Invariants of Finite and Discrete Group Actions via Moving Frames

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Abstract. A new, elementary algorithm for constructing complete, minimal sets of generating invariants for finite or, more generally, discrete group actions, both linear and nonlinear, is proposed. The resulting fundamental invariants are piecewise analytic and endowed with a rewrite rule that enables one to immediately express any other invariant (polynomial, rational, smooth, analytic, etc.) as a function thereof. The construction is inspired by the method of equivariant moving frames for Lie group actions.

The theory and computation of invariants of finite groups has a long and distinguished history, [**3**, **5**, **21**, **32**]. Most of the effort has gone into constructing and classifying polynomial invariants of linear actions, i.e., representations. The fundamental result is Noether's Theorem, [**22**], which guarantees the existence of a finite system of generating polynomial invariants, meaning that any other polynomial invariant can be expressed as a polynomial function of the generators. The existence of generating sets of rational invariants of linear actions was earlier investigated by Burnside, [**3**]. Extensions to discrete groups are also of interest, [**4**].

The present study of invariants of finite and discrete group actions was motivated by recent applications in the computational chemistry of polyatomic molecules, [**2**, **26**], where the proliferation of generating polynomial invariants is problematic for the design of efficient and accurate computational algorithms. Further potential applications include coding theory, [**29**], combinatorics, [**32**], algebraic topology, [**21**, **30**], number theory and geometry, [**20**], tilings, tessellations, and crystallography, [**4**], as well as discrete symmetry groups, [**12**], and conservation laws, [**28**], of differential equations.

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In this note, we explain how to construct minimal generating sets of invariants of finite and discrete group actions using techniques motivated by the method of equivariant moving frames for Lie group actions, [**6**, **23**, **25**]. The one cautionary note is that the resulting invariants are not, in general, algebraic or even smooth; they are piecewise analytic functions on the domain of definition of the group action, often, but not necessarily, continuous. There are several rebuttals to the inevitable objections by algebraists. First, the method is completely elementary, constructive, and algorithmic. Moreover, the generators are endowed with a rewrite rule, [**11**], that enables one to *immediately* and uniquely express any other invariant (polynomial, rational, analytic, smooth, etc.) in terms of our fundamental invariants. Finally, one should note the amazing successes of modern deep learning algorithms, [**7**, **31**], which are now based on compositions of elementary piecewise linear, non-analytic functions — namely those based on the so-called "ReLU function" $f(x) = \max\{0, x\}$ — after it was realized that constructions based on "nicer" analytic or smooth functions are more technically complicated and perform no better. Thus, one can envision similar efficient invariant computational algorithms based on the piecewise analytic fundamental invariants constructed here.

Minimality is key, especially in practical applications, because, even though Noether's Theorem, [**22**], guarantees a finite system of generating polynomial invariants, the minimal number of generators can be very large even for relatively small groups; see, for example, the tables in [**13**]. Rational invariants are more well behaved; indeed, Burnside, [**3**], proves that a linear action of a finite group on an m-dimensional space requires either m or $m+1$ generating rational invariants; in the former case they are algebraically independent, while in the latter they satisfy a single algebraic syzygy. Classifying in which case a given action belongs is known as Noether's Problem, on which significant progress has been made in recent years, [**30**, **33**]. However, I am unaware of any direct constructive algorithms for producing Burnside's rational generators. Another potential computational drawback is the apparently unknown nature of the required denominators. General methods for constructing rational invariants presented in Derksen and Kemper, [**5**], can presumably be applied, although they are rather inefficient in that they require an initial computation of the polynomial invariants and then analysis of the structure of the associated rational combinations based on the underlying syzygies, which must also be classified. A more promising approach might be an adaptation of the moving frame-based algorithm of Hubert and Kogan, [**11**], for directly computing rational invariants.

Let us now present the general construction. Let G be a finite group acting on an open subset[†] $M \subset \mathbb{R}^m$. (In the final examples, we show how the method can be straightforwardly adapted to infinite discrete groups.) The identity element/transformation is denoted by e. The action can be either linear or nonlinear; indeed, the method to be described can be applied, by working in local coordinates, to group actions on manifolds.

Given such an action, the *orbit* of a point $z = (z_1, \ldots, z_m) \in M$ is, by definition, the subset $\mathcal{O}_z = \{\, g \cdot z \mid g \in G \,\}$. The *isotropy subgroup* of a point z is the set of all group elements that fix it, denoted by $G_z = \{\, g \in G \mid g \cdot z = z \,\}$. The cardinality of its orbit is

[†] For specificity, we will work in the real category throughout, noting that the methods trivially extend to actions on complex spaces.

$\# \mathcal{O}_z = \# G / \# G_z$. Given $g \in G$, we set

$$S_g = \{ z \in M \mid g \cdot z = z \}$$

to be the closed subset consisting of all points fixed by g, whereby $g \in G_z$ if and only if $z \in S_g$. A group element $g \neq e$ is said to act *effectively* if $M_g = M \setminus S_g$ is an open dense subset of M; in other words g does not act trivially on any open subset of M. If G acts *effectively*, meaning that every $g \neq e$ acts effectively, then

$$M_0 = \bigcap_{e \neq g \in G} M_g = \{ z \in M \mid G_z = \{e\} \} \subset M \subset \mathbb{R}^m$$

is an open dense[†] subset of M, which we call the *regular subset*. Its complement,

$$S = M \setminus M_0 = \bigcup_{e \neq g \in G} S_g$$

will be called the *singular subset*. The group G acts *freely* on M_0, meaning that for all $z \in M_0$, the only group element that fixes z is the identity: $G_z = \{e\}$. This is equivalent to the requirement that the orbit through z has the same cardinality as the group: $\# \mathcal{O}_z = \# G$. We will primarily focus our attention on the regular subset M_0.

By a (global) *cross-section*, we mean a subset $K \subset M_0$ that intersects each orbit in a single point:

$$\# (K \cap \mathcal{O}_z) = 1 \quad \text{for all} \quad z \in M_0. \tag{1}$$

In the non-moving frame literature, a cross-section is usually referred to as a *fundamental domain* for the group action. As we will see, a cross-section is, in general, the union of an open subset and part of its boundary. Fixing the cross-section K, the point $k \in K \cap \mathcal{O}_z$ is known as the corresponding *canonical form* or *normal form* of the point $z \in M_0$. We note that we can write the regular subset as the disjoint union of the images of the cross-section under the individual group elements:

$$M_0 = \bigcup_{g \in G} g \cdot K. \tag{2}$$

As in [**6**], the choice of cross-section allows us to define an *equivariant moving frame* map $\rho\colon M_0 \to G$, so that, given $z \in M_0$, its image $g = \rho(z) \in G$ is the group element that maps z to its canonical form:

$$g \cdot z = \rho(z) \cdot z = k \in K \cap \mathcal{O}_z. \tag{3}$$

Freeness of the group action at $z \in M_0$ implies that $g = \rho(z)$ is uniquely determined by this requirement. Further, it is not hard to see that, as constructed, the moving frame map is right-equivariant:

$$\rho(g \cdot z) = \rho(z) \cdot g^{-1}, \qquad g \in G, \quad z \in M_0.$$

† This is trivial because G is finite. The result also holds for countable discrete groups by the Baire Category Theorem, [**14**].

(Its left-equivariant counterpart $\widehat{\rho}\colon M_0 \to G$ is obtained by composition with the group inversion: $\widehat{\rho}(z) = \rho(z)^{-1}$.)

An *invariant* is a function† $I\colon M_0 \to \mathbb{R}$ that is unaffected by the group action:

$$I(g \cdot z) = I(z) \quad \text{for all} \quad g \in G, \quad z \in M_0. \tag{4}$$

The moving frame map induces an *invariantization* process, [**5, 6**], which we denote by ι, that uniquely associates to any function $F\colon M_0 \to \mathbb{R}$ an invariant $I = \iota(F)\colon M_0 \to \mathbb{R}$, defined as the unique invariant function that agrees with F on the cross-section:

$$I(k) = F(k) \quad \text{for all} \quad k \in K. \tag{5}$$

Since I is invariant, it is constant on the orbits, and so, in view of (3), must be given by

$$I(z) = I(\rho(z) \cdot z) = F(\rho(z) \cdot z) \quad \text{for all} \quad z \in M_0.$$

In other words, to evaluate $I = \iota(F)$ at a point $z \in M_0$, one maps z to its canonical form $k = \rho(z) \cdot z \in K$ and then evaluates F at k. In particular, by construction, invariantization does not affect an invariant function: $\iota(I) = I$, which further implies that $\iota \circ \iota = \iota$. In other words, invariantization defines a projection from the space of functions to the space of invariants. Moreover, by construction, invariantization respects all algebraic operations; in other words, given functions $F_1(z), \ldots, F_l(z)$ and $\Phi(t_1, \ldots, t_l)$, we have

$$\iota\big[\,\Phi(F_1(z), \ldots, F_l(z))\,\big] = \Phi\big(\,\iota\big[\,F_1(z)\,\big], \ldots, \iota\big[\,F_l(z)\,\big]\,\big). \tag{6}$$

Note: Here and in the sequel we will interchangeably write $\iota[F(z)] = \iota(F)(z)$ as necessary.

In particular, invariantization of the individual coordinate functions $F_j(z) = z_j$ produces the $m = \dim M$ *fundamental invariants*

$$I_j(z) = \iota\big[\,F_j(z)\,\big] = \iota(z_j), \qquad\qquad j = 1, \ldots, m. \tag{7}$$

By the above remark, the value of $I_j(z)$ at a point $z \in M_0$ is given by the j^{th} coordinate of its canonical form $k = \rho(z) \cdot z \in K$. I claim that the fundamental invariants $I_1, \ldots, I_m$ form a generating set of functionally independent invariants for the group action, meaning that any other invariant can be uniquely expressed in terms of the fundamental invariants $I_1, \ldots, I_m$. To prove the claim, according to (6, 7),

$$\iota\big[\,F(z_1, \ldots, z_m)\,\big] = F\big(\iota(z_1), \ldots, \iota(z_m)\big) = F\big(I_1(z), \ldots, I_m(z)\big). \tag{8}$$

In other words, to invariantize a function $F(z) = F(z_1, \ldots, z_n)$, one merely replaces each z_j in its coordinate expression by the associated fundamental invariant $I_j(z) = \iota(z_j)$. In particular, if $F = J$ is an invariant, then, as noted above, $\iota(J) = J$, and hence formula (8) reduces to

$$J(z_1, \ldots, z_m) = J\big(I_1(z), \ldots, I_m(z)\big), \tag{9}$$

† More generally, an invariant can be defined on all of M, or on any G-invariant subset thereof.

which is known as the *Replacement Theorem*, [**6**]. It allows one to immediately express any invariant as a function of the fundamental invariants. In computer algebra applications, equation (9) is referred to as a *Rewrite Rule*, [**11**].

The one cautionary note is that the fundamental invariants so constructed are not, in general, algebraic or even smooth. They are obviously analytic on the interior of the fundamental domain K, and hence also on the interior of its images $g \cdot K$ for any $g \in G$. In favorable situations, they extend to their boundaries, which include the singular subset $S = M \setminus M_0$, to define continuous, piecewise analytic functions on all of M, although more generally they may have discontinuities across the boundaries.

This describes the basic construction. Serious applications will be deferred to future investigations. Let us finish up by looking at a few elementary examples.

Example 1. Consider the standard linear action of the symmetric group $G = S_2$ on $M = \mathbb{R}^2$ that permutes the coordinates $(x, y) = (z_1, z_2)$. Let $\pi_{12} \in S_2$ be the nonidentity element. The regular subset is the off-diagonal component: $M_0 = \{ x \neq y \}$, while $S = \{ x = y \}$. Suppose we choose the cross-section (fundamental domain)

$$K = \{ x < y \} \subset M_0.$$

Then, according to the above construction, the corresponding moving frame map is given by

$$\rho(x, y) = \begin{cases} e, & x < y, \\ \pi_{12}, & x > y, \end{cases} \qquad \text{for} \qquad (x, y) \in M_0.$$

The two fundamental invariants are obtained by invariantizing the two coordinate functions $z_1 = x,\ \ z_2 = y$, namely

$$\big(I_1(x, y), I_2(x, y) \big) = \rho(x, y) \cdot (x, y) = \begin{cases} (x, y), & x < y, \\ (y, x), & x > y, \end{cases}$$

so that

$$I_1(x, y) = \min\{ x, y \}, \qquad\qquad I_2(x, y) = \max\{ x, y \}.$$

If $J(x, y)$ is any invariant, then the Replacement Rule (9) is used to immediately rewrite J in terms of the fundamental invariants:

$$J(x, y) = J\big(I_1(x, y), I_2(x, y) \big). \tag{10}$$

For example, the standard elementary symmetric polynomial invariants are given by

$$\begin{aligned} J_1(x, y) &= x + y = I_1(x, y) + I_2(x, y) = \min\{ x, y \} + \max\{ x, y \}, \\ J_2(x, y) &= x\, y = I_1(x, y)\, I_2(x, y) = \min\{ x, y \} \times \max\{ x, y \}, \end{aligned}$$

as can easily be verified. Observe that in this case the fundamental invariants are continuous and piecewise analytic on all of $M = \mathbb{R}^2$.

Example 2. Generalizing the preceding example, consider the standard action of the symmetric group $G = S_m$ on $M = \mathbb{R}^m$ that permutes the coordinates of $z = (z_1, \ldots, z_m)$. The regular subset consists of all points with all unequal entries:

$$M_0 = \{\, z_i \neq z_j \;\big|\; 1 \le i < j \le m \,\}.$$

An evident cross-section is

$$K = \{\, z \mid \; z_1 < z_2 < \;\cdots\; < z_m \,\}.$$

Thus, given $z \in M_0$, the permutation $\pi = \rho(z) \in S_m$ defined by the moving frame map $\rho: M_0 \to S_m$ is the one that sorts the entries of z into ascending order. The value of the fundamental invariant $I_j = \iota(z_j)$ at a point $z = (z_1, \ldots, z_m)$ is the j^{th} smallest entry of z. Thus, $I_1(z) = \min\{z_1, \ldots, z_m\}$, while $I_2(z_1, \ldots, z_m)$ is the second smallest entry, and so on, up to $I_m(z) = \max\{z_1, \ldots, z_m\}$. The Replacement Rule is as usual, (9), allowing one to immediately rewrite any invariant, including the elementary symmetric polynomials, in terms of the fundamental invariants:

$$\begin{aligned}
J_1(z) &= z_1 + z_2 + \;\cdots\; + z_m = I_1(z) + I_2(z) + \;\cdots\; + I_m(z),\\
J_2(z) &= \prod_{i<j} z_i z_j = \prod_{i<j} I_i(z)\, I_j(z),\\
&\;\;\vdots\\
J_m(z) &= z_1\, z_2 \;\cdots\; z_m = I_1(z)\, I_2(z) \;\cdots\; I_m(z).
\end{aligned}$$

Example 3. Consider the linear action of $G = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \simeq S_2$ on $M = \mathbb{R}^2$ with the non-identity element $-e \in \mathbb{Z}_2$ acting by reflection: $(x, y) \mapsto (-x, -y)$. The regular subset is $M_0 = M \setminus \{\mathbf{0}\}$. Let us choose the cross-section

$$K = \{x > 0\} \cup \{\, (0, y) \mid \; y > 0 \,\}.$$

The moving frame map is then given by

$$\rho(x, y) = \begin{cases} e, & (x, y) \in K,\\ -e, & (x, y) \in M_0 \setminus K. \end{cases}$$

The resulting fundamental invariants are

$$I_1(x, y) = \iota(x) = |\, x \,|, \qquad I_2(x, y) = \iota(y) = \begin{cases} y, & x > 0,\\ -y, & x < 0,\\ |\, y \,|, & x = 0. \end{cases}$$

In this case the invariant I_2 is discontinuous across the y axis.

As in (10), we can immediately rewrite any invariant $J(x, y)$ in terms of the fundamental invariants. For example, the three basic polynomial invariants are

$$J_1(x, y) = x^2 = I_1(x, y)^2, \quad J_2(x, y) = x\, y = I_1(x, y)\, I_2(x, y), \quad J_3(x, y) = y^2 = I_2(x, y)^2.$$

Example 4. The following example arises in the chemistry of tetratomic molecules, [**2**, **26**], and also in distance geometry, [**1**, **16**]. Generalizations to more complicated molecules/point configurations are apparent and worth detailed investigation from both a theoretical and computational standpoint.

Let $1 \le d \in \mathbb{Z}$, with $d = 3$ the physically interesting case. Given a point $p \in \mathbb{R}^d$, let $\| p \|$ denote its Euclidean norm[†]. Given 4 points $p_1, p_2, p_3, p_4 \in \mathbb{R}^d$, let $r_{ij} = r_{ji} = \| p_i - p_j \|$ for $1 \le i, j \le 4$ be the $6 = \binom{4}{2}$ interpoint distances. We consider

$$r = (r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34})$$

as coordinates on a certain conical subset of the positive orthant, $M \subset \mathbb{R}^6_+$, called the *Euclidean distance cone* in [**8**], which, by Schoenberg's Theorem, [**27**, **34**], can be explicitly prescribed by the positive semidefiniteness of the associated reduced Euclidean distance matrix. The permutation group $G = S_4$ acts on the points p_1, p_2, p_3, p_4, and hence on their interpoint distances, inducing an action on M. We use the moving frame construction to deduce the fundamental invariants $I_1(r), \ldots, I_6(r)$.

Let us concentrate on the subset $\widehat{M}_0 \subset M_0$ where the 6 interpoint distances are all distinct. (The regular subset $M_0 \subset M$ includes that part of $\partial \widehat{M}_0$ determined by configurations with one or more repeated distances but no nontrivial permutational symmetries.) To construct a cross-section $K \subset \widehat{M}_0$, we need to fix a *canonical form* $p_{j^\star} = p_{\pi(j)}$, $j = 1, \ldots, 4$, for the point configuration through application of a distinguished permutation $\pi \in S_4$. Let us denote the interpoint distances of the canonical configuration by

$$k = (k_{12}, k_{13}, k_{14}, k_{23}, k_{24}, k_{34}) \qquad \text{with} \qquad k_{ij} = \| p_{i^\star} - p_{j^\star} \| = r_{i^\star j^\star} = r_{\pi(i)\,\pi(j)}.$$

One evident candidate is to first select $p_{1^\star}, p_{2^\star}$ so that k_{12} is the smallest distance; this choice also serves to fix $p_{3^\star}, p_{4^\star}$ up to interchanging the two members of each pair. We can further specify $p_{1^\star}$ and $p_{3^\star}$ so that k_{13} is the smallest among the distances $k_{13}, k_{14}, k_{23}, k_{24}$ between the two pairs. This choice serves to fix all 4 points $p_{1^\star}, p_{2^\star}, p_{3^\star}, p_{4^\star}$, and hence determines the canonical form of the original point configuration. The moving frame $\rho\colon \widehat{M}_0 \to S_4$ maps a set of distances $r \in \widehat{M}_0$ to the permutation $\pi = \rho(r) \in S_4$ that places them in the above canonical form.

The resulting fundamental invariants are readily constructed as follows. First,

$$I_{12}(r) = \min\{ r_{ij} \} = k_{12} = r_{1^\star 2^\star}$$

is the smallest interpoint distance. Next,

$$I_{13}(r) = k_{13} = \min \{ r_{1^\star k}, r_{2^\star k} \mid 1^\star \ne k \ne 2^\star \} = r_{1^\star 3^\star},$$

where we interchange the indices $1^\star \longleftrightarrow 2^\star$ prescribing k_{12} if necessary. This fixes the ordering of the 4 points, and hence prescribes the moving frame permutation $\pi = \rho(r)$ to

[†] Actually, the construction works for any norm.

maps the point configuration to its canonical form, $\rho(r) \cdot r = k$, so that $(1^\star, 2^\star, 3^\star, 4^\star) = \pi(1,2,3,4)$. In this manner, we have constructed all 6 fundamental invariants:

$$\begin{aligned} I_{12} &= k_{12} = r_{1^\star 2^\star}, & I_{13} &= k_{13} = r_{1^\star 3^\star}, & I_{14} &= k_{14} = r_{1^\star 4^\star}, \\ I_{23} &= k_{23} = r_{2^\star 3^\star}, & I_{24} &= k_{24} = r_{2^\star 4^\star}, & I_{34} &= k_{34} = r_{3^\star 4^\star}. \end{aligned}$$

The Replacement Rule works exactly as before, so that if $J(r)$ is any permutation-invariant function of the interpoint distances, then

$$J(r_{12}, \ldots, r_{34}) = J\big(I_{12}(r),\ \ldots\ , I_{34}(r)\big)$$

gives its explicit formula in terms of the 6 fundamental invariants.

In contrast, the computations in [**26**] for a tetratomic molecule require 6 algebraically independent polynomial invariants (a system of parameters) along with an additional 5 secondary invariants, dictated by the Cohen–Macaulay structure of the ring of polynomial invariants, [**9**]. Similar constructions can evidently be applied to more complicated polyatomic molecules, including those composed of different atoms that serve to reduce the allowable permutations. See also [**17**] for more general types of molecular symmetry groups.

Example 5. Finally, to see how the methods extend to infinite discrete group actions, let us investigate two simple examples. Let $G = \mathbb{Z}^2$ act on $M = \mathbb{R}^2$ by translation:

$$(x,y) \longmapsto (x+i, y+j) \qquad \text{for} \qquad i,j \in \mathbb{Z}.$$

In this case, every point is regular, so $M_0 = M$. A cross-section (fundamental domain) is provided by the unit square $K = \{0 \leq x, y < 1\}$, including the origin and 2 of its 4 sides. The fundamental invariants are

$$I_1(x,y) = \iota(x) = x \bmod 1, \qquad I_2(x,y) = \iota(y) = y \bmod 1,$$

which are discontinuous. As before, the Replacement Rule enables one to rewrite any $\mathbb{Z}^2$ invariant function in terms of the fundamental invariants; for example

$$J(x,y) = \sin\big(2\pi x + 6\pi y\big) = \sin\big(\, 2\pi I_1(x,y) + 6\pi I_2(x,y)\,\big).$$

Let us next supplement the preceding discrete translational group by the group $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ consisting of rotations around the origin through integer multiples of 90°. Thus, the underlying group is the semidirect product $G = \mathbb{Z}_4 \ltimes \mathbb{Z}^2$ containing the transformations that map $(x,y) \in M$ to the following points:

$$(x+i, y+j), \quad (y+i, -x+j), \quad (-x+i, -y+j), \quad (-y+i, x+j), \quad i,j \in \mathbb{Z}.$$

The singular subset S consists of the half integer lattice points $\big(\frac{1}{2}i, \frac{1}{2}j\big)$, $i,j \in \mathbb{Z}$. A convenient cross-section is provided by the square of size $\frac{1}{2}$ along with one of its 4 sides:

$$K = K_0 \cup K_1, \qquad \text{where} \qquad K_0 = \{\, 0 < x, y < \tfrac{1}{2}\,\}, \qquad K_1 = \big\{\, (x,0) \;\big|\; 0 < x < \tfrac{1}{2}\,\big\}.$$

Let $\widehat{M}_0 \subset M_0$ be the open subset that maps to K_0, namely the set of points (x,y) such that neither coordinate is an integer nor half integer. (We leave the treatment of the other regular points to the reader.) Given a point $z = (x, y) \in \widehat{M}_0$, its canonical form is the point $k = (x^\star, y^\star) \in K$ given by

$$k = (x^\star, y^\star) = \begin{cases} (\,\widehat{x}, \widehat{y}\,), & 0 < \widehat{x}, \widehat{y} < \frac{1}{2}, \\ (\,1 - \widehat{y}, \widehat{x}\,), & 0 < \widehat{x} < \frac{1}{2} < \widehat{y} < 1, \\ (\,\widehat{y}, 1 - \widehat{x}\,), & 0 < \widehat{y} < \frac{1}{2} < \widehat{x} < 1, \\ (\,1 - \widehat{x}, 1 - \widehat{y}\,), & \frac{1}{2} < \widehat{x}, \widehat{y} < 1, \end{cases} \qquad \text{where} \qquad \begin{aligned} \widehat{x} &= x \bmod 1, \\ \widehat{y} &= y \bmod 1. \end{aligned} \tag{11}$$

The values of the fundamental invariants are thus

$$I_1(x, y) = \iota(x) = x^\star, \qquad I_2(x, y) = \iota(y) = y^\star, \tag{12}$$

where $x^\star, y^\star$ are the canonical coordinates of (x, y) given in formula (11). As before, the Replacement Rule (10) allows one to rewrite any G-invariant function in terms of our two fundamental invariants (12).

Many additional examples can be straightforwardly analyzed using the preceding moving frame constructions. A general foundational question, then, is the existence and algorithmic construction of a suitable cross-section (fundamental domain) for a given finite or discrete group action. In the latter case, this requires imposing certain conditions, although a fully general existence theorem is apparently not known. For example, in the case of the discrete group action on $\mathbb{R}$ generated by the translation $x \mapsto x + 1$ and the scaling $x \mapsto x/2$, every orbit is dense[†], and so there is no fundamental domain of the type postulated above. Indeed, the only continuous invariant is a constant function, although the preceding moving frame construction could still be applied to produce non-constant totally discontinuous invariants. One sufficient condition for existence of "nice" fundamental domains that has been proposed is that the action be proper, or more fully, *properly discontinuous*, meaning that, for every compact subset $K \subset M$, the set $\{\, g \in G \mid g \cdot K \cap K \neq \varnothing \,\}$ is finite. However, there are examples of non-proper discrete actions which nevertheless possess suitable fundamental domains; see the `Mathoverflow` discussion thread [**10**] for details. See also [**20**] for the construction and use of fundamental domains for arithmetic subgroups of semisimple Lie groups, of importance in geometry and number theory, including modular forms.

Remark: It would be instructive to adapt the inductive/recursive moving frame constructions for Lie groups, [**15, 24**], to the present context, enabling one to relate invariants of subgroups to those of larger groups.

One final comment: I have shown how moving frame-based constructions can be adapted to discrete group actions on continuous spaces. In [**18, 19**], moving frames are

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[†] Indeed, two points $x, y \in \mathbb{R}$ belong to the same orbit if and only if their binary expansions are identical after a finite number of digits, modulo a shift, or, equivalently, modulo multiplication by an integer power of 2. In other words, if $x = \sum_i x_i 2^{-i}$, $y = \sum_i y_i 2^{-i}$, where $x_i, y_i \in \{0, 1\}$ are their binary digits, then we require $x_i = y_{i+n}$ for some fixed n and all sufficiently large i, with the usual technical proviso that we identify expansions that are eventually all 0's or all 1's.

applied to actions of continuous groups on discrete spaces. It would clearly be of interest to investigate applications of moving frame methods to finite and discrete group actions on discrete spaces.

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