Moving Frame Derivation of the Fundamental Equi-Affine Differential Invariants for Level Set Functions

Peter J. Olver[†]
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
olver@umn.edu
http://www.math.umn.edu/∼olver

The purpose of this note is to explain how to use the equivariant method of moving frames, [**2**], to straightforwardly and algorithmically derive the equi-affine differential invariants

$$H = u_{xx} u_{yy} - u_{xy}^2, \qquad J = u_y^2 \, u_{xx} - 2 \, u_x u_y u_{xy} + u_x^2 \, u_{yy}, \tag{1}$$

which were called "the two basic independent affine invariant descriptors" in [**6**]. In addition, we show that all higher order differential invariants can be generated from these by the process of invariant differentiation.

Remark: An alternative approach, advocated in [**4**], is to use the infinitesimal invariance criteria, which requires solving a linear system of first order partial differential equations based on the prolonged infinitesimal generators of the transformation group. In contrast, the moving frame method is completely algebraic, typically much simpler, and, moreover provides significantly more information, particularly the recurrence formulae to be presented below that completely prescribe the structure of the underlying algebra of differential invariants.

The starting point is the standard action of the equi-affine group SA(2) on the plane:

$$z = \alpha \, x + \beta \, y + a, \qquad w = \gamma \, x + \delta \, y + b, \tag{2}$$

where the coefficients (group parameters) are subject to the unimodularity constraint

$$\alpha \, \delta - \beta \, \gamma = 1. \tag{3}$$

Thus SA(2) has 5 independent parameters, and hence forms a five-dimensional Lie group.

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Remark: In [**6**] and much of the classical literature dating back to Blaschke, [**1**], the action (2) is referred to as "affine geometry" but, in view of the unimodularity constraint (3), the correct term is "equi-affine", meaning area-preserving affine transformations.

There is an induced group action on plane curves $C \subset \mathbb{R}^2$, but in the image processing applications considered in [**6**], the curves are viewed as level sets of a function $u = f(x, y)$. We are interested in the induced action of $\mathrm{SA}(2)$ on the derivatives of the level set function, known as the *prolonged action*. The differential invariants are, by definition, particular combinations of derivatives that are not changed by the group action. The *order* of a differential invariant is, by definition, the maximal order of derivative it depends upon. Of course, the function u itself is a differential invariant, of order 0 since it involves no derivatives.

The prolonged action on the derivatives of u is obtained by applying the operators of implicit differentiation that relate derivatives of the function u with respect to the new variable z, w to those with respect to x, y:

$$D_z = \delta\, D_x - \gamma\, D_y, \qquad\qquad D_w = -\,\beta\, D_x + \alpha\, D_y, \tag{4}$$

where D_x, D_y are the usual[†] differentiation operators. Thus the prolonged transformation rules for the first and second order derivatives are simply

$$\begin{aligned} u_z &= D_z u = \delta\, u_x - \gamma\, u_y, \\ u_w &= D_w u = -\,\beta\, u_x + \alpha\, u_y, \\ u_{zz} &= D_z^2 u = \delta^2\, u_{xx} - 2\,\gamma\,\delta\, u_{xy} + \gamma^2\, u_{yy}, \\ u_{zw} &= D_z D_w u = -\,\beta\,\delta\, u_{xx} + (\alpha\,\delta + \beta\,\gamma)\, u_{xy} - \alpha\,\gamma\, u_{yy}, \\ u_{ww} &= D_w^2 u = \beta^2\, u_{xx} - 2\,\alpha\,\beta\, u_{xy} + \alpha^2\, u_{yy}. \end{aligned} \tag{5}$$

To compute the equivariant moving frame, we must normalize the $5 = \dim \mathrm{SA}(2)$ independent group parameters by setting 5 of the transformed variables in (2), (5) equal to conveniently chosen constants — this corresponds to the choice of a *cross-section* to the prolonged group orbits. The only constraint is that the resulting system of algebraic equations be (locally) uniquely and smoothly solvable for a bona fide group element, which, in this case, means that the unimodularity constraint (3) holds. In practice, the most convenient normalization constants are usually 0 or, occasionally 1. In this case, we set

$$z = w = u_z = u_{zw} = 0, \qquad\qquad u_w = 1. \tag{6}$$

(The last one cannot be $u_w = 0$ as this would lead to expressions for $\alpha, \beta, \gamma, \delta$ whose determinant is 0, not 1.) Using the formulas for z, w in (2) and solving the first two normalization equations $z = w = 0$ for the translational parameters yields

$$a = -\,\alpha\, x - \beta\, y, \qquad\qquad b = -\,\gamma\, x - \delta\, y. \tag{7}$$

[†] Technically they are total derivatives, [**2**, **4**], which is the reason for the capital letters.

However, since these parameters do not occur in the prolonged transformation formulas, equations (7) will not be of significance for our purposes. Next, in view of (5), we can solve the normalization equations $u_z = 0$, $u_w = 1$, for

$$\beta = \frac{\alpha u_y - 1}{u_x}, \qquad \delta = \frac{\gamma u_y}{u_x}. \tag{8}$$

Substituting these expressions into the unimodularity constraint (3) yields

$$\alpha\,\delta - \beta\,\gamma = \frac{\gamma}{u_x} = 1, \qquad \text{hence} \qquad \gamma = u_x, \qquad \text{and so} \qquad \delta = u_y. \tag{9}$$

Finally, substituting (8), (9) into the last normalization equation $u_{zw} = 0$, as determined by (5), allows us to solve for

$$\alpha = \frac{u_y u_{xx} - u_x u_{xy}}{J}, \tag{10}$$

where the denominator J is the invariant (1). Replacing α, γ in (8) by their formulas (10), (9) yields

$$\alpha = \frac{u_y u_{xx} - u_x u_{xy}}{J}, \qquad \beta = \frac{u_y u_{xy} - u_x u_{yy}}{J}, \qquad \gamma = u_x, \qquad \delta = u_y. \tag{11}$$

The final formulas (11), combined with (7) after $\alpha, \beta, \gamma, \delta$ have been replaced by their expressions (11), serve to define the right equivariant moving frame, [**2**], for the prolonged action of SA(2) corresponding to our choice of normalization equations.

The next phase of the moving frame calculus is to determine the (differential) invariants through the process of *invariantization*. In general, given any object, e.g., a function of u and its derivatives, one produces the corresponding invariantized object by first transforming the object according to the group transformations and then replacing all occurrences of the group parameters in the transformed object by the previously derived moving frame formulae. The result is guaranteed to be invariant. Invariantization is denoted by ι. For example, to compute $\iota(u_x)$ — the invariantization of the function u_x — we first determine its transformation rule, namely the expression u_z in (5), and then substitute the formulas for γ, δ given in (11). Of course, in this case $\iota(u_x) = 0$ because we determined the moving frame formulas (11) by solving, among others, the equation $u_z = 0$. Indeed, invariantizing the variables used to normalize the group parameters merely recovers the constant values we prescribed during our normalization procedure. Thus,

$$\iota(x) = \iota(y) = \iota(u_x) = \iota(u_{xy}) = 0, \qquad \iota(u_y) = 1. \tag{12}$$

These trivial constant invariants are sometimes referred to as the *phantom invariants*. On the other hand, invariantizing any variable that was not used in the normalization equations produces a nontrivial differential invariant, and, moreover, we are guaranteed that these form a complete system of functionally independent differential invariants. In particular, we find

$$\iota(u) = u, \qquad \iota(u_{xx}) = J, \qquad \iota(u_{yy}) = \frac{H}{J}\,. \tag{13}$$

The first of these comes from the fact that u is already invariant. Indeed, invariantization does not affect a differential invariant, and, in fact, defines a projection from the algebra of functions of u and its derivatives to the algebra of differential invariants that, moreover, *respects all algebraic operations*. The second and third expressions are found, after some algebraic simplification, by substituting the moving frame formulae (11) directly into the formulae (5) for the transformed second derivatives u_{zz}, u_{ww}. The conclusion is that u, J, and H/J — and hence u, H, and J — form a complete system of second order differential invariants for the equi-affine action on level set functions, meaning that any other second order differential invariant can be written as a function thereof. This justifies the basic observation in [**6**]. In fact, by the Replacement Rule, which merely expresses the fact that invariantization does not affect differential invariants and respects all algebraic operations, if

$$I(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$$

is any second order differential invariant, then

$$\begin{aligned} I = \iota(I) &= I(\iota(x), \iota(y), \iota(u), \iota(u_x), \iota(u_y), \iota(u_{xx}), \iota(u_{xy}), \iota(u_{yy})) \\ &= I(0, 0, u, 0, 1, J, 0, H/J) \end{aligned} \tag{14}$$

immediately expresses I in terms of the fundamental differential invariants. (In computer algebra, this is known as a “rewrite rule”, [**3**].)

There is no reason to stop at order two, and so applying the same invariantization procedure produces a complete system of higher order differential invariants. For example, to invariantize the third order derivative u_{xxx} we first compute its transformation using the implicit differentiation operators (4):

$$u_{zzz} = D_z^3 u = D_z(u_{zz}) = \delta^3 u_{xxx} - 3\gamma\delta^2 u_{xxy} + 3\gamma^2 \delta u_{xyy} - \gamma^3 u_{yy}.$$

Replacing γ, δ by their moving frame formulas (11) produces the third order differential invariant

$$K = \iota(u_{xxx}) = u_y^3 u_{xxx} - 3 u_x u_y^2 u_{xxy} + 3 u_x^2 u_y u_{xyy} - u_x^3 u_{yyy}. \tag{15}$$

This, along with $\iota(u_{xxy}), \iota(u_{xyy}), \iota(u_{yyy})$, and the lower order differential invariants u, H, J form a complete system of third order differential invariants, as guaranteed by the third order version of the Replacement Rule (14).

However, there is a more powerful way to produce higher order differential invariants, namely, by applying operators of invariant differentiation. These are found by invariantizing the basic differential operators D_x, D_y. As with functions, this is accomplished by first transforming them by the group action, which are merely the implicit differentiation operators (4). Substituting the moving frame formulae (11) for the group parameters in (4) produces the two fundamental invariant differential operators:

$$\mathcal{D}_1 = \iota(D_x) = u_y D_x - u_x D_y, \qquad \mathcal{D}_2 = \iota(D_y) = \frac{1}{J}\widetilde{\mathcal{D}}_2, \tag{16}$$

where

$$\widetilde{\mathcal{D}}_2 = (u_x u_{yy} - u_y u_{xy}) D_x + (u_y u_{xx} - u_x u_{xy}) D_y. \tag{17}$$

This implies that if I is any differential invariant, so are $\mathcal{D}_1 I$ and $\mathcal{D}_2 I$. Observe that $\mathcal{D}_1 I$ is just minus the total Jacobian determinant of the level set function u and the invariant I, which we write as

$$\mathcal{D}_1 I = -\frac{D(u, I)}{D(x, y)}. \tag{18}$$

Further, since J is itself a differential invariant, $\widetilde{\mathcal{D}}_2 I$ is also a differential invariant, and so the operator $\widetilde{\mathcal{D}}_2$ is also an invariant differential operator, being the following total Jacobian determinant:

$$\widetilde{\mathcal{D}}_2 I = -\,u_x^2\,\frac{D(u_y/u_x, I)}{D(x, y)} = u_y^2\,\frac{D(u_x/u_y, I)}{D(x, y)}, \tag{19}$$

where u_y/u_x can be identified with the "projectivization" of the gradient or normal vector ∇u to the level set.

Consequently, starting with H and J, we can immediately produce an infinite hierarchy of higher and higher order differential invariants:

$$\mathcal{D}_1 H,\ \mathcal{D}_2 H,\ \mathcal{D}_1 J,\ \mathcal{D}_2 J,\ \mathcal{D}_1^2 H,\ \mathcal{D}_1\mathcal{D}_2 H,\ \mathcal{D}_2\mathcal{D}_1 H,\ \mathcal{D}_2^2 H,\ \mathcal{D}_1^2 J,\ \ldots\ .$$

The first 4 have order 3, the next 8 have order 4 and so on. Keep in mind that the invariant differential operators (16) do not in general commute. Here, it is not hard to show using the moving frame calculus, or directly form their explicit expressions (16), (17), that their commutator is given by

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = \mathcal{D}_1 \cdot \mathcal{D}_2 - \mathcal{D}_2 \cdot \mathcal{D}_1 = Y\,\mathcal{D}_1, \tag{20}$$

where the coefficient

$$Y = -\,\iota(u_{xxy})/J \tag{21}$$

is itself a differential invariant, known as a *commutator invariant*. Alternatively, one can replace $\mathcal{D}_2$ by $\widetilde{\mathcal{D}}_2$ to produce the higher order differential invariants. Note finally that

$$\mathcal{D}_1 u = 0, \qquad \mathcal{D}_2 u = 1, \qquad \widetilde{\mathcal{D}}_2 u = J. \tag{22}$$

While invariantization commutes with all algebraic operations, it does not commute with differentiation, and so, in general,

$$\iota(D_x F) \neq \iota(D_x)\,\iota(F) = \mathcal{D}_1 \iota(F).$$

However, the explicit relationships connecting the higher order differential invariants obtained by invariantization and those obtained by invariant differentiation are provided by the fundamental moving frame *recurrence formulae*. Remarkably, these can be derived using only *linear* algebra and differentiation without even knowing the explicit formulas for the differential invariants, or the moving frame, or even the group transformations! We only need to know which coordinates are being normalized to which constants, along with the well known formulas for the (prolonged) infinitesimal generators of the group action. We refer the reader to [**2**, **5**] for complete details on how the recurrence formulae are systematically found. In the present case, we merely state the results at third order:

$$\begin{aligned} \iota(u_{xxx}) &= \mathcal{D}_1 J, & \iota(u_{xxy}) &= \mathcal{D}_2 J - 2\,H = \frac{\widetilde{\mathcal{D}}_2 J}{J} - 2\,H, \\ \iota(u_{xyy}) &= \mathcal{D}_1 \left(\frac{H}{J} \right), & \iota(u_{yyy}) &= \mathcal{D}_2 \left(\frac{H}{J} \right) + 2\frac{H^2}{J^2} = \frac{J\,\widetilde{\mathcal{D}}_2 H - H\,\widetilde{\mathcal{D}}_2 J + 2\,H^2 J}{J^3}. \end{aligned} \tag{23}$$

They can, of course, be rather tediously checked by direct computation; however, the point is that one can find them directly, completely bypassing the explicit expressions for the invariantized third order derivatives and the moving frame.

Thus, the third order recurrence formulae (23) imply that we can generate all the third order differential invariants by invariantly differentiating the second order differential invariants H, J. The higher order recurrence formulae can be similarly constructed. However, in this case, since the moving frame has order 2, by a general theorem, [**2**], we immediately conclude:

Theorem 1. *A complete system of differential invariants for the action of the equi-affine group* $\mathrm{SA}(2)$ *on level set functions is provided by the zero-th order differential invariant* u*, the second order differential invariants* H, J *and the latter's invariant derivatives obtained by repeatedly applying the invariant differential operators* $\mathcal{D}_1, \mathcal{D}_2$*.*

However, we can do better. Note that, according to (20), (21), and (23),

$$\mathcal{D}_1 \mathcal{D}_2 J - \mathcal{D}_2 \mathcal{D}_1 J = Y\,\mathcal{D}_1 J = \big(2\,H - \mathcal{D}_2 J \big) \frac{\mathcal{D}_1 J}{J},$$

hence

$$H = \frac{J\big(\mathcal{D}_1 \mathcal{D}_2 J - \mathcal{D}_2 \mathcal{D}_1 J \big)}{2\,\mathcal{D}_1 J} + \tfrac{1}{2}\,\mathcal{D}_2 J = \frac{\mathcal{D}_1 \widetilde{\mathcal{D}}_2 J - \widetilde{\mathcal{D}}_2 \mathcal{D}_1 J}{2\,\mathcal{D}_1 J}\,, \tag{24}$$

the second expression following from (16). Thus we can obtain H by invariantly differentiating J. Moreover, since $\widetilde{\mathcal{D}}_2 u = J$, we deduce the following result:

Theorem 2. *A complete system of differential invariants for the action of the equi-affine group* $\mathrm{SA}(2)$ *on level set functions is provided by* u *and its invariant derivatives obtained by repeatedly applying the invariant differential operators* $\mathcal{D}_1, \widetilde{\mathcal{D}}_2$*.*

One final remark: the above calculations implicitly assumed that $J \neq 0$. However, since J is a differential invariant, it is not hard to see that one can clear denominators in the recurrence formulae to produce a complete system of polynomial differential invariants, generated by u and H by successively applying $\mathcal{D}_1, \widetilde{\mathcal{D}}_2$. On the other hand, to obtain H by invariantly differentiating u via equation (1) requires that $\mathcal{D}_1 J \not\equiv 0$, meaning that u and J are functionally independent.

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