum_{n=0}^{\infty} \frac{1}{n!} \mu_n^y H^n + K \sum_{n=0}^{\infty} \frac{1}{n!} \mu_{n+1}^x H^n, \\ \mu^u \llbracket H, K \rrbracket &= \sum_{n=0}^{\infty} \frac{1}{n!} \mu_{n+1}^y H^n + K \sum_{n=0}^{\infty} \frac{1}{n!} \mu_{n+2}^x H^n, \end{aligned} \quad (3.11)$$

into (2.13). Those involving  $\mu_{X^n}^x$  reduce to the structure equations

$$\begin{aligned} d\sigma^x &= \mu_X^x \wedge \sigma^x, & d\mu_{X^n}^x &= -\mu_{X^{n+1}}^x \wedge \sigma^x + \sum_{i=0}^{n-1} \binom{n}{i} \mu_{X^{i+1}}^x \wedge \mu_{X^{n-i}}^x \\ & & &= \sigma^x \wedge \mu_{X^{n+1}}^x - \sum_{j=0}^{[n+1/2]} \frac{n-2j+1}{n+1} \binom{n+1}{j} \mu_{X^j}^x \wedge \mu_{X^{n+1-j}}^x \end{aligned} \quad (3.12)$$

for  $\mathcal{D}(\mathbb{R})$  (cf. Cartan [12], eq. (48)), while

$$\begin{aligned} d\sigma^y &= \mu_X^y \wedge \sigma^x + \mu_X^x \wedge \sigma^y, & d\sigma^u &= \mu_{X^2}^y \wedge \sigma^x + \mu_{X^2}^x \wedge \sigma^y, \\ d\mu_{X^n}^y &= \sigma^x \wedge \mu_{X^{n+1}}^y + \sigma^y \wedge \mu_{X^{n+1}}^x + \sum_{j=0}^{n-1} \left[ \binom{n}{j} - \binom{n}{j+1} \right] \mu_{X^{j+1}}^y \wedge \mu_{X^{n-j}}^x. \end{aligned} \quad (3.13)$$

Additional examples of this procedure can be found in [14, 58]; see also [49] for a comparison with other approaches appearing in the literature.

## 4 Pseudo-Group Actions on Extended Jet Bundles

In this paper, our primary focus is on the induced action of our pseudo-group on submanifolds of a fixed dimension. For  $0 \leq n \leq \infty$ , let  $J^n = J^n(M, p)$  denote the  $n^{\text{th}}$  order (extended) jet bundle consisting of equivalence classes of  $p$ -dimensional submanifolds  $S \subset M$  under the equivalence relation of  $n^{\text{th}}$  order contact, cf. [52]. We use the standard local coordinates

$$z^{(n)} = (x, u^{(n)}) = (\dots, x^i, \dots, u_J^\alpha, \dots) \quad (4.1)$$

on  $J^n$  induced by a splitting of the local coordinates

$$z = (x, u) = (x^1, \dots, x^p, u^1, \dots, u^q)$$

on  $M$  into  $p$  independent and  $q = m - p$  dependent variables [51, 52].

The choice of independent and dependent variables brings about a decomposition of the differential one-forms on  $J^\infty$  into horizontal and vertical components. The basis *horizontal forms* are the differentials  $dx^1, \dots, dx^p$  of the independent variables, while the basis vertical forms are provided by the *contact forms*

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^p u_{J_i}^\alpha dx^i, \quad \alpha = 1, \dots, q, \quad \#J \geq 0. \quad (4.2)$$



This decomposition induces a splitting of the exterior derivative  $d = d_H + d_V$  on  $J^\infty$  into horizontal and vertical (or contact) components, and locally endows the algebra of differential forms on  $J^\infty$  with the structure of a variational bicomplex [1, 32, 70].

Local diffeomorphisms preserve the  $n^{\text{th}}$  order contact equivalence relation between submanifolds, and thus give rise to an action on the jet bundle  $J^n$ , known as the  $n^{\text{th}}$  prolonged action, which, by the chain rule factors through the diffeomorphism jet groupoid  $\mathcal{D}^{(n)}$ . It will be useful to combine the two bundles  $\mathcal{D}^{(n)}$  and  $J^n$  into a new groupoid  $\mathcal{E}^{(n)} \rightarrow J^n$  by pulling back  $\mathcal{D}^{(n)} \rightarrow M$  via the standard projection  $\tilde{\pi}_o^n: J^n \rightarrow M$ . Points in  $\mathcal{E}^{(n)}$  consists of pairs  $(z^{(n)}, g^{(n)})$ , where  $z^{(n)} \in J^n$  and  $g^{(n)} \in \mathcal{G}^{(n)}$  are based at the same point  $z = \hat{\pi}_o^n(g^{(n)}) = \tilde{\pi}_o^n(z^{(n)})$ .

Local coordinates on  $\mathcal{E}^{(n)}$  are written as  $\mathbf{Z}^{(n)} = (z^{(n)}, Z^{(n)})$ , where

$$z^{(n)} = (x, u^{(n)}) = (\dots, x^i, \dots, u_J^\alpha, \dots)$$

indicate submanifold jet coordinates, while

$$Z^{(n)} = (X^{(n)}, U^{(n)}) = (\dots, X_A^i, \dots, U_A^\alpha, \dots)$$

indicate diffeomorphism jet coordinates. The groupoid structure on  $\mathcal{E}^{(n)}$  is induced by the source map, which is merely the projection,  $\tilde{\sigma}^n(z^{(n)}, g^{(n)}) = z^{(n)}$ , and the target map  $\tilde{\tau}^n(z^{(n)}, g^{(n)}) = g^{(n)} \cdot z^{(n)}$ , which is defined by the prolonged action of  $\mathcal{D}^{(n)}$  on  $J^n$ . We let  $\varphi \in \mathcal{D}$  with domain  $\text{dom } \varphi = \mathcal{U} \subset M$  act on the set  $\mathcal{E}^{(n)}|_{\mathcal{U}} = \{(z^{(n)}, g^{(n)}) \in \mathcal{E}^{(n)} \mid \tilde{\pi}_o^n(z^{(n)}) \in \mathcal{U}\}$  by

$$\varphi \cdot (z^{(n)}, g^{(n)}) = (j_z^n \varphi \cdot z^{(n)}, g^{(n)} \cdot j_{\varphi(z)}^n \varphi^{-1}), \quad (4.3)$$

where  $\tilde{\pi}_o^n(z^{(n)}) = z$ . The action (4.3) obviously factors into an action of  $\mathcal{D}^{(n)}$  on  $\mathcal{E}^{(n)}$ .

The cotangent bundle  $T^*\mathcal{E}^\infty$  naturally splits into jet and group components, spanned, respectively, by the *jet forms*, consisting of the horizontal one-forms  $dx^i$  and contact one-forms  $\theta_j^\alpha$  from the submanifold jet bundle  $J^\infty$ , and by the contact one-forms  $\Upsilon_B^\alpha$  from the diffeomorphism jet bundle  $\mathcal{D}^{(\infty)}$ . We accordingly decompose the differential on  $\mathcal{E}^\infty$  into jet and group components, the former further splitting into horizontal and vertical components:

$$d = d_J + d_G = d_H + d_V + d_G. \quad (4.4)$$

The resulting operators satisfy the tricomplex relations [32],

$$d_J^2 = d_G^2 = d_H^2 = d_V^2 = 0, \\ d_J d_G = -d_G d_J, \quad d_H d_V = -d_V d_H, \quad d_H d_G = -d_G d_H, \quad d_V d_G = -d_G d_V.$$

The above splitting determines *lifted total derivative operators*

$$\mathbb{D}_{x^j} = \mathbb{D}_{x^j} + \sum_{\alpha=1}^q u_j^\alpha \mathbb{D}_{u^\alpha} + \sum_{\#J \geq 1} u_{Jj}^\alpha \frac{\partial}{\partial u_J^\alpha} \quad (4.5)$$

on  $\mathcal{E}^\infty$ , where  $\mathbb{D}_{x^j}, \mathbb{D}_{u^\alpha}$  are the standard total derivative operators (2.8) on  $\mathcal{G}^{(\infty)}$ . Proceeding in analogy with the construction of the invariant total derivative operators (2.7) on  $\mathcal{D}^{(\infty)}$ , we define *lifted invariant total derivative operators* on  $\mathcal{E}^\infty$  by

$$\mathbb{D}_{X^j} = \sum_{k=1}^p \widehat{W}_j^k \mathbb{D}_{x^k}, \quad \text{where} \quad \widehat{W}_j^k = (\mathbb{D}_{x^j} X^k)^{-1} \quad (4.6)$$

indicates the entries of the inverse total Jacobian matrix. With this, the chain rule formulas for the higher-order prolonged action of  $\mathcal{D}^{(n)}$  on  $J^n$ , i.e., coordinates  $\widehat{U}_j^\alpha$  of the target map  $\widetilde{\tau}^n: \mathcal{E}^{(n)} \rightarrow J^n$ , are obtained by successively differentiating the target dependent variables  $U^\alpha$  with respect to the target independent variables  $X^i$ , whereby

$$\widehat{U}_j^\alpha = \mathbb{D}_{X^j} U^\alpha = \mathbb{D}_{X^{j_1}} \cdots \mathbb{D}_{X^{j_k}} U^\alpha. \quad (4.7)$$

These are the multi-dimensional versions of the usual implicit differentiation formulas from calculus.

Given a Lie pseudo-group  $\mathcal{G}$ , let  $\mathcal{H}^{(n)} \subset \mathcal{E}^{(n)}$  denote the subgroupoid obtained by pulling back  $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$  via the projection  $\widetilde{\pi}_o^n: J^n \rightarrow M$ .

**Definition 4.1** A moving frame  $\rho^{(n)}$  of order  $n$  is a  $\mathcal{G}^{(n)}$  equivariant local section of the bundle  $\mathcal{H}^{(n)} \rightarrow J^n$ .

More explicitly, we require  $\rho^{(n)}: \mathcal{V}^n \rightarrow \mathcal{H}^{(n)}$ , where  $\mathcal{V}^n \subset J^n$  is open, to satisfy

$$\widetilde{\sigma}^n(\rho^{(n)}(z^{(n)})) = z^{(n)}, \quad \rho^{(n)}(g^{(n)} \cdot z^{(n)}) = g^{(n)} \cdot \rho^{(n)}(z^{(n)}), \quad (4.8)$$

for all  $g^{(n)} \in \mathcal{G}^{(n)}|_z$  near the jet  $\mathbb{I}_z^{(n)}$  of the identity transformation such that both  $z^{(n)}$  and  $g^{(n)} \cdot z^{(n)}$  lie in the domain of definition  $\mathcal{V}^n$  of  $\rho^{(n)}$ , where  $\mathcal{G}^{(n)}|_z$  denotes the source fibre of  $\mathcal{G}^{(n)}$  at  $z = \widetilde{\pi}_o^n(z^{(n)})$ . Then, with a moving frame at hand, the composition  $\widetilde{\tau}^n \circ \rho^{(n)}$ , due to equations (4.3), (4.8), is invariant under the action of  $\mathcal{G}$  on  $J^n$  and, as we will subsequently see, provide differential invariants for the action of  $\mathcal{G}$  in  $J^n$ .

A moving frame  $\rho^{(k)}: \mathcal{V}^k \rightarrow \mathcal{H}^{(k)}$  of order  $k > n$  is *compatible* with a moving frame  $\rho^{(n)}: \mathcal{V}^n \rightarrow \mathcal{H}^{(n)}$  of order  $n$  if  $\widehat{\pi}_k^n \circ \rho^{(k)} = \rho^{(n)} \circ \widetilde{\pi}_k^n$ , where defined. A *complete moving frame* is provided by a mutually compatible collection  $\rho^{(k)}: \mathcal{V}^k \rightarrow \mathcal{H}^{(k)}$  of moving frames of all orders  $k \geq n$  with domains  $\mathcal{V}^k = (\widetilde{\pi}_n^k)^{-1} \mathcal{V}^n$ .

As in the finite-dimensional construction [23], the (local) existence of a moving frame requires that the group action be free and regular.

**Definition 4.2** The pseudo-group  $\mathcal{G}$  acts *freely* at  $z^{(n)} \in J^n$  if its *isotropy subgroup* is trivial,  $\mathcal{G}_{z^{(n)}}^{(n)} = \{g^{(n)} \in \mathcal{G}^{(n)}|_z \mid g^{(n)} \cdot z^{(n)} = z^{(n)}\} = \{\mathbb{I}_z^{(n)}\}$ , and *locally freely* if  $\mathcal{G}_{z^{(n)}}^{(n)}$  is discrete.

According to the standard definition [23] any (locally) free action of a finite-dimensional Lie group satisfies the (local) freeness condition of definition 4.2, but the converse is *not* necessarily valid.

The pseudo-group acts locally freely at  $z^{(n)}$  if and only if the dimension of the prolonged pseudo-group orbit through  $z^{(n)}$  agrees with the dimension  $r_n = \dim \mathcal{G}^{(n)}|_z$  of the source fiber at  $z = \tilde{\pi}_o^n(z^{(n)})$ . Thus, freeness of the pseudo-group at order  $n$  requires, at the very least, that

$$r_n = \dim \mathcal{G}^{(n)}|_z \leq \dim J^n = p + (m - p) \binom{p + n}{p}. \quad (4.9)$$

Freeness thus provides an alternative and simpler means of quantifying the Spencer cohomological growth conditions imposed in [33, 34]. Pseudo-groups having too large a fiber dimension  $r_n$  will, typically, act transitively on (a dense open subset of)  $J^n$ , and thus possess no non-constant differential invariants. A key result of [60], generalizing the trivial finite-dimensional case, is the persistence of local freeness.

**Theorem 4.3** *Let  $\mathcal{G}$  be a Lie pseudo-group acting on an  $m$ -dimensional manifold  $M$ . If  $\mathcal{G}$  acts locally freely at  $z^{(n)} \in J^n$  for some  $n > 0$ , then it acts locally freely at any  $z^{(k)} \in J^k$  with  $\tilde{\pi}_n^k(z^{(k)}) = z^{(n)}$ , for  $k \geq n$ .*

As in the finite-dimensional version, [23], moving frames are constructed through a normalization procedure based on a choice of *cross-section* to the pseudo-group orbits, i.e., a transverse submanifold of the complementary dimension.

**Theorem 4.4** *Suppose  $\mathcal{G}^{(n)}$  acts freely on  $\mathcal{V}^n \subset J^n$  with its orbits forming a regular foliation. Let  $\mathcal{K}^n \subset \mathcal{V}^n$  be a local cross-section to the pseudo-group orbits. Given  $z^{(n)} \in \mathcal{V}^n$ , define  $\rho^{(n)}(z^{(n)}) \in \mathcal{H}^{(n)}$  to be the unique groupoid jet such that  $\tilde{\sigma}^n(\rho^{(n)}(z^{(n)})) = z^{(n)}$  and  $\tilde{\tau}^n(\rho^{(n)}(z^{(n)})) \in \mathcal{K}^n$  (when such exists). Then  $\rho^{(n)}: J^n \rightarrow \mathcal{H}^{(n)}$  is a moving frame for  $\mathcal{G}$  defined on an open subset of  $\mathcal{V}^n$  containing  $\mathcal{K}^n$ .*

In most practical situations, we select a coordinate cross-section of minimal order, defined by fixing the values of  $r_n$  of the individual submanifold jet coordinates  $(x, u^{(n)})$ . We write out the explicit formulas  $(X, U^{(n)}) = \mathbf{F}(x, u^{(n)}, g^{(n)})$  using expressions (4.5) for the prolonged pseudo-group action in terms of a convenient system of group parameters  $g^{(n)} = (g_1, \dots, g_{r_n})$ . The  $r_n$  components corresponding to our choice of cross-section variables serve to define the *normalization equations*

$$F_1(x, u^{(n)}, g^{(n)}) = c_1, \quad \dots, \quad F_{r_n}(x, u^{(n)}, g^{(n)}) = c_{r_n}. \quad (4.10)$$

Solving for the group parameters,

$$g_i = \gamma_i^{(n)}(x, u^{(n)}), \quad i = 1, \dots, r_n, \quad (4.11)$$

yields the formula

$$\rho^{(n)}(x, u^{(n)}) = (x, u^{(n)}, \gamma^{(n)}(x, u^{(n)}))$$

for a moving frame section.

The general invariantization procedure introduced in [32] in the finite-dimensional case adapts straightforwardly. To compute the invariantization of a function, differential form, differential operator, etc., one writes out how it explicitly transforms under the pseudo-group, and then replaces the pseudo-group parameters by their moving frame expressions (4.11). Invariantization thus defines a projection, depending upon the choice of cross-section or moving frame, from the spaces of general functions and forms to the spaces of invariant functions and forms. In particular, invariantizing the coordinate functions on  $J^\infty$  leads to the *normalized differential invariants*

$$H^i = \iota(x^i), \quad i = 1, \dots, p, \quad I_J^\alpha = \iota(u_J^\alpha), \quad \alpha = 1, \dots, q, \quad \#J \geq 0. \quad (4.12)$$

These naturally split into two species: those appearing in the normalization equations (4.10) will be constant, and are known as the *phantom differential invariants*. The remaining  $s_n = \dim J^n - r_n$  components, called the *basic differential invariants*, form a complete system of functionally independent differential invariants of order  $\leq n$ .

Secondly, invariantization of the basis horizontal one-forms leads to the invariant horizontal one-forms

$$\varpi^i = \iota(dx^i) = \omega^i + \kappa^i, \quad i = 1, \dots, p, \quad (4.13)$$

where  $\omega^i$ ,  $\kappa^i$  are, respectively, the horizontal and vertical (contact) components. If the pseudo-group acts projectably, then the contact components vanish,  $\kappa^i = 0$ . Otherwise, the two components are not individually invariant, although the horizontal forms  $\omega^1, \dots, \omega^p$  are, in the language of [52], a contact-invariant coframe on  $J^\infty$ .

The dual invariant differential operators  $\mathcal{D}_1, \dots, \mathcal{D}_p$  are uniquely defined by the formula

$$dF = \sum_{i=1}^p \mathcal{D}_i F \varpi^i + \dots, \quad (4.14)$$

valid for any differential function  $F$ , where we omit the contact components (although these do play an important role in the study of invariant variational problems, [32]). The invariant differential operators map differential invariants to differential invariants. In general, they do not commute, but are subject to linear commutation relations of the form

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^p Y_{ij}^k \mathcal{D}_k, \quad i, j = 1, \dots, p, \quad (4.15)$$

where the coefficients  $Y_{ij}^k$  are certain differential invariants that must also be determined. Finally, invariantizing the basis contact one-forms

$$\vartheta_K^\alpha = \iota(\theta_K^\alpha), \quad \alpha = 1, \dots, q, \quad \#K \geq 0, \quad (4.16)$$

provide a complete system of invariant contact one-forms. The invariant coframe serves to characterize the  $\mathcal{G}$  invariant variational complex in the domain of the complete moving frame, [32].

**Example 4.5** Consider the action of the pseudo-group (3.4) on surfaces  $u = h(x, y)$ . The pseudo-group maps the basis horizontal forms  $dx, dy$  to the one-forms

$$d_H X = f_x dx, \quad d_H Y = e_x dx + f_x dy. \quad (4.17)$$

By (4.7), the prolonged pseudo-group transformations are found by applying the dual implicit differentiations

$$D_X = \frac{1}{f_x} D_x - \frac{e_x}{f_x^2} D_y, \quad D_Y = \frac{1}{f_x} D_y$$

successively to  $U = u + e_x/f_x$ , so that

$$\begin{aligned} U_X &= \frac{u_x}{f_x} + \frac{e_{xx} - e_x u_y}{f_x^2} - 2 \frac{f_{xx} e_x}{f_x^3}, & U_Y &= \frac{u_y}{f_x} + \frac{f_{xx}}{f_x^2}, \\ U_{XX} &= \frac{u_{xx}}{f_x^2} + \frac{e_{xxx} - e_{xx} u_y - 2 e_x u_{xy} - f_{xx} u_x}{f_x^3} \\ &\quad + \frac{e_x^2 u_{yy} + 3 e_x f_{xx} u_y - 4 e_{xx} f_{xx} - 3 e_x f_{xxx}}{f_x^4} + 8 \frac{e_x f_{xx}^2}{f_x^5}, & (4.18) \\ U_{XY} &= \frac{u_{xy}}{f_x^2} + \frac{f_{xxx} - f_{xx} u_y - e_x u_{yy}}{f_x^3} - 2 \frac{f_{xx}^2}{f_x^4}, & U_{YY} &= \frac{u_{yy}}{f_x^2}, \\ U_{XYX} &= \frac{f_x u_{xyy} - e_x u_{yyy} - 2 f_{xx} u_{yy}}{f_x^4}, & U_{YYX} &= \frac{u_{yyy}}{f_x^3}, \end{aligned}$$

and so on. In these formulas, the diffeomorphism jet coordinates  $f, f_x, f_{xx}, \dots, e, e_x, e_{xx}, \dots$  are to be regarded as the independent pseudo-group parameters. The pseudo-group cannot act freely on  $J^1$  since  $r_1 = \dim \mathcal{G}^{(1)}|_z = 6 > \dim J^1 = 5$ . On the other hand,  $r_2 = \dim \mathcal{G}^{(2)}|_z = 8 = \dim J^2$ , and the action on  $J^2$  is, in fact, locally free and transitive on the sets  $\mathcal{V}_+^2 = J^2 \cap \{u_{yy} > 0\}$  and  $\mathcal{V}_-^2 = J^2 \cap \{u_{yy} < 0\}$ . Moreover, in accordance with Theorem 4.3,  $\mathcal{G}^{(n)}$  acts locally freely on the corresponding open subsets of  $J^n$  for any  $n \geq 2$ .

To construct the moving frame, we adopt the following cross-section normalizations:

$$\begin{aligned} X = 0 &\implies f = 0, \\ Y = 0 &\implies e = 0, \\ U = 0 &\implies e_x = -u f_x, \\ U_Y = 0 &\implies f_{xx} = -u_y f_x, \\ U_X = 0 &\implies e_{xx} = (u u_y - u_x) f_x, \\ U_{YY} = 1 &\implies f_x = \sqrt{u_{yy}}, \\ U_{XY} = 0 &\implies f_{xxx} = -\sqrt{u_{yy}} (u_{xy} + u u_{yy} - u_y^2), \\ U_{XX} = 0 &\implies e_{xxx} = -\sqrt{u_{yy}} (u_{xx} - u u_{xy} - 2 u^2 u_{yy} - 2 u_x u_y + u u_y^2). \end{aligned} \quad (4.19)$$

At this stage, we have normalized enough pseudo-group parameters to compute the first two basic differential invariants by substituting the normalizations (4.19) into the transformation rules for  $u_{xyy}$ ,  $u_{yyy}$  in (4.18), which yields the expressions

$$I_{12} = \iota(u_{xyy}) = \frac{u_{xyy} + uu_{yyy} + 2u_y u_{yy}}{u_{yy}^{3/2}}, \quad I_{03} = \iota(u_{yyy}) = \frac{u_{yyy}}{u_{yy}^{3/2}} \quad (4.20)$$

for the invariants. Higher order differential invariants are found by continuing this process, or by using the more effective Taylor series method of [59]. Further, substituting the pseudo-group normalizations into (4.17) fixes the invariant horizontal coframe

$$\omega^1 = \iota(dx) = \sqrt{u_{yy}} dx, \quad \omega^2 = \iota(dy) = \sqrt{u_{yy}} (dy - u dx).$$

The invariant contact forms are similarly constructed [59]. The dual invariant total derivative operators are

$$\mathcal{D}_1 = \frac{1}{\sqrt{u_{yy}}} (\mathbf{D}_x + u \mathbf{D}_y), \quad \mathcal{D}_2 = \frac{1}{\sqrt{u_{yy}}} \mathbf{D}_y. \quad (4.21)$$

The higher-order differential invariants can be generated by successively applying these differential operators to the pair of basic differential invariants (4.20). The total derivative operators satisfy the commutation relation

$$[\mathcal{D}_1, \mathcal{D}_2] = -\frac{1}{2} I_{03} \mathcal{D}_1 + \frac{1}{2} I_{12} \mathcal{D}_2. \quad (4.22)$$

Finally, there is a single basic syzygy

$$\mathcal{D}_1 I_{03} - \mathcal{D}_2 I_{12} = 2 \quad (4.23)$$

among the differentiated invariants from which all others can be deduced by invariant differentiation.

## 5 Recurrence Formulas

The recurrence formulas [23, 32] connect the differentiated invariants and invariant forms with their normalized counterparts. These formulas are fundamental, since they prescribe the structure of the algebra of (local) differential invariants, underlying a full classification of generating differential invariants, their syzygies (differential identities), as well as the structure of invariant variational problems and, indeed, the local structure of the entire invariant variational bicomplex. As in the finite-dimensional version, the recurrence formulas are established using purely infinitesimal information, requiring only linear algebra and differentiation. In particular, they do *not* require the explicit formulas for either the moving frame, or the Maurer–Cartan forms, or the normalized differential invariants and invariant forms, or even the invariant differential

operators! Beyond the formulas for the infinitesimal determining equations, the only additional information required is the specification of the moving frame cross-section. Under the moving frame map, the pulled-back Maurer–Cartan forms will be denoted  $\nu^\infty = (\rho^{(\infty)})^* \mu^\infty$ , with individual components

$$\nu_A^b = (\rho^{(\infty)})^*(\mu_A^b), \quad b = 1, \dots, m, \quad \#A \geq 0. \quad (5.1)$$

As such, they are invariant one-forms, and so are invariant linear combinations of our invariant coframe elements  $\omega^i$ ,  $\vartheta_K^\alpha$  defined in (4.13), (4.16), with coefficients that are certain differential invariants.

Fortunately, the precise formulas need not be established a priori, as they will be a direct consequence of the recurrence formulas for the phantom differential invariants. We will extend our invariantization process to vector field coefficient jet coordinates (2.15) by defining

$$\iota(\zeta_A^b) = \nu_A^b, \quad b = 1, \dots, m, \quad \#A \geq 0, \quad (5.2)$$

and, by extension, to their differential function and form-valued linear combinations,

$$\iota \left( \sum_{b=1}^m \sum_{\#A \leq n} \zeta_A^b \omega_b^A \right) = \sum_{b=1}^m \sum_{\#A \leq n} \nu_A^b \wedge \iota(\omega_b^A), \quad (5.3)$$

where the  $\omega_b^A$  are  $k$ -forms on  $J^\infty$ , so the result is an invariant differential  $(k+1)$ -form on  $J^\infty$ . With this interpretation, the pulled-back Maurer–Cartan forms  $\nu_A^b$  are subject to the linear relations

$$L^{(n)}(H, I, \nu^{(n)}) = \iota[L^{(n)}(z, \zeta^{(n)})] = 0, \quad n \geq 0, \quad (5.4)$$

obtained by invariantizing the original linear determining equations (3.2). Here

$$(H, I) = \iota(x, u) = \iota(z)$$

are the zero<sup>th</sup> order differential invariants in (4.12).

Given a locally defined vector field

$$\mathbf{v} = \sum_{a=1}^m \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \in \mathcal{X}(M), \quad (5.5)$$

let

$$\mathbf{v}^\infty = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_{n=\#J \geq 0} \widehat{\varphi}_J^\alpha(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha} \in \mathcal{X}(J^\infty) \quad (5.6)$$

denote its infinite prolongation. The coefficients are computed via the usual prolongation formula,

$$\widehat{\varphi}_J^\alpha = D_J Q^\alpha + \sum_{i=1}^p \xi^i u_{Ji}^\alpha, \quad \text{where} \quad Q^\alpha = \varphi^\alpha - \sum_{i=1}^p u_i^\alpha \xi^i, \quad \alpha = 1, \dots, q, \quad (5.7)$$

are the components of the *characteristic* of  $\mathbf{v}$ , [51, 52].

Consequently, each prolonged vector field coefficient

$$\widehat{\varphi}_J^\alpha = \Phi_J^\alpha(u^{(n)}, \zeta^{(n)}) \quad (5.8)$$

is a certain universal linear combination of the vector field jet coordinates (2.15), whose coefficients are polynomials in the submanifold jet coordinates  $u_K^\beta$  for  $1 \leq \#K \leq n$ . Therefore, the  $n^{\text{th}}$  order prolongation of vector fields factors through the  $n^{\text{th}}$  order vector field jet bundle. Let

$$\eta^i = \iota(\xi^i) = \nu^i, \quad \widehat{\psi}_J^\alpha = \iota(\widehat{\varphi}_J^\alpha) = \Psi_J^\alpha(I^{(n)}, \nu^{(n)}), \quad (5.9)$$

denote the invariantizations of the prolonged infinitesimal generator coefficients (5.8), which are linear combinations of the pulled-back Maurer–Cartan forms (5.2), with polynomial coefficients in the normalized differential invariants  $I_K^\beta$  for  $1 \leq \#K \leq \#J$ .

With all these in hand, the desired *universal recurrence formula* is as follows.

**Theorem 5.1** *If  $\omega$  is any differential form on  $J^\infty$ , then*

$$d\iota(\omega) = \iota(d\omega + \mathbf{v}^\infty(\omega)), \quad (5.10)$$

where  $\mathbf{v}^\infty(\omega)$  denotes the Lie derivative of  $\omega$  with respect to the prolonged vector field (5.6), and where we use (5.9) to invariantize the result.

Specializing  $\omega$  in (5.10) to be one of the coordinate functions  $x^i$ ,  $u_J^\alpha$  yields recurrence formulas for the normalized differential invariants (4.12),

$$\begin{aligned} dH^i &= \iota(dx^i + \xi^i) = \varpi^i + \eta^i, \\ dI_J^\alpha &= \iota(du_J^\alpha + \widehat{\varphi}_J^\alpha) = \iota\left(\sum_{i=1}^p u_{J_i}^\alpha dx^i + \theta_J^\alpha + \widehat{\varphi}_J^\alpha\right) = \sum_{i=1}^p I_{J_i}^\alpha \varpi^i + \vartheta_J^\alpha + \widehat{\psi}_J^\alpha, \end{aligned} \quad (5.11)$$

where  $\widehat{\psi}_J^\alpha$  is written in terms of the pulled-back Maurer–Cartan forms  $\nu_A^b$  as in (5.9), and are subject to the linear constraints (5.4). Each phantom differential invariant is, by definition, normalized to a constant value, and hence has zero differential. Consequently, the phantom recurrence formulas in (5.11) form a system of linear algebraic equations which can, as a result of the transversality of the cross-section, be uniquely solved for the pulled-back Maurer–Cartan forms.

**Theorem 5.2** *If the pseudo-group acts locally freely on  $\mathcal{V}^n \subset J^n$ , then the  $n^{\text{th}}$  order phantom recurrence formulas can be uniquely solved in a neighborhood of the cross section to express the pulled-back Maurer–Cartan forms  $\nu_A^b$  of order  $\#A \leq n$  as invariant linear combinations of the invariant horizontal and contact one-forms  $\varpi^i, \vartheta_J^\alpha$ .*



Substituting the resulting expressions into the remaining, non-phantom recurrence formulas in (5.11) leads to a complete system of recurrence relations, for both the vertical and horizontal differentials of all the normalized differential invariants.

As the prolonged vector field coefficients  $\widehat{\varphi}_j^\alpha$  are polynomials in the jet coordinates  $u_K^\beta$  of order  $\#K \geq 1$ , their invariantizations are polynomial functions of the differential invariants  $I_K^\beta$  for  $\#K \geq 1$ . Since the correction terms are constructed by solving a linear system for the invariantized Maurer–Cartan forms, the resulting coefficients are *rational functions* of these differential invariants. Thus, in most cases (including the majority of applications), the algebra of differential invariants for the pseudo-group is endowed with an entirely rational algebraic recurrence structure.

**Theorem 5.3** *If  $\mathcal{G}$  acts transitively on  $M$ , or, more generally, its infinitesimal generators depend polynomially on the coordinates  $z = (x, u) \in M$ , then the correction terms in the recurrence formulas (5.11) are rational functions of the normalized differential invariants.*

**Example 5.4** In this example we let the pseudo-group (3.4) act on surfaces in  $\mathbb{R}^3$ . We will illustrate how the above ideas can be used to uncover recursion formulas for derivatives of the normalized differential invariants with respect to the invariant total derivative operators (4.21). For this only the horizontal components of equations (5.11) are needed, and we write these as

$$\begin{aligned} d_H H^1 &= \omega^x + \chi^x, & d_H H^2 &= \omega^y + \chi^y, \\ d_H I_{ij} &= I_{i+1,j} \omega^x + I_{i,j+1} \omega^y + \varkappa_{ij} & i, j &\geq 0. \end{aligned} \quad (5.12)$$

In (5.12) each  $\varkappa_{ij}$  is a linear combination of the horizontal pulled-back Maurer–Cartan forms which we denote

$$\chi_{X^a Y^b U^c}^x, \quad \chi_{X^a Y^b U^c}^y, \quad \chi_{X^a Y^b U^c}^u, \quad a, b, c \geq 0. \quad (5.13)$$

These are again constrained by the invariantized infinitesimal determining equations  $L^{(n)}(H, \chi^{(n)}) = 0$ , and, as in Example 3.3, we see that a basis for the forms (5.13) is provided by  $\chi_{X^n}^x, \chi_{X^n}^y, n \geq 0$ , and, moreover, that

$$\chi_{X^n Y}^y = \chi_{X^{n+1}}^x, \quad \chi_{X^n}^u = \chi_{X^{n+1}}^y, \quad \chi_{X^n Y}^u = \chi_{X^{n+2}}^x, \quad n \geq 0, \quad (5.14)$$

while all the other horizontal pulled-back Maurer–Cartan forms vanish.

We choose a cross section by imposing, in addition to (4.19), the normalization equations

$$U_{X^n} = 0, \quad U_{X^{n-1} Y} = 0, \quad n \geq 3. \quad (5.15)$$

It now follows from (5.14) that the horizontal correction terms  $\varkappa_{ij}$  are precisely the coefficients of the horizontal invariantization of the vector field obtained by first prolonging the vector field

$$\mathbf{v} = \xi(x) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \eta_x(x, y) \frac{\partial}{\partial u}$$

and then applying the relations

$$\eta_y = \xi_x, \quad \xi_y = \xi_u = \eta_u = 0$$

and their differential consequences to express the resulting coefficient functions solely in terms of the repeated  $x$ -derivatives of  $\xi$  and  $\eta$ . With this, the  $u_{ij}$ -component  $\widehat{\varphi}_{ij}$  of  $\text{pr } \mathbf{v}$  becomes

$$\begin{aligned} \widehat{\varphi}_{ij} &= \delta_{0j} \eta_{x^{i+1}} + \delta_{1j} \xi_{x^{i+2}} \\ &\quad - \sum_{s=1}^{i+1} \frac{i+1+(j-1)s}{i+1} \binom{i+1}{s} \xi_{x^s} u_{i-s+1,j} - \sum_{s=1}^i \binom{i}{s} \eta_{x^s} u_{i-s,j+1}, \end{aligned}$$

yielding the invariantization

$$\begin{aligned} \varkappa_{ij} &= \delta_{0j} \chi_{X^{i+1}}^y + \delta_{1j} \chi_{X^{i+2}}^x \\ &\quad - \sum_{s=1}^{i+1} \frac{i+1+(j-1)s}{i+1} \binom{i+1}{s} I_{i-s+1,j} \chi_{X^s}^x - \sum_{s=1}^i \binom{i}{s} I_{i-s,j+1} \chi_{X^s}^y. \end{aligned}$$

Taking into account normalizations (4.19), (5.15), the phantom recurrence formulas for the horizontal derivatives reduce to

$$\begin{aligned} 0 &= d_H H^x = \omega^x + \chi^x, & 0 &= d_H H^y = \omega^y + \chi^y, \\ 0 &= d_H I = \chi_X^y, & 0 &= d_H I_{10} = \chi_{X^2}^y, \\ 0 &= d_H I_{01} = \omega^y + \chi_{X^2}^x, & 0 &= d_H I_{20} = \chi_{X^3}^y, \\ 0 &= d_H I_{11} = I_{12} \omega^y - \chi_X^y + \chi_{X^3}^x, & 0 &= d_H I_{02} = I_{12} \omega^x + I_{03} \omega^y - 2\chi_X^x, \\ 0 &= d_H I_{30} = \chi_{X^4}^y, & 0 &= d_H I_{21} = I_{22} \omega^y + \chi_{X^4}^x - \chi_{X^2}^y - 2I_{12} \chi_X^y, \end{aligned} \quad (5.16)$$

and so on. These can be solved for the Maurer–Cartan forms  $\chi_{X^n}^x, \chi_{X^n}^y$  to yield

$$\begin{aligned} \chi^x &= -\omega^x, & \chi^y &= -\omega^y, & \chi_X^x &= \frac{1}{2}(I_{12} \omega^x + I_{03} \omega^y), & \chi_X^y &= 0, \\ \chi_{X^2}^x &= -\omega^y, & \chi_{X^2}^y &= 0, & \chi_{X^3}^x &= -I_{12} \omega^y, & \chi_{X^3}^y &= 0, \\ \chi_{X^4}^x &= -I_{22} \omega^y, & \chi_{X^4}^y &= 0, & \dots & & \end{aligned} \quad (5.17)$$

Next we substitute the expressions (5.17) in the recurrence formulas for the non-phantom invariants in (5.12). Keeping in mind that

$$d_H I_{ij} = (\mathcal{D}_1 I_{ij}) \omega^x + (\mathcal{D}_2 I_{ij}) \omega^y,$$

the  $\omega^x$ -,  $\omega^y$ -components of the resulting equations yield the expressions

$$\begin{aligned}
\mathcal{D}_1 I_{12} &= I_{22} - \frac{3}{2} I_{12}^2, & \mathcal{D}_2 I_{12} &= I_{13} - \frac{3}{2} I_{12} I_{03} + 2, \\
\mathcal{D}_1 I_{03} &= I_{13} - \frac{3}{2} I_{12} I_{03}, & \mathcal{D}_2 I_{03} &= I_{04} - \frac{3}{2} I_{03}^2, \\
\mathcal{D}_1 I_{22} &= I_{32} - 2I_{12} I_{22}, & \mathcal{D}_2 I_{22} &= I_{23} - 2I_{03} I_{22} + 7I_{12}, \\
\mathcal{D}_1 I_{13} &= I_{23} - 2I_{12} I_{13}, & \mathcal{D}_2 I_{13} &= I_{14} - 2I_{03} I_{13} + 3I_{03}, \\
\mathcal{D}_1 I_{04} &= I_{14} - 2I_{12} I_{04}, & \mathcal{D}_2 I_{04} &= I_{05} - 2I_{03} I_{04}, \\
&\vdots
\end{aligned} \tag{5.18}$$

It becomes apparent from these formulas that the invariants  $I_{12}$  and  $I_{03}$  generate the algebra of differential invariants for the pseudo-group (3.4).

A derivation of the recurrence formulas (5.18) using Taylor series methods can be found in [59].

## 6 Algebra of Differential Invariants

Establishment of the theoretical results that underpin our constructive algorithms relies on the interplay between two algebraic structures associated with the two jet bundles: the first is defined by the symbols of the linearized determining equations for the infinitesimal generator jets; the second from a similar construction for the prolonged generators on the submanifold jet bundle. We begin with the former, which is the more familiar of the two.

We introduce algebraic variables  $t = t_1, \dots, t_m$  and  $T = T^1, \dots, T^m$ . Let

$$\mathcal{T} = \left\{ \eta(t, T) = \sum_{a=1}^m \eta_a(t) T^a \right\} \simeq \mathbb{R}[t] \otimes \mathbb{R}^m \tag{6.1}$$

denote the  $\mathbb{R}[t]$  module consisting of real polynomials that are linear in the  $T$ 's. We grade  $\mathcal{T} = \bigoplus_{n \geq 0} \mathcal{T}^n$ , where  $\mathcal{T}^n$  consists of the homogeneous polynomials of degree  $n$  in  $t$ . We set  $\mathcal{T}^{\leq n} = \bigoplus_{k=0}^n \mathcal{T}^k$  to be the space of polynomials of degree  $\leq n$ .

Given a subspace  $\mathcal{I} \subset \mathcal{T}$ , we set  $\mathcal{I}^n = \mathcal{I} \cap \mathcal{T}^n$ ,  $\mathcal{I}^{\leq n} = \mathcal{I} \cap \mathcal{T}^{\leq n}$ . The subspace is *graded* if  $\mathcal{I} = \bigoplus_{n \geq 0} \mathcal{I}^n$  is the sum of its homogeneous constituents. A subspace  $\mathcal{I} \subset \mathcal{T}$  is a *submodule* if the product  $\lambda(t) \eta(t, T) \in \mathcal{I}$  whenever  $\eta(t, T) \in \mathcal{I}$  and  $\lambda(t) \in \mathbb{R}[t]$ . A subspace  $\mathcal{I} \subset \mathcal{T}$  spanned by monomials  $t_A T^b = t_{a_1} \cdots t_{a_n} T^b$  is called a *monomial subspace*, and is automatically graded. In particular, a *monomial submodule* is a submodule that is spanned by monomials. A polynomial  $\eta \in \mathcal{T}^{\leq n}$  has *degree*  $n = \deg \eta$  and *highest order terms*  $\mathbf{H}(\eta) \in \mathcal{T}^n$  provided  $\eta = \mathbf{H}(\eta) + \tilde{\eta}$ , where  $\mathbf{H}(\eta) \neq 0$  and  $\tilde{\eta} \in \mathcal{T}^{\leq n-1}$  is of lower degree. By convention, only the zero polynomial has zero highest order term.

We will assume in the remaining part of the paper that all our functions, vector fields etc. are analytic. We can locally identify the dual bundle to the infinite order jet bundle as  $(\mathcal{X}^\infty)^* \simeq M \times \mathcal{T}$  via the pairing  $\langle j_z^\infty \mathbf{v}; t_A T^b \rangle = \zeta_A^b$ . A parametrized polynomial

$$\eta(z; t, T) = \sum_{b=1}^m \sum_{\#A \leq n} h_b^A(z) t_A T^b \quad (6.2)$$

corresponds to an  $n^{\text{th}}$  order linear differential function

$$L(z, \zeta^{(n)}) = \langle j_z^\infty \mathbf{v}; \eta(z; t, T) \rangle = \sum_{b=1}^m \sum_{\#A \leq n} h_b^A(z) \zeta_A^b. \quad (6.3)$$

Its *symbol* consists, by definition, of the highest order terms in its defining polynomial,

$$\Sigma(L(z, \zeta^{(n)})) = \mathbf{H}(\eta(z; t, T)) = \sum_{b=1}^m \sum_{\#A=n} h_b^A(z) t_A T^b. \quad (6.4)$$

The symbol of a linear differential function is well defined by our assumption that the coefficient functions  $h_b^A(z)$  be analytic.

Given a pseudo-group  $\mathcal{G}$ , let  $\mathcal{L} = (\mathfrak{g}^\infty)^\perp \subset (\mathcal{X}^\infty)^*$  denote the *annihilator subbundle* of its infinitesimal generator jet bundle. Each equation in the linear determining system (3.2) is represented by a parametrized polynomial (6.2), which cumulatively span  $\mathcal{L}$ . Let  $\mathcal{I} = \mathbf{H}(\mathcal{L}) \subset (\mathcal{X}^\infty)^*$  denote the associated *symbol subbundle* (assuming regularity). On the symbol level, total derivative corresponds to multiplication,

$$\Sigma(\mathbb{D}_{z^a} L) = t_a \Sigma(L), \quad a = 1, \dots, m. \quad (6.5)$$

Thus, formal integrability of (3.2) implies that each fiber  $\mathcal{I}|_z \subset \mathcal{T}$  forms a graded submodule, known as the *symbol module* of the pseudo-group at the point  $z \in M$ . On the other hand, the annihilator  $\mathcal{L}|_z \subset \mathcal{T}$  is typically *not* a submodule.

We now develop an analogous symbol algebra for the prolonged infinitesimal generators. We introduce variables  $s = s_1, \dots, s_p$ ,  $S = S^1, \dots, S^q$ , and let

$$\widehat{\mathcal{S}} = \left\{ \tilde{\sigma}(s, S) = \sum_{\alpha=1}^q \tilde{\sigma}_\alpha(s) S^\alpha \right\} \simeq \mathbb{R}[s] \otimes \mathbb{R}^q \subset \mathbb{R}[s, S] \quad (6.6)$$

be the  $\mathbb{R}[s]$  module consisting of polynomials which are linear in  $S$ . Let

$$\mathcal{S} = \mathbb{R}^p \oplus \widehat{\mathcal{S}} = \bigoplus_{n=-1}^{\infty} \widehat{\mathcal{S}}^n \quad (6.7)$$

consist of the “extended” polynomials

$$\sigma(s, S, \tilde{s}) = \sum_{i=1}^p c_i \tilde{s}_i + \tilde{\sigma}_\alpha(s, S) = \sum_{i=1}^p c_i \tilde{s}_i + \sum_{\alpha=1}^q \tilde{\sigma}_\alpha(s) S^\alpha, \quad (6.8)$$

where  $\tilde{s} = \tilde{s}_1, \dots, \tilde{s}_p$  are extra algebraic variables,  $c_1, \dots, c_p \in \mathbb{R}$ , and  $\tilde{\sigma}(s, S) \in \widehat{\mathcal{S}}$ . In coordinates, we can identify  $T^*J^\infty \simeq J^\infty \times \mathcal{S}$  via the pairing

$$\begin{aligned} \langle \mathbf{V}; \tilde{s}_i \rangle &= \xi^i, & \langle \mathbf{V}; S^\alpha \rangle &= Q^\alpha = \varphi^\alpha - \sum_{i=1}^p u_i^\alpha \xi^i, \\ \langle \mathbf{V}; s_J S^\alpha \rangle &= \widehat{\varphi}_J^\alpha, & \#J &\geq 1, \end{aligned} \quad (6.9)$$

for any tangent vector

$$\mathbf{v} = \sum_{i=1}^p \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_{k=\#J \geq 0} \widehat{\varphi}_J^\alpha \frac{\partial}{\partial u_J^\alpha} \in TJ^\infty.$$

Every one-form on  $J^\infty$  is thereby represented, locally, by a parametrized polynomial

$$\sigma(x, u^{(n)}; \tilde{s}, s, S) = \sum_{i=1}^p h_i(x, u^{(n)}) \tilde{s}_i + \sum_{\alpha=1}^q \tilde{\sigma}_\alpha(x, u^{(n)}; s) S^\alpha,$$

depending linearly on the variables  $(\tilde{s}, S) \in \mathbb{R}^m$ , polynomially on the variables  $s \in \mathbb{R}^p$ , and analytically on the jet coordinates  $(x, u^{(n)})$  of some finite order  $n < \infty$ .

Given  $z^{(\infty)} \in J^\infty$ , let

$$\mathbf{p} = \mathbf{p}^{(\infty)}: J^\infty TM|_z \longrightarrow T_{z^{(\infty)}} J^\infty, \quad \mathbf{p}(j_z^\infty \mathbf{v}) = \mathbf{v}^\infty|_{z^{(\infty)}}, \quad (6.10)$$

denote the *prolongation map* that takes the jet of a vector field at the base point  $z = \widehat{\pi}_o^\infty(z^{(\infty)}) \in M$  to its prolongation (5.6). Let  $\mathbf{p}^*: \mathcal{S} \rightarrow \mathcal{T}$  be the corresponding *dual prolongation map*, defined so that

$$\langle j_z^\infty \mathbf{v}; \mathbf{p}^*(\sigma) \rangle = \langle \mathbf{p}(j_z^\infty \mathbf{v}); \sigma \rangle \quad (6.11)$$

for all  $j_z^\infty \mathbf{v} \in J^\infty TM|_z$ ,  $\sigma \in \mathcal{S}$ . In general,  $\mathbf{p}^*$  is *not* a module morphism. However, on the symbol level it essentially is, as we now explain.

Consider the particular linear polynomials

$$\beta_i(t) = t_i + \sum_{\alpha=1}^q u_i^\alpha t_{p+\alpha}, \quad B^\alpha(T) = T^{p+\alpha} - \sum_{i=1}^p u_i^\alpha T^i, \quad (6.12)$$

for  $i = 1, \dots, p$ ,  $\alpha = 1, \dots, q$ , where  $u_i^\alpha = \partial u^\alpha / \partial x^i$  are the first order jet coordinates of our point  $z^{(\infty)}$ . Note that  $B^\alpha(T)$  is the symbol of  $Q^\alpha$ , the  $\alpha^{\text{th}}$  component of the characteristic of  $\mathbf{v}$ , cf. (6.9), while  $\beta_i(t)$  represents the symbol of the total derivative,  $\Sigma(\mathbf{D}_i L) = \beta_i(t) \Sigma(L)$ . The functions  $s_i = \beta_i(t)$ ,  $S^\alpha = B^\alpha(T)$  serve to define a linear map  $\beta: \mathbb{R}^{2m} \rightarrow \mathbb{R}^m$ .

Since  $\beta$  has maximal rank, the induced pull-back map

$$(\beta^* \sigma)(t_1, \dots, t_n, T^1, \dots, T^n) = \sigma(\beta_1(t), \dots, \beta_p(t), B^1(T), \dots, B^q(T)) \quad (6.13)$$

defines an injection  $\beta^*: \widehat{\mathcal{S}} \rightarrow \mathcal{T}$ . The algebraic structure of the vector field prolongation map at the symbol level is encapsulated in the following result.

**Lemma 6.1** *The symbols of the prolonged vector field coefficients are*

$$\Sigma(\xi^i) = T^i, \quad \Sigma(\varphi^\alpha) = T^{\alpha+p}, \quad \Sigma(Q^\alpha) = \beta^*(S^\alpha), \quad \Sigma(\widehat{\varphi}_J^\alpha) = \beta^*(s_J S^\alpha). \quad (6.14)$$

Now given a Lie pseudo-group  $\mathcal{G}$  acting on  $M$ , let  $\mathfrak{g}^\infty|_{z^{(\infty)}} = \mathbf{p}(J^\infty \mathfrak{g}|_z) \subset T_{z^{(\infty)}} J^\infty$  denote the subspace<sup>1</sup> spanned by its prolonged infinitesimal generators. Let

$$\mathcal{Z}_{|z^{(\infty)}} = (\mathfrak{g}^\infty|_{z^{(\infty)}})^\perp = (\mathbf{p}^*)^{-1}(\mathcal{L}_{|z}) \subset \mathcal{S} \quad (6.15)$$

denote the *prolonged annihilator subbundle*, containing those polynomials (6.8) that annihilate all prolonged infinitesimal generators  $\mathbf{v}^\infty \in \mathfrak{g}^\infty|_{z^{(\infty)}}$ . Further, let  $\mathcal{U}_{|z^{(\infty)}} = \mathbf{H}(\mathcal{Z}_{|z^{(\infty)}}) \subset \mathcal{S}$  be the subspace spanned by the highest order terms of the prolonged annihilators. In general  $\mathcal{U}_{|z^{(\infty)}}$  is *not* a submodule, as, for instance, is the case with the pseudo-group discussed in Example 4.5.

Let

$$\mathcal{J}_{|z^{(1)}} = (\beta^*)^{-1}(\mathcal{I}_{|z}) = \{\sigma(s, S) \mid \beta^*(\sigma)(t, T) = \sigma(\beta(t), B(T)) \in \mathcal{I}_{|z}\} \subset \widehat{\mathcal{S}}, \quad (6.16)$$

where  $z = \widetilde{\pi}_0^1(z^{(1)})$ , be the *prolonged symbol submodule*, the inverse image of the symbol module under the polynomial pull-back morphism (6.13). In view of (6.15),  $\mathcal{U}_{|z^{(\infty)}} \subset \mathcal{J}_{|z^{(1)}}$ , where  $z^{(1)} = \widetilde{\pi}_1^\infty(z^{(\infty)})$ . Moreover, assuming local freeness, these two spaces agree at sufficiently high order, thereby endowing  $\mathcal{U}_{|z^{(\infty)}}$  with the structure of an “eventual submodule”.

**Lemma 6.2** *If  $\mathcal{G}^{(n)}$  acts locally freely at  $z^{(n)} \in J^n$ , then  $\mathcal{U}^k|_{z^{(k)}} = \mathcal{J}^k|_{z^{(k)}}$  for all  $k > n$  and all  $z^{(k)} \in J^k$  with  $\widetilde{\pi}_n^k(z^{(k)}) = z^{(n)}$ .*

Let us fix a degree compatible term ordering on the polynomial module  $\widehat{\mathcal{S}}$ , which we extend to  $\mathcal{S}$  by making the extra monomials  $\widetilde{s}_i$  be ordered before all the others. At a fixed regular submanifold jet  $z^{(\infty)}$ , let  $\mathcal{N}_{|z^{(\infty)}}$  be the monomial subspace generated by the leading monomials of the polynomials in  $\mathcal{Z}_{|z^{(\infty)}}$ , or, equivalently, the annihilator symbol polynomials in  $\mathcal{U}_{|z^{(\infty)}}$ . Let  $\mathcal{K}_{|z^{(\infty)}}$  denote the *complementary monomial subspace* spanned by all monomials in  $\mathcal{S}$  that are *not* in  $\mathcal{N}_{|z^{(\infty)}}$ , which can be constructed by applying the usual Gröbner basis algorithm [17] to  $\mathcal{J}_{|z^{(\infty)}}$  and then possibly supplementing the resulting complementary (or standard) monomials by any additional ones required at orders  $\leq n^*$ , where  $J^{n^*}$  stands for the lowest order jet space on which the pseudo-group  $\mathcal{G}$  acts locally freely.

Each monomial in  $\mathcal{K}_{|z^{(\infty)}}$  corresponds to a submanifold jet coordinate  $x^i$  or  $u^a$ . For each  $0 \leq n \leq \infty$ , we let  $K^n \subset J^n$  denote the coordinate cross-section passing through  $z^{(n)} = \widetilde{\pi}_n^\infty(z^{(\infty)})$  prescribed by the monomials in  $\mathcal{K}^{\leq n}|_{z^{(\infty)}}$ . For the corresponding “algebraic” moving frame, each normalized differential invariant is indexed by a monomial in  $\mathcal{S}$ , with  $H^i$  corresponding to  $\widetilde{s}_i$ , and  $I_J^\alpha$  to  $s_J S^\alpha$ . The monomials in  $\mathcal{N}_{|z^{(\infty)}}$  index the complete system of functionally independent basic differential invariants, whereas the complementary monomials in  $\mathcal{K}_{|z^{(\infty)}}$  index the constant phantom differential invariants.

<sup>1</sup>In general,  $\mathfrak{g}^\infty$  may only be a regular subbundle on a (dense) open subset of jet space. We restrict our attention to this subdomain throughout.

However, at this stage a serious complication emerges: Because the invariantized Maurer–Cartan forms are found by solving the recurrence formulas for the phantom invariants, their coefficients may depend on  $(n + 1)^{\text{th}}$  order differential invariants, and hence the correction term in the resulting recurrence formula for  $dI_J^\alpha$  can have the same order as the leading term; see, for example, the recurrence formulas for the KP symmetry algebra derived in [15]. As a consequence, the recurrence formulas (5.11) for the non-phantom invariants do not in general directly yield the leading order normalized differential invariants in terms of lower order invariants and their invariant derivatives. However, this complication, which does not arise in the finite dimensional situation [23], can be circumvented by effectively invariantizing all the constructs in this section and by introducing an alternative collection of generating invariants that is better adapted to the underlying algebraic structure of the prolonged symbol module.

To each parametrized symbol polynomial

$$\tilde{\sigma}(\mathbf{I}^{(k)}; s, S) = \sum_{\alpha=1}^q \sum_{\#J \leq n} h_\alpha^J(\mathbf{I}^{(k)}) s_J S^\alpha \in \widehat{\mathcal{S}}, \quad (6.17)$$

whose coefficients depend on differential invariants  $\mathbf{I}^{(k)} = \iota(x, u^{(k)})$  of order  $\leq k$ , we associate a differential invariant

$$I_{\tilde{\sigma}} = \sum_{\alpha=1}^q \sum_{\#J \leq n} h_\alpha^J(\mathbf{I}^{(k)}) I_J^\alpha. \quad (6.18)$$

Moreover, let  $\tilde{\mathcal{J}} = \iota(\mathcal{J}_{|z^{(1)}})$  denote the *invariantized prolonged symbol submodule* obtained by invariantizing all the polynomials in  $\mathcal{J}_{|z^{(1)}}$ . If  $\mathcal{G}$  acts transitively on an open subset of  $J^1$ , then  $\tilde{\mathcal{J}}$  is a fixed module, since the map  $\beta$  only depends on first order jet coordinates.

As a consequence of Lemma 6.2, every *homogeneous* polynomial  $\tilde{\sigma}(\mathbf{I}^{(1)}; s, S) \in \tilde{\mathcal{J}}$ , for  $n > n^*$ , is the leading term of an annihilating polynomial

$$\tilde{\tau}(\mathbf{I}^{(1)}; s, S) = \tilde{\sigma}(\mathbf{I}^{(1)}; s, S) + \tilde{\nu}(\mathbf{I}^{(1)}; s, S) \quad (6.19)$$

contained in the invariantized version  $\tilde{\mathcal{Z}}$  of the prolonged annihilator bundle  $\mathcal{Z}$ . For such polynomials, the recurrence formula (5.11) reduces to

$$d_H I_{\tilde{\sigma}} = \sum_{i=1}^p (I_{s_i} \tilde{\sigma} + I_{\mathcal{D}_i} \tilde{\sigma}) \omega^i - \langle \psi^\infty; \tilde{\nu} \rangle. \quad (6.20)$$

In contrast to the original recurrence formulas, for  $n > n^*$ , the *correction term*

$$\sum_{i=1}^p I_{\mathcal{D}_i} \tilde{\sigma} \omega^i - \langle \psi^\infty; \tilde{\nu} \rangle \quad (6.21)$$

in the algebraically adapted recurrence formula (6.20) is of lower order than the leading term  $I_{s_i} \tilde{\sigma}$ . Equating the coefficients of the forms  $\omega^i$  in (6.20) leads to individual recurrence formulas

$$\mathcal{D}_i I_{\tilde{\sigma}} = I_{s_i} \tilde{\sigma} + R_{\tilde{\sigma}, i}, \quad (6.22)$$

in which, as long as  $n = \deg \sigma_i > n^*$ , the *leading term*  $I_{s_i \bar{\sigma}}$  is a differential invariant of order  $= n + 1$ , while the *correction term*  $R_{\bar{\sigma}, i}$  is of strictly lower order.

With this in hand, we arrive at the Constructive Basis and Syzygy Theorems governing the differential invariant algebra of an eventually locally freely acting pseudo-group [60].

**Theorem 6.3** *Let  $\mathcal{G}$  be a Lie pseudo-group that acts locally freely on the submanifold jet bundle at order  $n^*$ . Then the following constitute a finite generating system for its differential invariant algebra:*

1. *the differential invariants  $I_\nu = I_{\sigma_\nu}$ , where  $\sigma_1, \dots, \sigma_l$  form a Gröbner basis for the submodule  $\tilde{\mathcal{J}}$  relative to our chosen term ordering, and, possibly,*
2. *a finite number of additional differential invariants of order  $\leq n^*$ .*

**Theorem 6.4** *Every differential syzygy among the generating differential invariants is either a syzygy among those of order  $\leq n^*$ , or arises from an algebraic syzygy among the Gröbner basis polynomials in  $\tilde{\mathcal{J}}$ . Therefore, the differential syzygies are all generated by a finite number of rational syzygies, corresponding to the generators of the syzygy module of  $\tilde{\mathcal{J}}$  plus possibly a finite number of additional syzygies of order  $\leq n^*$ .*

Applications of these results can be found in our papers listed in the references.

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**Author information**

Peter Olver, School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA.  
Email: [olver@math.umn.edu](mailto:olver@math.umn.edu)

Juha Pohjanpelto, Department of Mathematics, Oregon State University, Corvallis, OR 97331, USA.  
Email: [juha@math.oregonstate.edu](mailto:juha@math.oregonstate.edu)