Classification of Integrable One-Component Systems on Associative Algebras

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Abstract

This paper is devoted to the complete classification of integrable one-component evolution equations whose field variable takes its values in an associative algebra. The proof that the list of noncommutative integrable homogeneous evolution equations is complete relies on the symbolic method. Each equation in the list has infinitely many local symmetries and these can be generated by its recursion (recursive) operator or master symmetry.

1 Introduction

This paper can be viewed as a continuation of the work of [22], where the authors began the classification of integrable evolution equations in which the field variables taking their values in an associative algebra. The analysis provided a list of one-component (and certain two-component) evolution equations which admit at least one higher order local matrix symmetry. These equations can be regarded as quantized analogs of classical integrable systems; see [12] for a discussion of the noncommutative KdV equation from this point of view. Some analysis of the solution by inverse scattering and explicit soliton solutions for certain examples can be found in [14, 18].

In this paper, we rigorously prove that the list of integrable one-component evolution equations found in [22] is complete, and, moreover, each equation in their list has infinitely many symmetries. Furthermore, we give their recursion

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(recursive) operators or master symmetries to produce such symmetries. The proofs are based on methods introduced in [26], where it was shown that the integrability of a homogeneous evolution equation

$$u_t = u_n + f(u, \cdots, u_{n-1}), \qquad \text{where} \qquad u_n = D_x^n u, \tag{1.1}$$

with f a polynomial starting with terms that are at least quadratic is determined by the existence of one nontrivial higher order symmetry. This led to the proof of a long-standing conjecture that "in all known cases the existence of one generalized symmetry implies the existence of infinitely many", [10], under fairly relaxed conditions. In particular, for homogeneous scalar evolution equations, to prove the integrability of an equation of order 2 we need a symmetry of order 3, for an equation of order 3 we need a symmetry of order 5, for an equation of order 5 we need a symmetry of order 7, and for an equation of order 7 we need a symmetry of order 13. These a priori bounds enable us to symbolically compute the complete list of integrable homogeneous equations.

Here we generalize the results to noncommutative polynomial evolution equations of the form (1.1) in which the field variable u takes its values in an associative, non-commutative algebra $\mathbb{A}$. Examples include matrix and operator algebras, [18, 17], Clifford algebras (including the quaternions), [23], and the group algebras appearing in the representation theory of finite-dimensional groups and group algebras, [5]; see [22] for details. We use the notation

$$L_u(v) = uv, \qquad\qquad R_u(v) = vu,$$

for the operators of left and right multiplication in $\mathbb{A}$. (L_u is *not* a Lie derivative.) The commutator and anti-commutator are denoted by

$$C_u = L_u - R_u, \qquad\qquad A_u = L_u + R_u.$$

2 Complete Classification Results

In almost all interesting integrable evolution equations, the right-hand side of equation is a homogeneous differential polynomial under a suitable weighting scheme. The differential equation (1.1) is said to be λ*–homogeneous* of *weight* μ if it admits the one-parameter group of scaling symmetries

$$(x, t, u) \longmapsto (a^{-1}x, a^{-\mu}t, a^{\lambda}u), \qquad a \in \mathbb{R}^+.$$

For example, the Korteweg–deVries equation $u_t = u_{xxx} + uu_x$ is 2–homogeneous of weight 3.

A second evolution equation is called a (generalized) symmetry of (1.1) if the corresponding vector fields commute. An equation is called *integrable* if it has at infinitely many higher order symmetries. In [22], the following noncommutative λ–homogeneous equations were shown to possess one higher order symmetry. The main goal of this paper is to rigorously prove that for $\lambda > 0$ all the equations in this list are integrable, and, moreover, the list of integrable equations is complete in the sense that every other $\lambda > 0$ integrable noncommutative equation is contained in the symmetry hierarchy of one of these equations.

2.1 Second order equations

The only possible integrable case is for $\lambda = 1$.

Burgers' Equation. There are two versions:

$$u_t = u_2 + uu_1, \qquad \text{or} \qquad u_t = u_2 + u_1 u.$$

They both have the same master symmetry, [20]:

$$\mathfrak{M} = xu_t + \tfrac{1}{2}u^2$$

Burgers' equation has symmetries of every order, while the third order equations only have odd order symmetries. As far as we can tell, the noncommutative Burgers' equation does not have a recursion operator.

Remark 2.1. The potential Burgers' equation $u_t = u_2 + u_1^2$ does not appear because it has $\lambda = 0$ and we are restricting our attention to positive λ.

2.2 Third order equations

For $\lambda = 2$, we have only one case.

Korteweg–deVries equation

$$u_t = u_3 + uu_1 + u_1 u,$$

with hereditary recursion operator [8], [12],

$$\begin{aligned} \mathfrak{R} &= D_x^2 + \tfrac{2}{3}A_u + \tfrac{1}{3}A_{u_1}D_x^{-1} + \tfrac{1}{9}C_u D_x^{-1} C_u D_x^{-1} \\ &= D_x\left(D_x + \tfrac{1}{3}A_u D_x^{-1} + \tfrac{1}{3}D_x^{-1}A_u + \tfrac{1}{9}D_x^{-1}C_u D_x^{-1} C_u D_x^{-1}\right). \end{aligned} \tag{2.1}$$

For $\lambda = 1$, there are five cases. The first two are the third order symmetries of the two Burgers' equations. The other three cases are:

Potential KdV

$$u_t = u_3 + u_1^2.$$

The hereditary recursion operator is

$$\begin{aligned} \mathfrak{R} &= D_x^2 + \tfrac{2}{3}A_{u_1} + \tfrac{1}{9}C_u D_x^{-1} C_{u_1} - \tfrac{1}{9}D_x^{-1}(C_u C_{u_1} + 3A_{u_2}) \\ &= \left(D_x + \tfrac{1}{3}A_{u_1}D_x^{-1} + \tfrac{1}{3}D_x^{-1}A_{u_1} + \tfrac{1}{9}D_x^{-1}C_{u_1}D_x^{-1}C_{u_1}D_x^{-1}\right)D_x. \end{aligned} \tag{2.2}$$

Modified KdV, case 1

$$u_t = u_3 + u^2 u_1 + u_1 u^2,$$

with hereditary recursion operator

$$\begin{aligned} \mathfrak{R} &= D_x^2 + \tfrac{2}{3}A_{u^2} + \tfrac{1}{3}A_{u_1}D_x^{-1}A_u + \tfrac{1}{9}C_u D_x^{-1}(C_{u^2}D_x^{-1}A_u - 3C_{u_1}) \\ &= \left(D_x + \tfrac{1}{3}C_u D_x^{-1} C_u\right)\left(D_x + \tfrac{1}{3}A_u D_x^{-1} A_u\right). \end{aligned} \tag{2.3}$$

Modified KdV, case 2

$$u_t = u_3 + u u_2 - u_2 u - \tfrac{2}{3} u u_1 u.$$

The recursion operator for this equation,

$$\mathfrak{R} = \left(D_x + \tfrac{2}{3}C_u\right)\left(D_x - \tfrac{2}{3}R_u\right)\left(D_x + \tfrac{1}{3}C_u\right)^{-1}\left(D_x + \tfrac{2}{3}L_u\right)D_x\left(D_x + \tfrac{1}{3}C_u\right)^{-1}, \tag{2.4}$$

was very recently found by Gürses, Karasu and Sokolov, [15]. Two other methods of constructing the higher order symmetries are discussed in Section 6.3 below.

There are two main results in our paper. The first is that the higher order symmetries of these noncommutative integrable systems are all local. The recursion operator, when it exists, will operate on an individual symmetry to give the next symmetry in the hierarchy. Master symmetries accomplish the same things through the bracketing operation. A *recursive operator* requires more than one symmetry in order to construct the next symmetry; examples will appear below.

Theorem 2.2. *Each of the integrable equations listed above admits a hierarchy of higher order local symmetries, constructed by applying the recursion operator, recursive operator, or master symmetry.*

The second result shows that this list is complete for positive λ.

Theorem 2.3. *Let $\lambda > 0$. A noncommutative λ-homogeneous equation* (1.1) *admits a higher order symmetry if and only if $\lambda = 1$ or 2 and the equation belongs to one the preceding six symmetry hierarchies: two Burgers', KdV, potential KdV, MKdV1, or MKdV2.*

In particular, unlike the commutative case, there are no direct analogs of the fifth order Sawada–Kotera or Kaup–Kupershmidt equations, as noted in [22]. The only integrable fifth order noncommutative $\lambda = 2$ equation is the fifth order KdV equation. Also, the third order commutative Calogero equation

$$u_t = u_3 + 3u^2 u_2 + 9 u u_1^2 + 3u^4 u_1,$$

with $\lambda = \frac{1}{2}$, has no noncommutative analog. On the other hand, there are two different versions of the MKdV equation. It seems that noncommutativity both increases and decreases the number of integrable equations.

3 Noncommutative Differential Polynomials and Differential Operators

We begin by assembling basic facts about noncommutative differential polynomials. For simplicity, we restrict our attention to the case of a single independent variable $x \in \mathbb{R}$ and a single dependent field variable u which takes its values in an associative algebra. Extensions of the basic ideas to several (commutative) independent variables and (noncommutative) dependent variables are immediate. The derivatives of u with respect to x are denoted by $u_n = D_x^n u$. A differential monomial takes the form

$$u_I = u_{i_1} u_{i_2} \cdots u_{i_k}.$$

The sequence of the factors *is* important. We call k the *degree* of the monomial, and $\#I = i_1 + \cdots + i_k$ the *order*. We let $\mathcal{U}_n^k$ denote the set of differential polynomials of degree k and order n. Let $\mathcal{U}^k = \bigoplus_n \mathcal{U}_n^k$, and $\mathcal{U}_n = \bigoplus_k \mathcal{U}_n^k$. The algebra of all differential polynomials is denoted by $\mathcal{U} = \bigoplus_{n,k} \mathcal{U}_n^k$.

Any linear differential operator $H: \mathcal{U} \to \mathcal{U}$ can be written as a linear combination of the operators

$$D_{IJ}^k = L_{u_I} \circ R_{u_J} \circ D_x^k, \qquad \text{so that} \qquad D_{IJ}^k(v) = u_I \cdot D_x^k v \cdot u_J = u_I v_k u_J.$$

We call k the *order* of the operator D_{IJ}^k. Let $\mathcal{D}_{ij}^k$ denote the space of differential operators spanned by the D_{IJ}^k for $\#I = i$, $\#J = j$. Let $\mathcal{D}^k = \bigoplus_{i,j} \mathcal{D}_{ij}^k$ denote the space of k-th order differential operators, and let $\mathcal{D}_{ij} = \bigoplus_k \mathcal{D}_{ij}^k$. The space of all linear differential operators is denoted by $\mathcal{D} = \bigoplus_k \mathcal{D}^k = \bigoplus_{i,j} \mathcal{D}_{ij}$. In particular, a linear operator $H: \mathcal{U} \to \mathcal{U}$ means an order $k = 0$ linear differential operator, so $H[v] = \sum u_I \cdot v \cdot u_J$ does not involve derivatives of v. In the commutative case, any order 0 linear operator is given by multiplication by a differential polynomial, $v \mapsto Q[u]\, v$, and so one can identify $\mathcal{D}^0 \simeq \mathcal{U}$. However, this is *not* true in the noncommutative case, since the v can appear in different positions.

Definition 3.1. *The* Fréchet derivative *of $K \in \mathcal{U}$ is the differential operator $D_K \in \mathcal{D}$ defined so that,*

$$D_K[v] = \frac{d}{d\epsilon}\, K[u + \epsilon v] \Big|_{\epsilon=0} \qquad \text{for every} \qquad v \in \mathcal{U}.$$

For example, the Fréchet derivative of $K = u_i u_j$ is

$$D_{u_i u_j}[v] = (R_{u_j} D_x^i + L_{u_i} D_x^j) v = v_i u_j + u_i v_j.$$

For any $K, Q \in \mathcal{U}$, we define the *bracket*

$$[K, Q] = D_Q[K] - D_K[Q]. \tag{3.1}$$

This bracket makes $\mathcal{U}$ into a bigraded Lie algebra, i.e. a Lie algebra with two different gradings — order and degree.

Definition 3.2. Two evolution equations $u_t = K$ and $u_t = Q$ are symmetries of each other if and only if

$$[K, Q] = 0. \tag{3.2}$$

On an intuitive level, the symmetry condition (3.2) means that the two flows defined by the evolution equations commute; however, this interpretation is only formal, since we do not have the analytical results to establish the existence of a flow for general evolution equations. We let

$$\mathcal{S}_K = \{ Q \in \mathcal{U} \mid [K, Q] = 0 \}$$

denote the space of symmetries of the equation $u_t = K$. As we shall see, even for noncommutative evolution equations, the symmetry space forms an abelian Lie algebra under the bracket (3.1) thanks to the Jacobi identity. (Note that we are only considering constant coefficient differential polynomial symmetries.) The symmetry approach to integrability requires characterizing evolution equations $u_t = K$ with nontrivial symmetry algebras $\mathcal{S}_K$.

In the definition of noncommutative functionals in [22], we assumed the existence of a trace operation $\mathrm{tr}\colon \mathbb{A} \to \mathbb{R}$ on our algebra, satisfying $\mathrm{tr}(uv) = \mathrm{tr}(vu)$ for all $u, v \in \mathbb{A}$. In fact, the explicit trace operation is not required, and we only need its basic formal properties of linearity and symmetry

$$\mathrm{tr}\big[\, u_{i_1} u_{i_2} \cdots u_{i_k} \,\big] = \mathrm{tr}\big[\, u_{i_j} \cdots u_{i_k} u_{i_1} \cdots u_{i_{j-1}} \,\big], \qquad j = 1, \ldots, k,$$

under cyclic permutation of our noncommutative variables. In other words, the cyclic group $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ will act linearly on the homogeneous subspace $\mathcal{U}^k$ by cyclic permutations

$$u_{i_1} u_{i_2} \cdots u_{i_k} \;\longmapsto\; u_{i_j} \cdots u_{i_k} u_{i_1} \cdots u_{i_{j-1}}, \qquad j = 1, \ldots, k,$$

of monomials We then identify the space of *trace forms* of degree k as the subspace $\mathcal{R}^k = \mathcal{U}^k/\mathbb{Z}_k$ of cyclically symmetric differential polynomials. The space of all trace forms, $\mathcal{R} = \bigoplus_k \mathcal{R}^k$, is only a vector space since multiplication does not preserve cyclic symmetry. The *trace* itself can be formally identified with the group averaging or projection operator:

$$\mathrm{tr}\, P \;\simeq\; \frac{1}{k} \sum_{g \in \mathbb{Z}_k} g \cdot P, \qquad P \in \mathcal{U}^k.$$

For example, we can (formally) identify $\mathrm{tr}(u u_1 u_2)$ with $\frac{1}{3}\big[u u_1 u_2 + u_2 u u_1 + u_1 u_2 u\big]$.

Remark 3.3. Let $\mathbb{S}^k$ denote the full symmetric group on k letters, which acts on monomials by permuting the factors. We can formally identify the space of commutative differential polynomials of degree k with the subspace $\mathcal{S}^k = \mathcal{U}^k/\mathbb{S}^k$ of fully symmetric differential polynomials. The algebra structure of $\mathcal{S} = \bigoplus_k \mathcal{S}^k$ is obtained by first multiplying and then symmetrizing.

The total derivative operator D_x acts on the space of trace forms $\mathcal{R}$ in the obvious fashion. The cokernel $\mathcal{G} = \mathcal{R}/D_x\mathcal{R}$ is the vector space of (noncommutative) functionals, [22], whose elements are denoted by $\int \mathrm{tr}\, P\, dx$ where $P \in \mathcal{U}$. We introduce the natural nondegenerate pairing

$$\langle\, P\, ; Q\, \rangle = \int \mathrm{tr}(P \cdot Q)\, dx \tag{3.3}$$

on the space $\mathcal{U}$. This allows us to define the adjoint $H^* \in \mathcal{D}$ to a linear differential operator $H \in \mathcal{D}$ by the usual formula

$$\langle\, v\,; H[w]\,\rangle = \langle\, H^*[v]\,; w\,\rangle \qquad \text{for all} \qquad v, w \in \mathcal{U}. \tag{3.4}$$

For example, $C_u^* = -C_u$ is skew-adjoint.

The *variational derivative* or *Euler operator* associates with each functional $\mathcal{I}[u] = \int \operatorname{tr} P[u]\, dx$ its Euler-Lagrange expression $\delta(\mathcal{I}) = \mathbf{E}(\operatorname{tr} P) \in \mathcal{U}$, defined so that

$$\langle\, \mathbf{E}(\operatorname{tr} P)\,; v\,\rangle = \int \operatorname{tr}(\mathbf{E}(\operatorname{tr} P) \cdot v)\, dx = \frac{d}{d\epsilon}\, \mathcal{I}[u + \epsilon v]\, \bigg|_{\epsilon=0}. \tag{3.5}$$

We note the important formula

$$\delta\,\langle\, P\,; Q\,\rangle = \mathbf{E}(\operatorname{tr}(PQ)) = D_P^*(Q) + D_Q^*(P), \tag{3.6}$$

cf. equation (5.80) in [20]. In particular, $\mathbf{E}(P) = D_P^*(1)$ is obtained by applying the adjoint to the Fréchet derivative operator to the constant function 1. The characterization of the kernel of the Euler operator $\mathbf{E}$ as the image of the total derivative follows as in the commutative case.

Theorem 3.4. *A trace form* $\operatorname{tr} P \in \mathcal{R}$ *lies in the image of the total derivative,* $\operatorname{tr} P = \operatorname{tr} D_x A$, *if and only if* $\mathbf{E}(\operatorname{tr} P) = 0$.

This result is an immediate consequence of the symbolic method; see below. As discussed in [20], this forms one term in an entire complex — the (noncommutative) variational complex. See [11] for further development of the noncommutative variational complex.

The development of the full noncommutative variational complex is an open (and rather urgent) problem.

Theorem 3.4 characterizes the image of the total derivative on the space of trace forms. We also need to describe the image of D_x on the full algebra of differential polynomials $\mathcal{U}$. Define the space $\mathcal{J} = \mathcal{U}/D_x\mathcal{U}$ of algebra-valued functionals $\mathcal{P}[u] = \int P[u]\, dx$ where $P \in \mathcal{U}$. Note that there is no trace in this case, and so we cannot cyclically permute the monomials, although we are still permitted to integrate by parts. For example,

$$\int \operatorname{tr}(uu_1)\, dx = -\int \operatorname{tr}(u_1 u)\, dx = -\int \operatorname{tr}(uu_1)\, dx,$$

so that $\int \operatorname{tr}(uu_1)\, dx = 0$. Indeed,

$$\tfrac{1}{2} D_x \operatorname{tr} u^2 = \tfrac{1}{2} \operatorname{tr}[uu_1 + u_1 u] = \operatorname{tr} uu_1.$$

On the other hand,

$$\int uu_1\, dx = -\int u_1 u\, dx \neq 0.$$

If $H \in \mathcal{D}$ is a linear differential operator, we define its pairing with a differential polynomial $v \in \mathcal{U}$ by integrating its action on v

$$\langle\, H\,;v\,\rangle = \langle\, v\,;H\,\rangle = \int H[v]\,dx. \tag{3.7}$$

(There is no longer a trace in this formula.) Integration by parts shows that we can always replace H by an order 0 differential operator $M \in \mathcal{D}^0$, so that $\langle\, H\,;v\,\rangle = \langle\, M\,;v\,\rangle$ for all $v \in \mathcal{U}$. The pairing $\langle\, M\,;v\,\rangle$ between $\mathcal{D}^0$ and $\mathcal{U}$ is nondegenerate. Note that, in the commutative case, $\mathcal{D}^0 \simeq \mathcal{U}$ and so this pairing reduces to the previous one, but this fact is no longer true in the noncommutative category.

The variational derivative of an algebra-valued functional $\mathcal{P} = \int P\,dx \in \mathcal{J}$ will be a linear operator $\delta\mathcal{P} = \mathbf{E}(P) \in \mathcal{D}^0$. It is defined by the usual variational rule

$$\langle\, \mathbf{E}(P)\,;v\,\rangle = \int \big\{\mathbf{E}(P)\cdot v\big\}\,dx = \frac{d}{d\epsilon}\,\mathcal{P}[u+\epsilon v]\,\bigg|_{\epsilon=0}. \tag{3.8}$$

For example, if

$$P = uu_1u_2,$$

then, integrating by parts twice,

$$\begin{aligned}\langle\, \mathbf{E}(P)\,;v\,\rangle &= \int \big[vu_1u_2 + uv_1u_2 + uu_1v_2\big]\,dx\\ &= \int \big[vu_1u_2 - u_1vu_2 - uvu_3 - u_1u_1v_1 - uu_2v_1\big]\,dx\\ &= \int \big[vu_1u_2 - u_1vu_2 - uvu_3 + u_2u_1v + 2u_1u_2v + uu_3v\big]\,dx\end{aligned}$$

Therefore,

$$\mathbf{E}(P) = L_{u_2u_1+2u_1u_2+uu_3} - L_{u_1}\cdot R_{u_2} - L_u\cdot R_{u_3} + R_{u_1u_2}. \tag{3.9}$$

There is a direct analog to the previous Theorem 3.4.

Theorem 3.5. *A differential polynomial $P \in \mathcal{U}$ lies in the image of the total derivative $P = D_xA$ if and only if $\mathbf{E}(P) = 0$.*

If $P \in \mathcal{U}^k$ is homogeneous of degree k, then

$$\mathbf{E}(P)\,v = \sum_{i=1}^{k}\mathbf{E}_i(P)\,v = \sum_{i=1}^{k}\sum_{\nu}\ \mathbf{E}_i^{\nu,1}(P)\cdot v\cdot \mathbf{E}_i^{\nu,2}(P),$$

where $\mathbf{E}_i^{\nu,1}(P) \in \mathcal{U}^{i-1}$ and $\mathbf{E}_i^{\nu,2}(P) \in \mathcal{U}^{k-i}$ are homogeneous differential polynomials of respective degrees $i-1$ and $k-i$, and so $\mathbf{E}_i(P) \in \mathcal{D}_{i-1,k-i}$. We further define the differential polynomial

$$\mathbf{F}_i(P) = \sum_{\nu}\ \mathbf{E}_i^{\nu,1}(P)\cdot\mathbf{E}_i^{\nu,2}(P) \in \mathcal{U}^{k-1} \qquad \text{for} \qquad P \in \mathcal{U}^k. \tag{3.10}$$

Note that the differential operator $\mathbf{E}_i(P)$ is uniquely determined by the differential polynomial $\mathbf{F}_i(P)$. For our preceding example (3.9),

$$\mathbf{F}_1(P) = u_1u_2, \qquad \mathbf{F}_2(P) = -u_1u_2 - uu_3, \qquad \mathbf{F}_3(P) = u_2u_1 + 2u_1u_2 + uu_3.$$

Note that one can compute each $\mathbf{F}_i(P)$ directly by just varying the i-th factor in each term in P.

Theorem 3.6. *Let $P \in \mathcal{U}^k$. Then $P = D_xQ$ if and only if $\mathbf{F}_i(P) = 0$ for any single $1 \leq i \leq k$.*

The remarkable fact is that we only need to check that *one* of the $\mathbf{F}_i(P)$ vanishes in order to conclude that $P \in \operatorname{Im} D_x$, which implies that *all* $\mathbf{F}_i(P) = 0$. Clearly Theorem 3.5 is a corollary of Theorem 3.6. We will prove Theorem 3.6 in the following section.

4 The Symbolic Method

As in the commutative classification in [26], the proof of Theorem 2.3 relies on the symbolic method first introduced by Gel'fand and Dikii, [13]. The method was generalized by Shakiban, [27, 28], who used it to apply the invariant theory of finite groups to the study of conservation laws of evolution equations, and Ball, Currie, and Olver, [3, 19], to classify null Lagrangians arising in nonlinear elasticity. In [19] the connections with the symbolic method of classical invariant theory were first recognized; see [21] for the full details. We also note the applications of Anderson and Pohjanpelto, [1, 2], in the calculus of variations.

The basic idea is simply to replace u_i, where i is an index — in our case counting the number of derivatives — by ξ^i, where ξ is now a symbol. We see that the basic operation of differentiation, that is, replacing u_i by u_{i+1}, is now replaced by multiplication with ξ, as is the case in Fourier transform theory. For higher degree terms with multiple u's, one uses different symbols to denote differentiation; for example, the noncommutative binomial u_iu_j has symbolic form $\xi_1^i\xi_2^j$. In the commutative case, [21], one needs to average over permutations of the differentiation symbols so that u_iu_j and u_ju_i have the same symbolic form. However, in the noncommutative case under consideration here, this is no longer necessary. In other words, the noncommutative symbolic method works with general polynomials, while in the commutative case one restricts to (multi-)symmetric polynomials.

Let $\mathcal{A}^k = \mathbb{R}[\xi_1, \ldots, \xi_k]$ denote the algebra of polynomials in k variables. Let $\mathcal{A}_n^k$ be subspace of homogeneous polynomials of degree n. The *transform* or *symbolic form* defines a linear isomorphism between the space $\mathcal{U}^k$ of homogeneous, noncommutative differential polynomials of degree k and the space $\mathcal{A}^k$ of algebraic polynomials in k variables. It is uniquely defined by its action on monomials.

Definition 4.1. *The* symbolic form *of a differential monomial is defined as*

$$u_{i_1}u_{i_2}\cdots u_{i_k} \in \mathcal{U}_n^k \quad \longmapsto \quad \xi_1^{i_1}\xi_2^{i_2}\cdots\xi_k^{i_k} \in \mathcal{A}_n^k.$$

In general, in analogy with Fourier transforms, we denote the symbolic form of $P \in \mathcal{U}^k$ by $\widehat{P} \in \mathcal{A}^k$. The transform has two basic properties:

$$\begin{aligned} \widehat{D_x P}(\xi_1, \ldots, \xi_k) &= (\xi_1 + \cdots + \xi_k)\, \widehat{P}(\xi_1, \ldots, \xi_k), \\ \widehat{\frac{\partial P}{\partial u_i}}(\xi_1, \ldots, \xi_{k-1}) &= \frac{1}{i!} \sum_{j=1}^{k} \frac{\partial^i \widehat{P}}{(\partial \xi_j)^i}(\xi_1, \ldots, \xi_{j-1}, 0, \xi_j, \ldots, \xi_{k-1}) \end{aligned} \tag{4.1}$$

for $P \in \mathcal{U}^k$. The following key result is a consequence of these formulae. It is proved in the same manner as in the commutative case, [26].

Proposition 4.2. *Let $K \in \mathcal{U}_r^m$ and $Q \in \mathcal{U}_s^n$, then $D_K(Q) \in \mathcal{U}_{r+s}^{m+n-1}$, where D_K is the Fréchet derivative of K, and*

$$\widehat{D_K[Q]} = \sum_{\tau=1}^{m} \widehat{K}\left(\xi_1, \ldots, \xi_{\tau-1}, \sum_{\kappa=0}^{n-1} \xi_{\tau+\kappa}, \xi_{\tau+n}, \ldots, \xi_{m+n-1}\right) \widehat{Q}(\xi_\tau, \ldots, \xi_{\tau+n-1}).$$

The symbolic forms of the algebra-valued version of our two different Euler operators are also based on (4.1). Again, the proof proceeds as in the commutative case, [13].

Proposition 4.3. *Let $P \in \mathcal{U}^k$ then*

$$\widehat{\mathbf{F}_i(P)} = \widehat{P}(\xi_1, \ldots, \xi_{i-1}, -\xi_1 - \cdots - \xi_{k-1}, \xi_i, \ldots, \xi_{k-1})$$

Theorem 3.6 is now an immediate consequence of Proposition 4.3 and the formula (4.1) for the symbolic form of the total derivative.

For trace forms, we need to quotient by the action of the cyclic group $\mathbb{Z}_k$, which acts on $\mathcal{A}^k$ by cyclically permuting the variables; the generator is

$$(\xi_1, \xi_2, \ldots, \xi_k) \longmapsto (\xi_2, \ldots, \xi_k, \xi_1).$$

Let $\mathcal{B}^k = \mathcal{A}/\mathbb{Z}_k$ be the space of cyclically-symmetric polynomials. The transform defines a linear isomorphism from $\mathcal{R}^k = \operatorname{tr} \mathcal{A}^k$ to $\mathcal{B}^k$. Given $P \in \mathcal{U}^k$, let $\operatorname{tr} P$ be the corresponding trace form, and $\mathbf{E}(\operatorname{tr} P) \in \mathcal{U}^{k-1}$ its Euler-Lagrange expression. Then

$$\widehat{\mathbf{E}(\operatorname{tr} P)} = \sum_{i=1}^{k} \widehat{P}(\xi_i, \ldots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1}, \xi_1, \ldots, \xi_{i-1}).$$

Theorem 3.4 follows immediately.

Incidentally, the commutative case is done in a similar fashion, but one quotients by the full symmetric group $\mathbb{S}^k$ instead of the cyclic subgroup $\mathbb{Z}_k$. See [21] for details.

Theorem 4.4. *Let $P \in \mathcal{U}$ be an arbitrary differential polynomial. Then the differential polynomial*

$$C_u \mathbf{E}(\operatorname{tr} P) = u\,\mathbf{E}(\operatorname{tr} P) - \mathbf{E}(\operatorname{tr} P)\,u$$

lies in the image of the total derivative, that is, there exists $A \in \mathcal{U}$ such that $C_u \mathbf{E}(\operatorname{tr} P) = D_x A$.

Proof: Without losing generality, we take $P \in \mathcal{U}^k$, so

$$\widehat{C_u \mathbf{E}(\operatorname{tr} P)} = \sum_{i=1}^{k} \widehat{P}(\xi_{i+1}, \ldots, \xi_k, -\xi_2 - \cdots - \xi_k, \xi_2, \ldots, \xi_i) - \\ - \sum_{i=1}^{k} \widehat{P}(\xi_i, \ldots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1}, \xi_1, \ldots, \xi_{i-1}).$$

According to Theorem 3.6, we only need to check that $\mathbf{F}_1(C_u \mathbf{E}(\operatorname{tr} P)) = 0$. Its symbolic form is

$$\mathbf{F}_1(\widehat{C_u \mathbf{E}(\operatorname{tr} P)}) = \sum_{i=1}^{k} \widehat{P}(\xi_i, \ldots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1}, \xi_1, \ldots, \xi_{i-1}) - \\ - \sum_{i=1}^{k} \widehat{P}(\xi_{i-1}, \ldots, \xi_{k-2}, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1}, \xi_1, \ldots, \xi_{i-2}) \\ = 0.$$

Thus the statement is proved.

5 Symmetries of λ-Homogeneous Equations

In this section, we apply the symbolic method to prove our basic classification Theorem 2.3. A key observation is that it suffices to compute the linear and quadratic or cubic terms of a nontrivial odd order symmetry in order to guarantee its existence. This observation speeds up the classification process, since any obstructions to the existence of symmetries have to show up early in the computation. The computations are remarkably similar to the commutative case, [26]. The key differences are *a*) the polynomials arising in the symbolic computations are not required to be symmetric under permutations, and *b*) while the bounds on the orders of the equation and the symmetry happen to be the same as in the commutative case, the final symbolic computation used to complete the classification relies on whether or not the variables commute.

In [26], we gave extensive results about the mutual divisibility of certain particular multivariate polynomials, called "G-functions", which play a crucial role in proving the (non-)existence of symmetries. We will show that the *same* (commutative) G-functions appear in the computation for noncommutative evolution equations, and so all the results for the commutative case, as discussed in section 5 of [26], are immediately applicable to the noncommutative case.

Definition 5.1. *The G-functions are the (commutative) polynomials*

$$G_k^{(m)} = \xi_1^k + \cdots + \xi_{m+1}^k - \big(\xi_1 + \cdots + \xi_{m+1}\big)^k.$$

The key fact is the following formula for the bracket of a differential polynomial with a linear differential polynomial:

$$\widehat{[u_k, Q]} = G_k^{(m)}\,\widehat{Q}, \qquad \text{whenever} \qquad Q \in \mathcal{U}^{m+1}. \tag{5.1}$$

This follows immediately from (4.2) and the fact that u_k has symbolic form $\widehat{u_k} = \xi_1^k$. An immediate application is the following known result that classifies the symmetries of linear evolution equations.

Proposition 5.2. *Consider an n-th order linear evolution equation*

$$u_t = K = \sum_{j=1}^{n} c_j u_j,$$

where the c_j are constants and $c_n \neq 0$. Then its space of symmetries is

- $\mathcal{S}_K = \mathcal{U}$ *if and only if* $n = 1$*;*
- $\mathcal{S}_K = \mathcal{U}^1$ *if and only if* $n > 1$.

Proof: We can assume, without loss of generality, that the symmetry $Q \in \mathcal{U}^{m+1}$ is homogeneous of degree $m+1$. We merely transform the symmetry condition $[K, Q] = 0$ and use (5.1), obtaining

$$\widehat{[K, Q]} = \left(\sum_{j=1}^{n} c_j G_j^{(m)}\right)\,\widehat{Q} = 0.$$

Therefore

$$\sum_{j=1}^{n} c_j(\xi_1^j + \cdots + \xi_{m+1}^j) = \sum_{j=1}^{n} c_j(\xi_1 + \cdots + \xi_{m+1})^j.$$

Under the assumption, this holds if and only if either $n = 1$ or $n \neq 1$ and $m = 0$.

We recall the divisibility properties of the G-functions, which were proved in [4, 26].

Proposition 5.3. *We have $G_k^{(m)} = T_k^m H_k^{(m)}$, where $(H_k^{(m)}, H_l^{(m)}) = 1$ for all $k < l$, and T_k^m is one of the following polynomials*:

- $m = 1$:
  - $k = 0 \pmod 2$: $\xi_1\xi_2$
  - $k = 3 \pmod 6$: $\xi_1\xi_2(\xi_1 + \xi_2)$
  - $k = 5 \pmod 6$: $\xi_1\xi_2(\xi_1 + \xi_2)(\xi_1^2 + \xi_1\xi_2 + \xi_2^2)$

  - $k = 1 \pmod 6$: $\xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_1\xi_2+\xi_2^2)^2$

- $m = 2$:
  - $k = 0 \pmod 2$: 1
  - $k = 1 \pmod 2$: $(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)$

- $m > 2$: 1

Any λ–homogeneous evolution equation of order n can be broken up into its homogeneous components, and so takes the form

$$u_t = \sum_{i\geq 0} K^i_{n-\lambda i}, \qquad \text{where} \qquad K^i_{n-\lambda i} \in \mathcal{U}^{i+1}_{n-\lambda i}. \tag{5.2}$$

We assume that $K^0_n = u_n$ and $0 < \lambda \in \mathbb{Q}$. When $i\lambda \notin \mathbb{N}$, $K^i_{n-i\lambda} = 0$. This reduces the number of relevant λ to a finite set.

Let $S \in \mathcal{U}$ be a symmetry of order m of the evolution equation (5.2). We break up the bracket condition $[S, K] = 0$ into its homogeneous summands, leading to the series of successive symmetry equations

$$\sum_{i+j=r} [S^j_{m-\lambda j}, K^i_{n-\lambda i}] = 0, \qquad \text{for} \qquad r = 0, 1, 2, \dots\,. \tag{5.3}$$

According to Proposition 5.2, S must have nontrivial linear term, $S^0_m \neq 0$, and we can set $S^0_m = u_m$ without loss of generality. Clearly we have $[S^0_m, K^0_n] = 0$. The next equation to be solved is

$$[S^0_m, K^1_{n-\lambda}] + [S^1_{m-\lambda}, K^0_n] = 0. \tag{5.4}$$

Condition (5.4) is trivially satisfied if K has no quadratic terms: $K^1_{n-\lambda} = 0$. Let us concentrate on the other case $K^1_{n-\lambda} \neq 0$. We use Proposition 5.3 to rewrite (5.4) in symbolic form:

$$\widehat{K}^1_{n-\lambda} = \frac{\widehat{S}^1_{m-\lambda}}{G^{(1)}_m}\, G^{(1)}_n = \frac{p(\xi_1,\xi_2)}{\xi_1\xi_2(\xi_1+\xi_2)}\, G^{(1)}_n, \tag{5.5}$$

where $\lim_{\xi_1+\xi_2\to 0} p(\xi_1,\xi_2)$ exists. We next set $r = 2$ in (5.3), and find

$$\widehat{S}^2_{m-2\lambda} \quad = \quad \frac{\widehat{K}^2_{n-2\lambda} G^{(2)}_m + \widehat{M}}{G^{(2)}_n}, \tag{5.6}$$

where $\widehat{M}$ is the symbolic form of the commutator

$$M = [S^1_{m-\lambda}, K^1_{n-\lambda}] \tag{5.7}$$

between the quadratic terms.

We use the notation $q \mid p$ to indicate that the polynomial q divides the polynomial p, while $q \not\mid p$ indicates that q does not evenly divide p. Consider the set

$$\mathcal{I} = \{\ p(\xi_1, \xi_2)\ :\ (\xi_1 + \xi_2) \mid p(\xi_1, \xi_2) \quad \text{or} \quad \xi_1 \xi_2 \mid p(\xi_1, \xi_2)\ \}$$

consisting of bivariate polynomials $p(\xi_1, \xi_2)$ which have either $\xi_1 + \xi_2$ or $\xi_1 \xi_2$ as a factor.

Proposition 5.4. *Suppose m and n are odd. Let $\widehat{M}, p$ be given by (5.7), (5.5), respectively. Then $(\xi_1 + \xi_2)(\xi_2 + \xi_3)(\xi_1 + \xi_3)$ divides $\widehat{M}$ if and only if $p \in \mathcal{I}$.*

Proof: Using formula (5.5), we compute $\widehat{M}$ to be

$$\widehat{M} = \frac{p(\xi_1 + \xi_2, \xi_3) p(\xi_1, \xi_2) F_{\xi_2, \xi_3}(\xi_1 + \xi_2)}{\xi_1 \xi_2 \xi_3 (\xi_1 + \xi_2)^2 (\xi_1 + \xi_2 + \xi_3)} + \frac{p(\xi_1, \xi_2 + \xi_3) p(\xi_2, \xi_3) F_{\xi_2, \xi_1}(\xi_2 + \xi_3)}{\xi_1 \xi_2 \xi_3 (\xi_2 + \xi_3)^2 (\xi_1 + \xi_2 + \xi_3)},$$

where

$$F_{\xi_i, \xi_j}(\eta) = G_m^{(1)}(\eta, \xi_j) G_n^{(1)}(\eta - \xi_i, \xi_i) - G_n^{(1)}(\eta, \xi_j) G_m^{(1)}(\eta - \xi_i, \xi_i).$$

Notice that $\xi_1 + \xi_3$ is a factor. We now prove that $\lim_{\xi_1 + \xi_2 \to 0}$ of this expression is zero. The second summand has

$$\begin{aligned}
&\lim_{\xi_1 + \xi_2 \to 0} F_{\xi_2, \xi_1}(\xi_2 + \xi_3) = \\
&\qquad = G_m^{(1)}(-\xi_2, \xi_2 + \xi_3) G_n^{(1)}(\xi_2, \xi_3) - G_n^{(1)}(-\xi_2, \xi_2 + \xi_3) G_m^{(1)}(\xi_2, \xi_3) \\
&\qquad = -G_m^{(1)}(\xi_2, \xi_3) G_n^{(1)}(\xi_2, \xi_3) + G_n^{(1)}(\xi_2, \xi_3) G_m^{(1)}(\xi_2, \xi_3) = 0.
\end{aligned}$$

As for the first, a straightforward computation shows that

$$F_{\xi_2, \xi_3}(0) = 0 = \frac{d}{d\eta} F_{\xi_2, \xi_3}(0)$$

Moreover,

$$\begin{aligned}
&\frac{d^2}{d\eta^2} F_{\xi_2, \xi_3}(0) = \\
&\quad = 2 \frac{d}{d\eta} G_m^{(1)}(\xi_3, \eta) \frac{d}{d\eta} G_n^{(1)}(\eta - \xi_2, \xi_2) - 2 \frac{d}{d\eta} G_m^{(1)}(\eta - \xi_2, \xi_2) \frac{d}{d\eta} G_n^{(1)}(\xi_3, \eta) \Bigg|_{\eta = 0} \\
&\quad = 2nm \left[(-1)^n \xi_3^{m-1} \xi_2^{n-1} - (-1)^m \xi_3^{n-1} \xi_2^{m-1} \right] \neq 0.
\end{aligned}$$

This implies that

$$\lim_{\xi_1 + \xi_2 \to 0} \frac{F_{\xi_2, \xi_3}(\xi_1, \xi_2)}{(\xi_1 + \xi_2)^2} \neq 0$$

and therefore $(\xi_1 + \xi_2) \not\mid \widehat{M}$ unless $(\xi_1 + \xi_2) | p(\xi_1 + \xi_2, \xi_3) p(\xi_1, \xi_2)$, or, equivalently, $(\xi_1 + \xi_2) \,|\, p(\xi_1, \xi_2)$ or $\xi_1 | p(\xi_1, \xi_2)$. Similarly, when we deal with the factor $\xi_2 + \xi_3$, we obtain $(\xi_2 + \xi_3) \not\mid \widehat{M}$ unless $(\xi_2 + \xi_3) \,|\, p(\xi_1, \xi_2 + \xi_3) p(\xi_2, \xi_3)$, or, equivalently, $(\xi_1 + \xi_2) \mid p(\xi_1, \xi_2)$ or $\xi_2 \mid p(\xi_1, \xi_2)$. Therefore, the statement of the proposition follows.

Corollary 5.5. *Assume m and n are odd. Then $(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)$ divides $\widehat{K}^2_{n-2\lambda}G^{(2)}_m+\widehat{M}$ if and only if $\widehat{K}^1_{n-\lambda}(\xi_1,\xi_2)\in\mathcal{I}$.*

We next state a result that says that the symmetry algebra of a noncommutative polynomial evolution equation is commutative. Moreover, every symmetry is uniquely determined by its quadratic terms. The proof is the same as Theorem 5.5 in [26].

Theorem 5.6. *Suppose the evolution equation* (5.2) *has a nonzero symmetry of order $m\geq 2$. Suppose $Q^1_{q-\lambda}\neq 0$ is a nonzero quadratic differential polynomial, where $q\geq\lambda$, with $q\neq m,n$, and q odd if n is odd, that satisfies the leading order symmetry condition $[K^0_n,Q^1_{q-\lambda}]+[K^1_{n-\lambda},Q^0_q]=0$, cf.* (5.4). *Then there exists a unique symmetry of the form $Q=\sum_{i\geq 0}Q^i_{q-\lambda i}$. Moreover, the symmetries Q and S commute.*

We make a very interesting observation. Suppose Q is a nontrivial q-th odd order symmetry of (5.2) with odd n, whose quadratic terms have symbolic form (5.5):

$$\widehat{Q}^1_{q-\lambda} \quad = \quad \frac{\widehat{K}^1_{n-\lambda}\;(\xi_1^2+\xi_1\xi_2+\xi_2^2)^{s-s'}\;H^{(1)}_q}{H^{(1)}_n}.$$

Proposition 5.3 implies that $\lambda\leq 3+2\min(s,s')$, where $s'=\frac{n+3}{2} \pmod 3$ and $s=\frac{q+3}{2} \pmod 3$. Then Theorem 5.6 implies that

$$\widehat{Q}^1_{2s+3-\lambda} \quad = \quad \frac{\widehat{K}^1_{n-\lambda}\;(\xi_1^2+\xi_1\xi_2+\xi_2^2)^{s-s'}\;H^{(1)}_{2s+3}}{H^{(1)}_n},$$

gives rise to a symmetry $Q=Q^0_{2s+3}+Q^1_{2s+3-\lambda}+\cdots$ of the original equation. (Of course, one can use this argument to generate an entire hierarchy of symmetries.) This implies that the evolution equations defined by Q and K have the same symmetries, so instead of considering K we may consider the equation given by Q, which is of order $q=2s+3$ for $s=0,1,2$. It follows that we only need to find the symmetries of λ-homogeneous equations (with $\lambda\leq 7$) of order at most 7 in order to obtain the complete classification of symmetries of λ-homogeneous scalar polynomial equations starting with linear terms.

A similar observation can be made for even $n>2$. Suppose we have found a nontrivial symmetry with quadratic term

$$\widehat{Q}^1_{q-\lambda}=\frac{\widehat{K}^1_{n-\lambda}\,G^{(1)}_q}{\xi_1\xi_2\,H^{(1)}_n}$$

This immediately implies $\lambda\leq 2$. Then $\widehat{Q}^1_{2-\lambda}=2\widehat{K}^1_{n-\lambda}/H^{(1)}_n$ gives rise to a symmetry $Q=Q^0_2+Q^1_{2-\lambda}+\cdots$ of the original equation. Therefore, we only need to find the symmetries of equations of order 2 to get the complete classification of symmetries of λ-homogeneous scalar polynomial equations (with $\lambda \leq 2$) starting with an even linear term.

Finally, we must analyze the case when K has no quadratic terms. Assuming K is not a linear differential polynomial, we have $K^i_{n-\lambda i} = 0$ for $i = 1, \ldots, j-1$, and $K^j_{n-\lambda j} \neq 0$ for some $j > 1$. In place of (5.4), we now need to solve the leading order equation

$$[S^0_m, K^j_{n-j\lambda}] + [S^j_{m-j\lambda}, K^0_n] = 0.$$

Using (5.1), the symbolic form of this condition is

$$\widehat{S}^j_{m-j\lambda} = \frac{\widehat{K}^j_{n-j\lambda} G^{(j)}_m}{G^{(j)}_n}. \tag{5.8}$$

Proposition 5.3 implies that this polynomial identity has no solutions when $j \geq 3$, or when $j = 2$ and n is even, since $G^{(j)}_m$ and $G^{(j)}_n$ have no common factors, and the degree of $K^j_{n-j\lambda}$ is $n - j\lambda < n$, which is the degree of $G^{(j)}_n$. This implies that there are no symmetries for such equations. When $j = 2$ and n is odd, the equation can only have odd order symmetries. If the equation (5.8) can be solved for any m, it can also be solved for $m = 3$.

By now, we have proved the noncommutative counterpart of Theorem 5.7 in [26].

Theorem 5.7. *A nontrivial symmetry of a λ-homogeneous equation is part of a hierarchy starting at order* 2, 3, 5 *or* 7.

Only an equation with nonzero quadratic or cubic terms can have a nontrivial symmetry. For such λ, we must find a third order symmetry for a second order equation, a fifth order symmetry for a third order equation, a seventh order symmetry for a fifth order equation with quadratic terms, and the thirteenth order symmetry for a seventh order equation with quadratic terms. It remains to analyze each of these particular cases in detail. A straightforward computation, done with the help of MAPLE, completes the proof of our fundamental Theorem 2.3. The details of this final symbolic computation are completed as in the commutative case described in [25].

6 Construction of Recursion Operators

In the paper [24], the structure of recursion operators was studied in the commutative case. The nonlocal terms in the recursion operators take the form $PD_x^{-1}Q$, where $P, Q \in \mathcal{U}$. The same method can be easily adapted to the noncommutative case, when the recursion operators are in the same form as (2.2), the recursion operator for noncommutative potential KdV. Here we apply these methods, as far as we are able, to discuss the recursion operators for the other noncommutative integrable systems.

First we need to extend our differential polynomial calculus. If $H[u] \in \mathcal{D}$ is a linear differential operator, we define its *Fréchet derivative* by the formula

$$D_H[w] = \frac{d}{d\epsilon}\, H[u+\epsilon w] \,\bigg|_{\epsilon=0} \qquad \text{for every} \qquad w \in \mathcal{U}.$$

Note that $D_H : \mathcal{U} \to \mathcal{D}$ is a linear map from differential polynomials to linear differential operators; thus $D_H[w]v$ is a bilinear operator on $v, w \in \mathcal{U}$. The *Lie derivative* of H with respect to a differential polynomial $Q \in \mathcal{U}$ is given by $\mathfrak{L}_Q(H) = D_H[Q]$. Intuitively, the Lie derivative represents the infinitesimal change under the flow $u_t = Q$.

The adjoint of a (bi-)linear operator $B : \mathcal{U} \to \mathcal{D}$ is defined in analogy with (3.4) using the pairing (3.7) between linear operators and differential polynomials, so that

$$\langle\, v\,; B[w]\,\rangle = \langle\, B^*[v]\,; w\,\rangle \qquad \text{for all} \qquad v, w \in \mathcal{U}.$$

We can use this to define the action $F[H]$ of a linear operator $F = F[u] \in \mathcal{D}$ on a second linear operator $H = H[u] \in \mathcal{D}$ so that

$$\langle\, F[H]\,; v\,\rangle = \langle\, H\,; F^*[v]\,\rangle.$$

The action $F[H]$ is a form of Lie differentiation. We can then generalize the product formula (3.6) for the variational derivative:

$$\delta\,\langle\, H\,; Q\,\rangle = \mathbf{E}(H[Q]) = D_H^*[Q] + D_Q^*[H]. \tag{6.1}$$

Example 6.1. We compute $D^*_{C_{u^2}}$ and $D^*_{C_{u_1}}$. For any $g, h \in \mathcal{U}$, we have

$$\langle\, D_{C_{u^2}}[g]\,; h\,\rangle = \langle\, C_{ug+gu}\,; h\,\rangle = \langle\, -C_h A_u\,; g\,\rangle;$$

$$\langle\, D_{C_{u_1}}[g]\,; h\,\rangle = \langle\, C_{D_x g}\,; h\,\rangle = \langle\, C_{D_x h}\,; g\,\rangle.$$

The nondegeneracy of the pairing implies that

$$D^*_{C_{u^2}}(h) = -C_h A_u, \qquad \text{and} \qquad D^*_{C_{u_1}}(h) = C_{D_x h}.$$

Recall that a *recursion operator* satisfies the basic condition

$$\mathfrak{L}_K[\![\mathfrak{R}]\!] \equiv \mathfrak{L}_K(\mathfrak{R}) + [\mathfrak{R}, D_K] = 0. \tag{6.2}$$

In [20], the left hand side of (6.2) is called the $(1,1)$–*Lie derivative* of the recursion operator $\mathfrak{R}$, since it represents the infinitesimal change in $\mathfrak{R}$ under the flow $u_t = K$ when $\mathfrak{R}$ is considered as a $(1,1)$–tensor.

6.1 KdV equation

We begin with the recursion operator (2.1) for the noncommutative KdV equation. We prove that the higher order symmetries produced by $\mathfrak{R}$ are all local.

Theorem 6.2. *Let $\mathfrak{R}$ be the recursion operator* (2.1) *for the KdV equation. If the "seed" symmetry $Q_0 \in \mathcal{U}$ satisfies $\mathfrak{L}_{Q_0}[\![\mathfrak{R}]\!] = 0$, then the higher order symmetries $Q_l = \mathfrak{R}^l Q_0$ are all local.*

Proof: In the proof, only the nonlocal terms are important. We use $\sim$ instead of $=$ to throw away the local terms of a operator. Thus, we can write

$$\mathfrak{R} \sim \tfrac{1}{3} A_{u_1} D_x^{-1} + \tfrac{1}{9} C_u D_x^{-1} C_u D_x^{-1}$$

for the KdV recursion operator (2.1).

We first note that $\mathfrak{R}$ is a *hereditary operator*, which means that it satisfies the additional condition

$$\mathfrak{L}_{\mathfrak{R}[P]}[\![\mathfrak{R}]\!] = \mathfrak{R}\,\mathfrak{L}_K[\![\mathfrak{R}]\!] = 0. \tag{6.3}$$

(See Definition 5.33 in [20].) The hereditary property (6.3) can be proved directly, or by using the fact that $\mathfrak{R}$ is recursion operator arising from the biHamiltonian structure of the KdV equation, [22]. As a consequence of the hereditary property,

$$\mathfrak{L}_{Q_l}[\![\mathfrak{R}]\!] = 0, \tag{6.4}$$

provided $Q_l = \mathfrak{R}^l(Q_0)$ is local. The goal is to prove that we can apply $\mathfrak{R}$ to Q_l and obtain a local differential polynomial $Q_{l+1} = \mathfrak{R}(Q_l) \in \mathcal{U}$. Let us write out the nonlocal terms in (6.4):

$$\begin{aligned}\mathfrak{L}_{Q_l}[\![\mathfrak{R}]\!] \sim{}& \tfrac{1}{3} A_{D_x Q_l} D_x^{-1} - \tfrac{1}{3} D_{Q_l} A_{u_1} D_x^{-1} + \tfrac{1}{3} A_{u_1} D_x^{-1} D_{Q_l} + \tfrac{1}{9} C_{Q_l} D_x^{-1} C_u D_x^{-1} + \\ &+ \tfrac{1}{9} C_u D_x^{-1} C_{Q_l} D_x^{-1} - \tfrac{1}{9} D_{Q_l} \left(C_u D_x^{-1} C_u D_x^{-1} \right) + \tfrac{1}{9} \left(C_u D_x^{-1} C_u D_x^{-1} \right) D_{Q_l}.\end{aligned}$$

We first consider the terms involving two D_x^{-1}:

$$C_{Q_l} D_x^{-1} C_u D_x^{-1} + C_u D_x^{-1} C_{Q_l} D_x^{-1} - D_{Q_l}[C_u] D_x^{-1} C_u D_x^{-1} + C_u D_x^{-1} C_u D_x^{-1} D_{Q_l}^*[1],$$

which must equal zero. Independence of the summands implies that $C_{Q_l} = D_{Q_l}[C_u]$ and $D_{Q_l}^*[1] = 0$, i.e. $\mathbf{E}(Q_l) = 0$, which implies that there exists $T_l \in \mathcal{U}$ such that $D_x T_l = Q_l$. Therefore,

$$C_u D_x^{-1} C_{Q_l} D_x^{-1} + C_u D_x^{-1} C_u D_x^{-1} D_{Q_l} = C_u C_{T_l} D_x^{-1} + C_u D_x^{-1} \left(D_{T_l}^*[C_u] - C_{T_l} \right),$$

and so (6.4) implies that $D_{T_l}^*[C_u] = C_{T_l}$. Therefore, by (6.1),

$$\delta\langle\, C_u\,; T_l\,\rangle = \mathbf{E}(C_u[T_l]) = D_{C_u}^*[T_l] + D_{T_l}^*[C_u] = -C_{T_l} + C_{T_l} = 0$$

since $D_{C_u}^*[T_l] = -C_{T_l}$. We have therefore proved that $Q_l = D_x T_l \in \operatorname{Im} D_x$ and also $C_u T_l \in \operatorname{Im} D_x$. This implies that $\mathfrak{R}[Q_l]$ is local, and completes the proof.

Remark 6.3. Since $\mathfrak{R}$ does not explicitly depend on x, we can take $Q_0 = u_1$, which is a trivial symmetry of any equation which does not explicitly depend on x. Therefore, each $Q_l = \mathfrak{R}^l(u_1)$ defines a local symmetry of the noncommutative KdV equation.

6.2 Modified KdV, case 1

We now implement the same proof for the operator (2.3), which gives the local symmetries for MKdV1.

Theorem 6.4. *Let $\mathfrak{R}$ be the recursion operator* (2.3) *for the MKdV1 equation. If the "seed" symmetry $Q_0 \in \mathcal{U}$ satisfies $\mathfrak{L}_{Q_0}[\![\mathfrak{R}]\!] = 0$, then the higher order symmetries $Q_l = \mathfrak{R}^l Q_0$ are all local.*

Proof: The MKdV1 recursion operator $\mathfrak{R}$ is also hereditary. Therefore,

$$\begin{aligned}\mathfrak{L}_{Q_l}[\![\mathfrak{R}]\!] \sim\ & \tfrac{1}{3}(A_{D_xQ_l} - D_{Q_l}[A_{u_1}])D_x^{-1}A_u + \tfrac{1}{3}A_{u_1}D_x^{-1}(A_{Q_l} + D^*_{Q_l}[A_u]) - \\ & - \tfrac{1}{9}D_{Q_l}\left(C_uD_x^{-1}C_{u^2}D_x^{-1}A_u\right) + \tfrac{1}{9}\left(C_uD_x^{-1}C_{u^2}D_x^{-1}A_u\right)D_{Q_l} + \\ & + \tfrac{1}{9}C_{Q_l}D_x^{-1}C_{u^2}D_x^{-1}A_u + \tfrac{1}{9}C_uD_x^{-1}C_{u^2}D_x^{-1}A_{Q_l} + \\ & + \tfrac{1}{9}C_uD_x^{-1}C_{uQ_l+Q_lu}D_x^{-1}A_u - \\ & - \tfrac{1}{3}(C_{Q_l} - D_{Q_l}[C_u])D_x^{-1}C_{u_1} - \tfrac{1}{3}C_uD_x^{-1}(C_{D_xQ_l} + D^*_{Q_l}[C_{u_1}]).\end{aligned}$$

Using the independence of the different summands, we conclude that

$$C_{Q_l} - D_{Q_l}[C_u] = 0 \qquad \text{and} \qquad A_{Q_l} + D^*_{Q_l}[A_u] = 0.$$

Therefore, using (6.1),

$$\delta\langle\, A_u\,; Q_l\,\rangle = D^*_{Q_l}[A_u] + D^*_{A_u}[Q_l] = 0,$$

so there exists $T_l \in \mathcal{U}$ such that $D_xT_l = A_uQ_l$. Notice that $A_uD_{Q_l} = D_xD_{T_l} - A_{Q_l}$. Hence

$$\begin{aligned}\mathfrak{L}_{Q_l}[\![\mathfrak{R}]\!] \sim\ & \tfrac{1}{3}(A_{D_xQ_l} - D_{Q_l}[A_{u_1}])D_x^{-1}A_u - \tfrac{1}{9}D_{Q_l}\left(C_uD_x^{-1}C_{u^2}D_x^{-1}A_u\right) + \\ & + \tfrac{1}{9}C_{Q_l}D_x^{-1}C_{u^2}D_x^{-1}A_u + \tfrac{1}{9}C_uC_{T_l}D_x^{-1}A_u + \\ & + \tfrac{1}{9}C_uD_x^{-1}\left(D^*_{T_l}[C_{u^2}] - C_{T_l}A_u - 3C_{D_xQ_l} - 3D^*_{Q_l}[C_{u_1}]\right).\end{aligned}$$

We conclude that

$$D^*_{T_l}[C_{u^2}] - C_{T_l}A_u - 3C_{D_xQ_l} - 3D^*_{Q_l}[C_{u_1}] = 0.$$

Therefore,

$$\mathbf{E}(C_{u^2}T_l - 3C_{u_1}Q_l) = D^*_{C_{u^2}}[T_l] + D^*_{T_l}[C_{u^2}] - 3(D^*_{C_{u_1}}[Q_l] + D^*_{Q_l}[C_{u_1}]) = 0,$$

since, according to Example 6.1,

$$D^*_{C_{u^2}}(T_l) = -C_{T_l}A_u, \qquad D^*_{C_{u_1}}(Q_l) = C_{D_xQ_l}.$$

Therefore, $C_{u^2}T_l - 3C_{u_1}Q_l \in \operatorname{Im} D_x$, which finishes the proof.

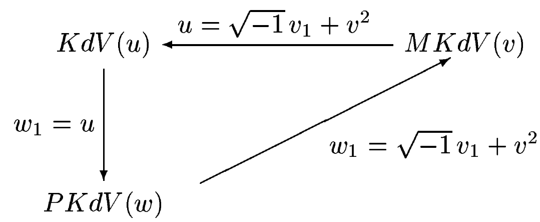
6.3 Modified KdV, case 2

The recursion operator (2.4) was recently constructed in [15] using the Lax operator, [16],

$$L = D_x^2 + \tfrac{2}{3} u D_x. \tag{6.5}$$

In the commutative case, the Lax operator (6.5) can be transformed into the Lax operator for MKdV1, $\widetilde{L} = D_x^2 - \frac{1}{3}u_1 - \frac{1}{9}u^2$, by a gauge transformation. In the noncommutative case, we could not find an operator to transform one to the other. This, perhaps, explains the existence of two versions of the noncommutative MKdV.

The Miura transformation can also be used to construct recursion operators, [29], p. 52. In the commutative case, the diagram



indicates the connections between the KdV and MKdV equations. The same holds between the noncommutative KdV, MKdV1 and PKdV. However, we do not know a Miura transformation for MKdV2, and so have been unable to put MKdV2 into this diagram.

Construction of the local symmetry hierarchy can also be based on the method of fractional powers of the associated Lax operator; see [6] for the commutative case, and [9] for the noncommutative KdV. The MKdV2 hierarchy is given by

$$L_{t_n} = [L_+^{\frac{2n+1}{2}}, L],$$

where $L_+^{\frac{2n+1}{2}}$ stands for all terms with positive powers of D_x. We compute

$$\begin{aligned} L^{\frac{1}{2}} = D_x &+ \tfrac{1}{3}u - \left(\tfrac{1}{6}u_1 + \tfrac{1}{18}u^2\right) D_x^{-1} + \left(\tfrac{1}{12}u_2 + \tfrac{1}{18}u_1 u + \tfrac{1}{18}u u_1 + \tfrac{1}{54}u^3\right) D_x^{-2} \\ &- \left(\tfrac{1}{24}u_3 + \tfrac{1}{24}u_2 u + \tfrac{1}{24}u u_2 + \tfrac{7}{72}u_1^2 + \tfrac{5}{216}u_1 u^2 \right. \\ &\qquad \left. + \tfrac{1}{36}u u_1 u + \tfrac{7}{216}u^2 u_1 + \tfrac{5}{648}u^4\right) D_x^{-3} + \cdots. \end{aligned}$$

Then

$$\begin{aligned} L_+^{\frac{5}{2}} = D_x^5 &+ \tfrac{5}{3}u D_x^4 + \left(\tfrac{5}{2}u_1 + \tfrac{5}{6}u^2\right) D_x^3 + \left(\tfrac{25}{12}u_2 + \tfrac{5}{18}u_1 u + \tfrac{25}{18}u u_1 + \tfrac{5}{54}u^3\right) D_x^2 \\ &+ \left(\tfrac{5}{8}u_3 + \tfrac{5}{72}u_2 u + \tfrac{5}{8}u u_2 + \tfrac{25}{72}u_1^2 - \tfrac{5}{216}u_1 u^2 \right. \\ &\qquad \left. + \tfrac{5}{108}u u_1 u + \tfrac{25}{216}u^2 u_1 - \tfrac{5}{648}u^4\right) D_x, \end{aligned}$$

which leads to the fifth order symmetry

$$\begin{aligned} &u_5 + \tfrac{5}{3}\left(uu_4 - u_4u + u_1u_3 - u_3u_1\right) \\ &\quad + \tfrac{5}{9}\left(u^2u_3 + u_3u^2 - 4uu_3u - u_2uu_1 - u_1uu_2 - 2u_1^3 - 3u_1u_2u - 3uu_2u_1\right) \\ &\quad + \tfrac{5}{27}\left(u_2u^3 - u^3u_2 + 3uu_2u^2 - 3u^2u_2u\right) \\ &\quad + \tfrac{10}{27}\left(u_1uu_1u - uu_1uu_1 + u_1^2u^2 - u^2u_1^2\right) + \tfrac{10}{81}\left(uu_1u^3 + u^2u_1u^2 + u^3u_1u\right). \end{aligned}$$

In [16], the recursive operator

$$T_{n+1} = D_xT_n + \tfrac{2}{3}uT_n + \sum_{j=0}^{n} T_jT_{n-j}, \qquad T_0 = -\tfrac{1}{3}u,$$

where $T_n \in \mathcal{U}$ was constructed. The trace forms $\mathrm{tr}(T_0), \mathrm{tr}(T_{2n+1}) \in \mathcal{R}$, where $n = 0, 1, \cdots$, are conserved densities. Therefore, one can construct infinitely many symmetries via the Hamiltonian flows obtained by applying the Hamiltonian operator $\mathfrak{H} = D_x + C_u + \dfrac{2}{9}C_uD_x^{-1}C_u$ to the associated cosymmetry $C_l = \mathbf{E}(\mathrm{tr}\, T_l)$. The resulting symmetries $Q_l = \mathfrak{H}C_l$ are all local due to Theorem 4.4. For example, the fifth order symmetry Q_5 is produced by applying $\mathfrak{H}$ to the cosymmetry

$$\begin{aligned} C_5 &= u_4 + \tfrac{2}{3}\left(uu_3 - u_3u\right) + \left(u_1u_2 - u_2u_1\right) - \\ &\quad - \tfrac{1}{9}\left(3u^2u_2 + 4uu_2u + 3u_2u^2\right) - \tfrac{2}{9}\left(2uu_1^2 + u_1uu_1 + 2u_1^2u\right) + \tfrac{2}{27}u^5. \end{aligned}$$

Remark 6.5. Following Dorfman, [7, 8], we call $D_tT + D_xX = 0$ a "conventional" conservation law of a differential equation if $T \in \mathcal{U}$ instead of $T \in \mathcal{R}$. For the KdV equation, $u_t = D_x(u_2 + u^2)$ is a "conventional" conservation law, which leads to cosymmetry 1 appearing in the D_x^{-1} term. For MKdV1, $D_tu^2 = D_x(uu_2 + u_2u - u_1^2 + u^4)$ is a "conventional" conservation law, which guarantees that $A_uu_t \in \mathrm{Im}\, D_x$. Interestingly, for MKdV2, none of the known conservation laws are conventional.

7 Operator Symmetries

Having extended our calculus of Fréchet and Lie derivatives from differential polynomials to differential operators, we can formally define the notion of a differential operator being a symmetry of a given evolution equation. However, it is not entirely clear what this really means in terms of the traditional interpretation of a symmetry generating a flow that maps solutions to solutions.

Definition 7.1. A differential operator $H \in \mathcal{D}$ will be called a *symmetry* of the evolution equation $u_t = K$ if it satisfies the bracket condition

$$[H, K] = D_H[K] - D_K[H] = 0.$$

A *cosymmetry* of the evolution equation $u_t = K$ is a differential operator $C \in \mathcal{U}$ satisfying

$$D_C[K] + D_K^*[C] = 0.$$

Example 7.2. Consider the commutator C_u. We find

$$D_{u_iu_j}[C_u] = (R_{u_j}D_x^i + u_iD_x^j)(C_u) = R_{u_j}C_{u_i} + u_iC_{u_j} = C_{u_iu_j}.$$

This reflects the fact that C_u is a trivial symmetry of any noncommutative evolution equation.

In the commutative case, the nonlocal terms of a recursion operator always take the form $PD_x^{-1}Q$, where P is a symmetry and Q is a cosymmetry of the equation, [24]. In the noncommutative case, this statement will be valid if we extend our notion of symmetry and cosymmetry to include linear (differential) operators. Furthermore, in the recursion operators (2.1) and (2.3), the D_x^{-1} appears twice in the nonlocal terms, and so one must use products of operators of the form $PD_x^{-1}Q$.

In the KdV recursion operator (2.1), we read the term $C_uD_x^{-1}C_uD_x^{-1}$ as $C_uD_x^{-1}1 \cdot C_uD_x^{-1}1$, where C_u is a trivial symmetry and 1 is a cosymmetry of the KdV equation. Similarly, in the MKdV1 recursion operator (2.3), we read $C_uD_x^{-1}C_{u^2}D_x^{-1}A_u$ as $C_uD_x^{-1}A_u \cdot C_uD_x^{-1}A_u$, where C_u is again a trivial symmetry and A_u is a cosymmetry of MKdV1. These observations deserve further investigation.

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