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INTEGRABILITY OF KLEIN-GORDON EQUATIONS*

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Abstract. Using the Painlevé test, it is shown that the only integrable nonlinear Klein-Gordon equations $u_{xt}=f(u)$ with f a linear combination of exponentials are the Liouville, sine-Gordon (or sinh-Gordon) and Mikhailov equations. In particular, the double sine-Gordon equation is not integrable.

Key words. completely integrable, Painlevé property, Klein-Gordon equation

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In [7], two of the present authors (J. B. M. and P. J. O.) considered the problem of which nonlinear Klein-Gordon equations

$$u_{xt}=f(u) \tag{1}$$

are completely integrable. They referred to the Ablowitz-Ramani-Segur (ARS) conjecture, [2], [3] which states that if a partial differential equation is integrable by the inverse scattering transform (IST) method, then all its reductions to ordinary differential equations have the Painlevé property, i.e., all their moveable singularities are poles. It was shown in [7] that if $f(u)$ is a linear combination of exponentials, the only equations of type (1) whose corresponding ordinary differential equation for travelling wave solutions

$$u(x,t)=w(\xi)=w(x-ct),$$

arising from the invariance of (1) under the group of translations, has the Painlevé property, are those of the form

$$u_{xt}=c_2e^{2\beta u}+c_1e^{\beta u}+c_{-1}e^{-\beta u}+c_{-2}e^{-2\beta u} \tag{2}$$

for constants $c_2,\cdots,c_{-2}$. In fact the singularities of u are not really poles, but rather "pure logarithms" in the sense that u_x, u_t and $\exp(\beta u)$ have only poles. This extension was included in the ARS conjecture as originally stated.

A paradox apparently remained; namely that the form (2), which does include the well-known Liouville equation (only one nonzero c_i), the sine-Gordon equation ($c_2=c_{-2}=0$, $c_1=-c_{-1}$, $\beta=i$) and the Mikhailov equation ($c_1=c_{-2}=0$), [8], [9], [12] all of which are known to be completely integrable, also includes the double sine-Gordon equation ($c_2=-c_{-2}$, $c_1=-c_{-1}$, $\beta=i$), which is *not* integrable. Indeed numerical studies have shown that its travelling wave solutions do not behave like solitons under collisions, [1]. This apparent problem, however, is easily resolved if one considers a second one-parameter group of symmetries of (1),

$$(x,t)\rightarrow(\lambda x,\lambda^{-1}t),\qquad \lambda>0,$$

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leading to a different form

$$u(x,t)=w(xt)=w(\xi)$$

for the group-invariant solutions. Then w satisfies the ordinary differential equation

$$\xi w''+w'=f(w) \tag{3}$$

where $\xi=xt$.

In order to apply the ARS conjecture, we need to analyze the singularities of solutions of (3) in the case f has the form (2). To eliminate logarithmic singularities, set $v=\exp(\beta w)$, so v satisfies

$$v''=\frac{v'^2}{v}-\frac{v'}{\xi}+\frac{c_2v^3+c_1v^2+c_{-1}+c_{-2}v^{-1}}{\xi}. \tag{4}$$

All second order ordinary differential equations with the Painlevé property have been classified by Painlevé and Gambier and can be reduced, through a change of variables, to one of fifty canonical forms—see [5]. An obvious candidate to reduce (4) to is the equation

$$w''=\frac{w'^2}{w}-\frac{w'}{z}+\frac{\alpha w^2+\beta}{z}+\gamma w^3+\frac{\delta}{w}, \tag{5}$$

which is canonical form number 13 in [5, p. 335]. Thus we need to determine when (4) can be reduced to the canonical form (5). The change of variables

$$\xi=z^p, \qquad v=z^q w \tag{6}$$

reduces (4) to

$$w''=\frac{w'^2}{w}-\frac{w'}{z}+\sum b_n w^{n+1}z^{nq+p-2}, \tag{7}$$

where the sum is on $n=2,1,-1$ and -2 (but not 0!) and the b_n's and c_n's are related by irrelevant powers of p.

In order for (7) to agree with (5), we need to have all of the following four conditions to hold:

a) either $b_2=0$ or $2q+p-2=0$;
b) either $b_1=0$ or $q+p-2=-1$;
c) either $b_{-1}=0$ or $-q+p-2=-1$;
d) either $b_{-2}=0$ or $-2q+p-2=0$.

Clearly this is not possible if all the b_n's are nonzero. If only one b_n is nonzero, there are no difficulties. The original equation was the Liouville equation $u_{xt}=e^{\beta u}$, which is integrable using the Bäcklund transformation

$$w_x=u_x+\exp\frac{\beta}{2}(u+w), \qquad w_t=-u_t-\frac{2}{\beta}\exp\frac{\beta}{2}(u-w)$$

with w satisfying $w_{xt}=0$, [4]. Alternatively, we can set

$$u=\frac{1}{\beta}\log\left[2v_xv_te^v/(e^v-1)^2\beta\right],$$

leading to $v_{xt}=0$. If $b_2=b_{-2}=0$, $b_1b_{-1}\neq 0$ then (4) is already in the canonical form (6) with $\gamma=\delta=0$, $\alpha\beta\neq 0$, so no change of variable is required, i.e., $p=1$, $q=0$. This is the case of the sine (and sinh-) Gordon equations, which are integrable by inverse scattering methods. If $b_1=b_{-1}=0$, $b_2b_{-2}\neq 0$, then we again have the sine-Gordon equation, but we have made a different choice for defining v in terms of u. This should not alter the Painlevé character of the equation, and indeed $p=2$, $q=0$ will satisfy conditions a)−d). Curiously enough this reduces (4) to a canonical form (5) with $\alpha=\beta=0$, $\gamma\delta\neq 0$, which is *not* the same as above. This shows that the same equation can be reduced by different changes of variables to *different* canonical forms.

If $b_1=b_{-2}=0$, $b_2b_{-1}\neq 0$ (or, respectively, $b_{-1}=b_2=0$, $b_1b_{-2}\neq 0$), then we have the Mikhailov equation

$$u_{xt}=be^{2\beta u}+b'e^{-\beta u},$$

which was shown to be integrable by a 3×3 matrix scattering problem, [8], [9], [12]. In this case, conditions a)–d) have the solution $p=\frac{4}{3}$, $q=\frac{1}{3}$ (respectively $p=\frac{4}{3}$, $q=-\frac{1}{3}$), and hence this reduction of Mikhailov's equation does have the Painlevé property.

Finally, if $b_1b_2\neq 0$, even if $b_{-2}=b_{-1}=0$, one would need $p=0$, $q=1$ for a)–d) to be satisfied. But this is not an acceptable change of variables as ξ would not really depend on z. Thus we cannot reduce (4) to the canonical form (5) if $b_1b_2\neq 0$ whatever the values of b_{-1} and b_{-2}. By symmetry, the same holds if $b_{-1}b_{-2}\neq 0$ no matter what values b_1 and b_2 have. Of course, this does not completely prove that (4) in this case does not have the Painlevé property since (6) is not the only possible choice of change of variables and it may be possible to reduce (4) to some other canonical form. Indeed, we have just seen that starting with $b_1=b_{-1}=0$, $b_2b_{-2}\neq 0$, the change of variables (6) with $p=2$, $q=0$ is a rather contrived way to reduce (4) to the canonical form (5) compared with the more obvious choice $v=w^2$. To check that if $b_1b_2\neq 0$, equation (4) does not have the Painlevé property, one could study the behavior of its singularities and show that they are not pure poles, or, alternatively, follow through Painlevé's deviation of the fifty canonical forms, as in [5], and see that it does not fall into one of these categories.

Instead of doing this, however, it is just as easy to check the Painlevé property for the partial differential equation (2) directly, using the method introduced by Weiss et al. [10], [11], and improved by Kruskal [6]. First set $v=\exp(\beta u)$, so (2) becomes

$$vv_{xt}=v_xv_t+c_2v^4+c_1v^3+c_{-1}v+c_{-2}. \tag{8}$$

Suppose $v(x,t)$ is singular along the curve

$$\psi(x,t)=x+\varphi(t)=0$$

with φ arbitrary. Let us expand v near this curve in a Laurent series

$$v=\psi^r\sum_{n=0}^{\infty}\alpha_n(t)\psi^n. \tag{9}$$

Without loss of generality, we can suppose $c_2\neq 0$. (If $c_2=0$, $c_{-2}\neq 0$ change variables by replacing v by $1/v$; if $c_2=c_{-2}=0$, change v to v^2 if $c_1\neq 0$ and v^{-2} if only $c_{-1}\neq 0$.) Balancing the lowest powers of ψ in both sides of (8), we have one possible solution $r=-1$. Equating the coefficients of ψ^{-4}, we get

$$2\alpha_0^2\frac{d\varphi}{dt}=\alpha_0^2\frac{d\varphi}{dt}+c_2\alpha_0^4,$$

so

$$c_2\alpha_0^2=\frac{d\varphi}{dt}. \tag{10}$$

Substituting (9) into (8) and identifying the coefficients of ψ^{n-4} gives an equation for all of the α_n's except α_2 which does not appear when one equates the coefficients of ψ^{-2}. Indeed $n=2$ is the "resonance" in the ARS terminology, [3]. More precisely, the coefficient of ψ^{-3} is

$$2\alpha_0\alpha_1\frac{d\varphi}{dt}-\alpha_0\frac{d\alpha_0}{dt}=-\alpha_0\frac{d\alpha_0}{dt}+4c_2\alpha_0^3\alpha_1+c_1\alpha_0^3,$$

which by (10) gives the expression

$$\alpha_1=-c_1/2c_2 \tag{11}$$

for α_1. At order ψ^{-2}, we find

$$2\alpha_0\alpha_2\frac{d\varphi}{dt}-\alpha_1\frac{d\alpha_0}{dt}=-2\alpha_0\alpha_2\frac{d\varphi}{dt}+6c_2\alpha_0^2\alpha_1^2+4c_2\alpha_0^3\alpha_2+3c_1\alpha_0^2\alpha_1.$$

By (10), (11) all the terms cancel except for $\alpha_1 d\alpha_0/dt$, the value of which is

$$\alpha_1\frac{d\alpha_0}{dt}=-\frac{c_1}{4c_2^{3/2}}\frac{d^2\varphi/dt^2}{\sqrt{d\varphi/dt}}.$$

If this quantity does not vanish, one cannot find an expansion for v in the form (9). Terms of the form

$$\psi(\alpha_2+\tilde{\alpha}_2\log\psi)$$

are needed at that order, and at higher and higher orders in ψ one will need higher and higher powers of logarithms of ψ. Such an expansion is not of Painlevé type.

In [7] the proof of the ARS conjecture was done by first showing that the solution $v(x,t)$ must be meromorphic when both x and t are allowed to assume complex values. This was then specialized to gain the required Painlevé property of group-invariant solutions. It also immediately gives the modification of the ARS conjecture by Weiss et al. [10], [11] which predicts that an equation will not be integrable if some nonpolar singularity exists on a line $\psi(x,t)=x+\varphi(t)=0$ for φ arbitrary. In particular, if $d^2\varphi/dt^2\neq 0$, then c_1 must vanish. This leads to an understanding of the result of [7]. For translation-invariant solutions, we have only straight lines $x=\lambda t+k$, so $d^2\varphi/dt^2=0$ in this case, and a nonvanishing α_1 does not pose any difficulty. Alternatively, the original ARS conjecture for the scale-invariant solutions would also lead to the same conclusion $c_1=0$.

So far we did not find any restrictions on c_{-1} and c_{-2}. However, if c_{-2} does not vanish, a similar argument shows that $c_{-1}=0$, namely we look at solutions with the

$$v=\alpha_0\psi+\alpha_1\psi^2+\cdots,$$

which, because of the coefficient v multiplying the highest derivative v_{xt} in (9) may be singular when $v=0$. Indeed, if $c_{-1}\neq 0$ one finds that it is a singular logarithmic point, with logarithms first entering at order ψ^3.

In conclusion the analysis of the singular behavior of solutions shows that the solutions are meromorphic along arbitrary curves if and only if $b_1b_2=0$ and $b_{-1}b_{-2}=0$. We conclude that the only possible integrable cases are the Liouville, sine-Gordon (sinh-Gordon) and Mikhailov equations, in perfect agreement with the known integrable character of these equations and the nonintegrable character of the double sine-Gordon equation, as suggested by numerical studies of its solutions.

REFERENCES

[1] M. J. ABLOWITZ, M. D. KRUSKAL AND J. F. LADIK, *Solitary wave collisions*, SIAM J. Appl. Math., 36 (1979), pp. 428–437.

[2] M. J. ABLOWITZ, A. RAMANI AND H. SEGUR, *Nonlinear evolution equations and ordinary differential equations of Painlevé type*, Lett. Nuovo Cimento, 23 (1978), pp. 333–338.

[3] ______, *A connection between nonlinear evolution equations and ordinary differential equations of P-type* I, J. Math. Phys., 21 (1980), pp. 715–721.

[4] R. L. ANDERSON AND N. H. IBRAGIMOV, *Lie–Bäcklund Transformations in Applications*, SIAM Studies 1, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1979.

[5] E. L. INCE, *Ordinary Differential Equations*, Dover, New York, 1956.

[6] M. D. KRUSKAL, private communication.

[7] J. B. MCLEOD AND P. J. OLVER, *The connection between partial differential equations soluble by inverse scattering and ordinary differential equations of Painlevé type*, this Journal, 14 (1983), pp. 488–506.

[8] A. V. MIKHAILOV, *Integrability of a two-dimensional generalization of the Toda chain*, Soviet Phys. JETP Lett., 30 (1979), pp. 414–418.

[9] ______, *The reduction problem and the inverse scattering method*, Physica, 3D (1981), pp. 73–117.

[10] J. WEISS, M. TABOR AND G. CARNEVALE, *The Painlevé property for partial differential equations*, J. Math. Phys., 24 (1983), pp. 522–526.

[11] J. WEISS, *The Painlevé property for partial differential equations.* II: *Bäcklund transformation, Lax pairs and the Schwarzian derivative*, J. Math. Phys., 24 (1983), pp. 1405–1413.

[12] A. P. FORDY AND J. D. GIBBONS, *Integrable nonlinear Klein–Gordon equations and Toda lattices*, Comm. Math. Phys., 77 (1980), pp. 21–30.