On The Construction of Deformations of Integrable Systems

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1. Introduction

In Gardner, Green, Kruskal and Miura's pioneering study of the remarkable properties of the Korteweg-de Vries (KdV) equation, the original proof of the existence of infinitely many conservation laws was based on a one-parameter family of transformations due to Gardner, generalizing a transformation of Miura between solutions of the KdV and modified KdV equations, [22]. Chen, [3], showed how the Gardner transformation could be combined with an obvious discrete symmetry of the modified KdV equation to rederive the Bäcklund transformations for the KdV equation originally due to Wahlquist and Estabrook, [27]. In the same paper, Chen introduced a general method for constructing Gardner-like transformations and consequently Bäcklund transformations for other "completely integrable" evolution equations associated with the AKNS scheme of inverse scattering, [1]. In essence, Chen's method consists of the introduction of projective coordinates for the associated AKNS scattering problem, followed by a somewhat ad hoc change of coordinates made so as to "simplify" the resulting equations. The distinguishing feature of Chen's transformations was that they were nonlinear transformations, involving derivatives of the dependent variable, between two different evolution equations. The present paper grew out of an attempt to understand Chen's change of coordinates from a group theoretic standpoint. The term "deformation" for all of these generalizations of the Gardner transformation was introduced by Kupershmidt, [18], since in all examples a parameter μ can be introduced so that the deformation reduces to the identity when $\mu = 0$.

The appropriate framework in which to place the present results is the reinterpretation of the Wahlquist-Estabrook prolongation structure for an evolution equation, [28], in terms of a flat connection on a principal bundle. (See Kobayshi-Nomizu, [13], or Sternberg, [26], for the necessary differential geometric concepts.) This view point was originally put forth by Hermann, [10], [11], and, implicitly, by Corones, [4]. In this paper, it will be shown that each system of evolution equations with decomposable prolongation structure (as defined below) has a nontrivial deformation. The corresponding modified equations arise naturally in terms of "additive subgroup coordinates." In this fashion, Chen's method finds its group-theoretic interpretation, and the above changes of coordinates no longer seem mysterious - they are just those that change the projective coordinate into the additive subgroup coordinate. Bäcklund transformations, as before, arise from discrete symmetries of the modified equations, or, as in Fordy's recent preprint, [5], automorphisms of an associated Lie algebra. Not all systems with nontrivial deformations, though, possess Bäcklund transformations.

In section two, the connection / principal bundle formulation of prolongation structures is presented in a form amenable to our construction. Each evolutionary system with nontrivial prolongation can be interpreted as the integrability conditions for a connection in a principal bundle with Lie group G. For systems in one space and one time variable, x and t, respectively, the base manifold is two dimensional with coordinates x,t . A connection is then described by a pair of differential operators (vector fields)

$$L = D_x + \beta(u) + \epsilon(\lambda) \quad ,$$
$$M = D_t + \tilde{\beta}(u,\lambda) \quad , \qquad (1.1)$$

where D_x , D_t are (total) derivatives, λ the spectral parameter (s) , and β , ϵ and $\tilde{\beta}$ maps into the Lie algebra $\mathfrak{g}$ of G . The connection is flat (integrable) if the Zakharov-Shabat, [30], equations

$$[L,M]=0 \tag{1.2}$$

hold. This provides a prolongation structure for a system of evolution equations provided the integrability conditions (1.2) are equivalent to the given system.

The prolongation structure is decomposable if there is a direct sum decomposition $\mathfrak{g}=\mathfrak{K}+\mathfrak{H}$, where $\mathfrak{K}$, $\mathfrak{H}$ are subalgebras, such that $\beta(u)\in\mathfrak{K}$, $\epsilon(\lambda)\in\mathfrak{H}$ for all u,λ. Reinterpret the scattering problem

$$L\Gamma=0$$

as a differential equation on the Lie group G (rather than using a representation of G) so that $\Gamma(x,t)\in G$. For Γ near the identity, we can "separate variables"

$$\Gamma=\Theta\,\Psi\ ,$$

with $\Theta\in K$, $\Psi\in H$, where K,H are the Lie subgroups associated with $\mathfrak{K}$, $\mathfrak{H}$ respectively. The aformentioned additive subgroup coordinates are found by setting

$$\Theta=\exp(\theta)\ ,\ \theta\in\mathfrak{K}\ .$$

The assumption of decomposability ensures that θ satisfies a system of differential equations involving only θ, u, λ, and these provide the spatial half of the deformation equations. The temporal half is found by performing a similar decomposition on the system $M\Gamma=0$. Elimination of u between these two halves results in the modified evolutionary system, bearing the same relation to the original system that the modified KdV equation bears to the KdV equation. This constitutes our group-theoretic generalization of Chen's construction, which is now entirely systematic and natural.

The well known AKNS examples associated with $SL(2,\mathbb{R})$, along with a couple of novel equations, are reworked using our methods in section 5. This includes a derivation of the most general form of the deformation equations for evolution equations having $SL(2,\mathbb{R})$ prolongation structures. An application to the three wave interaction equations, [12], whose prolongation structure involves $SL(3,\mathbb{R})$, is presented in the final section. The main object is the construction of a new type of deformation equation.

Unfortunately, not all examples of evolutionary systems known to possess deformations and / or Bäcklund transformations can be assimilated into the present framework. The most notable ommissions are the sytems associated with higher order Lax operators whose modifications were found by Kupershmidt and Wilson, [19], and the generalized Klein-Gordon equations modelled on Toda lattices for semisimple Lie algebras, [15], or, more generally, Kac-Moody algebras, [2], discussed by Mikhailov, Olshanetsky and Perelomov, [21], Fordy and Gibbons, [6], [7] and Sattinger, [25]. The key difference is that the deformations now involve higher order derivatives. This is presumably reflected in some nilpotent generalization of our decomposability assumption on the prolongation, but a complete explanation must await a future publication.

For simplicity, we work entirely in the real domain, although generalizations to complex equations and groups are readily apparent. All Lie groups are assumed to be connected, and all manifolds, bundles, etc. to be smooth.

2. Prolongation Structures

Let $\pi : Z \to Y$ be a smooth fiber bundle with q-dimensional fiber over a p-dimensional base manifold X . Let $J_k Z \to Y$ be the corresponding k-jet bundle. A system of k th order differential equations for sections of Z is defined by a subbundle (or, more generally, subvariety) $\Delta \subset J_k Z$, the solutions of which are local sections $u: Y \to Z$ with $j_k u \subset \Delta$.

According to Hermann, [11], a Wahlquist-Estabrook prolongation structure for the system Δ can be formulated as follows. Let $\hat{\pi} : P \to X$ be a principal bundle with structure group G and Lie algebra $\mathfrak{g}$. A connection on P , [13], [26], is a p-dimensional "horizontal" differential system $H \subset TP$ transversal to the tangent spaces to the fibers and invariant under the induced action of right multiplication of G on P . The corresponding connection form is a $\mathfrak{g}$-valued one form ω on P defined via projection of TP onto the tangent space to the fiber along H . Thus ω is a section of the bundle.

$$T^*P \otimes \mathfrak{g}_P \to P$$

which is invariant under the induced action of G . (Here $\mathfrak{g}_P \to P$ is the bundle obtained by identifying $\mathfrak{g}$ as the space of right-invariant vector fields on G .) The curvature of the connection form ω is the $\mathfrak{g}$-valued two-form, (i.e. section of $\wedge_2 T^*P \otimes \mathfrak{g}_P$)

$$\Omega = D\omega = d\omega + \frac{1}{2}[\omega,\omega] \quad .$$

The connection is flat if $\Omega = 0$, the curvature vanishes. This is equivalent to the integrability of the horizontal differential system H via Frobenius theorem and the formula, [13],

$$\omega([v,w]) = -2\Omega(v \wedge w), \tag{2.1}$$

valid for horizontal vector fields $v, w : P \to H$.

Let Λ be an r-dimensional manifold. In practice $\Lambda = \mathbb{R}$, and the coordinate λ is the spectral parameter appearing in the scattering problem for a completely integrable system.

Definition 2.1 A (P,Λ)-prolongation structure for the system of differential equations $\Delta \subset J_k Z$ is defined by a bundle map

$$\beta : J_{k-1}Z \times \Lambda \to T^*P \otimes \mathfrak{g}_P$$

such that

i) (Horizontality)

For any section $u : Y \to Z$, any $\lambda \in \Lambda$, $\beta(j_{k-1}u , \lambda)$ defines a connection form on P .

ii) (Integrability) Let

$$D\beta : J_k Z \times \Lambda \to \Lambda_2 T^*P \otimes \mathfrak{g}_P$$

be the induced map giving the curvature of the connection. Then

$$(D\beta)^{-1}\{0\} = \Delta \quad ,$$

where 0 is the zero section of $\Lambda_2 T^*P \otimes \mathfrak{g}_P$. In other words, for any section $u : Y \to Z$, any $\lambda \in \Lambda$, the induced connection $\beta(j_{k-1}u , \lambda)$ is flat if and only if u is a solution to the system of differential equations defined by Δ .

To see that the above definition is consistent with Corones' construction, [4], let $Y_0 \subset Y$ be a coordinate chart with coordinates $y = (y^1, \ldots, y^p)$ so that $P|Y_0 \simeq Y_0 \times G$, the identification depending on the choice of a local section of P over Y_0 . Since the connection is G-invariant, it suffices to describe it on $Y_0 \times \{e\}$, e being the identity of G . For any $\rho = (y,e) \in P|Y_0$, note that $TP_\rho \simeq TY_y \otimes g$, $T^*P_\rho \simeq T^*Y_y \otimes g^*$, g^* being the dual of g . At each ρ the horizontal subspace H_ρ is spanned by vectors of the form

$$v_i|_\rho = \partial_i|_y + \alpha_i(y), \ i = 1,\dots,p \ ,$$

where $\partial_i = \partial/\partial y^i$, $\alpha_i(y) \in g$. The corresponding connection form at ρ - $\omega_\rho \in (T^*Y_y \oplus g^*) \otimes g$ - is thus given by

$$\omega_\rho = \mathbb{1} - \sum_{i=1}^{p} \alpha_i dy^i \ , \qquad (2.2)$$

where $\mathbb{1}$ is the identity element of $g^* \otimes g \simeq \mathrm{Hom}(g\ ,\ g)$. The connection is flat if and only if the commutation relations

$$0 = [v_i\ , v_j] = \partial_i \alpha_j - \partial_j \alpha_i + [\alpha_i\ ,\ \alpha_j]\ ,\ i,j = 1,\dots,p \qquad (2.3)$$

hold for each $y \in Y_o$.

If $Z|Y_o = Y_o \times U$, where $U \approx \mathbb{R}^q$ has coordinates $u = (u^1,\dots,u^q)$, then a k-th order system of differential equations

$$\Delta(\ j_k u) = 0 \qquad (2.4)$$

has a prolongation structure if and only if there are functions

$$\beta_i(\ j_{k-1}u, \lambda) \in g\ ,\ i = 1,\dots,n\ ,$$

such that the corresponding connection form

$$\omega = \mathbb{1} - \sum \beta_i dy^i \qquad (2.5)$$

is flat whenever u satisfies the system (2.4). In other words, the commutation relations

$$D_j\beta_i - D_i\beta_j = [\beta_i, \beta_j]\ ,\ i,j = 1,\dots,p \qquad (2.6)$$

are equivalent to the system (2.4). (Here D_i denotes the total derivative with respect to y^i.) These are Corones' generalizations of the standard Wahlquist-Estabrook prolongation equations.

In practice, however, interesting examples where the above scheme yields important results have thus far only been found when the base manifold Y is two dimensional. (The Kadontsev-Petviashvili / two dimensional Korteweg-de Vries equation, while possessing a Lax equation of Zakharov - Shabat type, [16], [30] does not appear to fit into a finite-dimensional form as above.) Thus let $Y = X \times T$, $X = \mathbb{R}$ with "spatial" coordinate x and $T = \mathbb{R}$ with "temporal" coordinate t. In this case, the integrability relations assume the more familiar Zakharov-Shabat form

$$D_t\beta - D_x\tilde{\beta} = [\beta, \tilde{\beta}] \quad . \tag{2.7}$$

If we write $L = D_x + \beta$, $P = -\tilde{\beta}$, we recover the Lax representation

$$L_t = [P, L] \tag{2.7'}$$

for the system (2.4). This can be written even more succinctly as

$$[L, M] = 0 \ , \tag{2.7''}$$

where $M = D_t + \tilde{\beta}$.

In most cases of interest to date, β and $\tilde{\beta}$ only depend on the spatial derivatives u. Moreover, the Lax operator separates:

$$L = D_x + \beta(u^{(m)}) + \varepsilon(\lambda) \ ,$$

where $u^{(m)}$ denotes the spatial derivatives of u of orders $\leq m$. The system (2.7) thus assumes quasi-evolutionary form

$$D_t\beta(u^{(m)}) = K(x, t, u^{(k)}) \quad .$$

Here we restrict our attention to linear Lax operators, so $\beta(u_m)$ is a linear isomorphism from the vector space of m-th order derivatives of u, u_m, to $\mathfrak{g}$. In this case, we get evolution equations of the form

$$\frac{\partial^{m+1} u}{\partial t \partial x^m} = K(x,t,u^{(k)}) \quad . \tag{2.8}$$

(For the Korteweg-de Vries equation and its generalizations $m=0$, whereas for the sine-Gordon and related equations $m=1$.).

If $u(x,t)$ is a solution of this evolution equation, then for each $\lambda \in \Lambda$ the corresponding connection is flat in $P \approx Y \times G$. Thus for each $\rho_o = (x_o, t_o, g_o) \in P$ there is a section S_{ρ_o} of P passing through ρ_o, whose tangent space at each point is spanned by the vector fields

$$\begin{aligned} L(u) &= \partial_x + \beta(u_m(x,t)) + \epsilon(\lambda) \quad , \\ M(u) &= \partial_t + \tilde{\beta}(u^{(k-1)}(x,t),\lambda) \quad . \end{aligned} \tag{2.9}$$

(S_{ρ_o} is global since Y is simply connected.) Realizing S_{ρ_o} as the graph of a function $\Gamma : Y \to G$, the corresponding scattering equations are given by

$$\Gamma_x + (\beta(u_m) + \epsilon(\lambda))\Gamma = 0 \quad , \tag{2.10}$$

$$\Gamma_t + \beta(u^{(k-1)},\lambda)\Gamma = 0 \quad . \tag{2.11}$$

Since these equations hold on G, they can be realized as matrix ordinary differential equations by any convenient faithful representation $\rho : G \to GL(n)$ of G. The first equation (2.10) realizes $\rho(\Gamma)$ as the fundamental matrix of eigenfunctions for the first order matrix equation with potentials $\rho(\beta(u_m))$ and eigenvalues (spectral parameters) $\rho(\epsilon(\lambda))$. Similarly, the second equation (2.11) governs the time evolution of the eigenfunctions. In this form, the nonlinear evolution equation appears as the integrability condition for these two matrix equations. Its solution is amenable to an inverse scattering analysis of (2.10) along the lines of Ablowitz, Kaup, Newell and Segur, [1], for $G = SL(2,\mathbb{R})$ and Kaup, [12], for $G = SL(3,\mathbb{R})$. We will not pursue this direction here.

The equations (2.10-11) can also be interpreted on the Lie algebra level, so that, for instance, (2.10) becomes

$$\Gamma_x\Gamma^{-1} + \beta(u_m) + \varepsilon(\lambda) = 0 ,$$

where $\Gamma_x\Gamma^{-1} \in \mathfrak{g}$. Suppose $\Gamma(x,t)$ lies in the image of $\exp \mathfrak{g} \subset G$, so we can write

$$\Gamma(x,t) = \exp(\gamma(x,t))$$

for some smooth function $\gamma : Y \to \mathfrak{g}$. (As long as the initial values of Γ lie in $\exp(\mathfrak{g})$ this is possible for (x,t) in a small enough subdomain of Y.).

<u>Lemma</u> 2.2 Suppose $\Gamma(x) = \exp(\gamma(x))$, where $\gamma : \mathbb{R} \to \mathfrak{g}$ is smooth. Then

$$\frac{d\Gamma}{dx} \cdot \Gamma^{-1} = \varphi(\operatorname{ad} \gamma(x)) \cdot \frac{d\gamma}{dx} ,$$

where

$$\varphi(t) = (e^t - 1)/t .$$

<u>Proof</u> In a matrix representation,

$$\Gamma = \sum_{n=0}^{\infty} \frac{1}{n!} \gamma^n ,$$

hence

$$\Gamma_x = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j=0}^{n-1} \gamma^j \gamma_x \gamma^{n-j-1}$$

Moreover, $\Gamma^{-1} = \exp(-\gamma)$, hence

$$\Gamma_x\Gamma^{-1} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{n!m!} \sum_{j=0}^{n-1} \gamma^j \gamma_x \gamma^{n+m-j-1}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left[\sum_{i=j+1}^{k+1} \frac{(-1)^i}{i!(k+1-i)!} \right] \gamma^j \gamma_x \gamma^{k-j}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \gamma^j \gamma_x \gamma^{k-j}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)!} (\mathrm{ad}\ \gamma)^k \gamma_x$$

$$= \varphi(\mathrm{ad}\ \gamma)\gamma_x ,$$

where we have used the power series for φ and an elementary identity for binomial coefficients. (It is easily checked that all the above series converge absolutely for all γ .)

Therefore the scattering equations can be recast into the equivalent Lie algebraic form

$$\begin{aligned} &\varphi(\mathrm{ad}\ \gamma(x,t)) \frac{\partial \gamma}{\partial x} + \beta(u_m) + \epsilon(\lambda) = 0 , \\ &\varphi(\mathrm{ad}\ \gamma(x,t)) \frac{\partial \gamma}{\partial t} + \tilde{\beta}(u^{(k-1)}, \lambda) = 0 . \end{aligned} \qquad (2.12)$$

The use of this transformation will become apparent in the following section. Analysis of the Jordan form of $\mathrm{ad}\ \gamma$ shows that $\varphi(\mathrm{ad}\ \gamma)$ is invertible provided $\mathrm{ad}\ \gamma$ has no complex eigenvalues of the form $2n\pi i$ for n a non-zero integer. Thus for nice initial data $\gamma(0,0)$ the ordinary differential equations (2.12) will be regular for (x,t) small, but in general have singular points when $\mathrm{ad}(\gamma)$ has an eigenvalue of this form.

These are precisely the points where $\Gamma = \exp\gamma$ passes out of the range of the exponential map:

Proposition 2.3 Let $G_o = \exp(\mathfrak{g}) \subset G$. Then $G_o \cap \partial G_o = \{\exp\gamma \mid \varphi(\text{ad }\gamma) \text{ is not invertible}\}$.

Proof

Let $e_1, \ldots, e_n$ be a basis for $\mathfrak{g}$, so each $\alpha \in \mathfrak{g}$ is coordinatized by $\alpha = \alpha^1 e_1 + \ldots + \alpha^n e_n$. By smoothness $g = \exp(\gamma)$ lies on the boundary ∂G_o if and only if the tangents to the curves $\exp\alpha(t)$, $-\epsilon < t < \epsilon$, $\alpha(0) = \gamma$, corresponding to curves in $\mathfrak{g}$ passing through γ , do not span the tangent space TG_g , since otherwise g would lie in the interior of G_o . In other words, there is a linear relation among tangent vectors

$$\sum_{i=1}^{n} c_i \frac{\partial}{\partial\alpha^i} [\exp\alpha] \Big|_{\alpha=\gamma} = 0 \quad .$$

Multiplying by g^{-1} and using lemma 2.2,

$$\varphi(\text{ad }\alpha)c = 0$$

for $c = \Sigma\, c_i e_i \neq 0$. This proves the proposition.

Thus the "infinitesimal" scattering equations (2.12) have a singularity, i.e. either $|\gamma(x,t)| \to \infty$ or $\det[\text{ad }\varphi(\gamma(x,t))] \to 0$ as $(x,t) \to (x_o, t_o)$ if and only if $\Gamma(x,t) \to \partial G_o$ as $(x,t) \to (x_o, t_o)$. (Note that $\Gamma(x,t)$ is defined globally.) Nevertheless, the solutions of (2.12) are globally defined for many examples of interest.

3. Deformations

Consider an evolution equation of the form (2.8) with prolongation structure defined by the operators

$$L = D_x + \beta(u_m) + \epsilon(\lambda) \ ,$$
$$M = D_t + \tilde{\beta}(u^{(k-1)}, \lambda) \ , \tag{3.1}$$

where $u_m = \partial^m u/\partial x^m$, and β , $\tilde{\beta}$, δ lie in the Lie algebra $\mathfrak{g}$ of the prolongation group G . To implement our construction of deformations we make the following assumption on the Lax operator L .

The Lie algebra $\mathfrak{g}$ decomposes as a direct sum of two subalgebras (not necessarily ideals)

$$\mathfrak{g} = \mathfrak{K} + \mathfrak{H} \ . \tag{3.2}$$

Furthermore, we require the maps β , ϵ in (3.1) to split:

$$\beta(u_m) \in \mathfrak{K} \ , \qquad \epsilon(\lambda) \in \mathfrak{H}$$

for all u_m, λ . The following lemma due to Harish-Chandra, [8], provides a mechanism for "separating variables" in this situation.

Lemma 3.1 Given a decomposition of $\mathfrak{g}$, (3.2), let K,H be the analytic subgroups of G corresponding to the Lie algebras $\mathfrak{K},\mathfrak{H}$ respectively. Then the map $\mu(k,h) = kh$, $k \in K$, $h \in H$, is regular from $K \times H$ into G .

Let G^* be the image of μ in G . Modulo the covering group of $K \times H \to G^*$, each $g \in G^*$ can be uniquely written as a product $g = kh$ In many important cases $G^* = G$ is simply connected and the representation is unique.

If $\Gamma(x,t)$, the solution of the scattering equations (2.10-11), lies in G^* , it factors

$$\Gamma(x,t) = \Theta(x,t)\ \Psi(x,t) \ , \tag{3.3}$$

with $\Theta \in K$, $\Psi \in H$. Substituting (3.3) into the scattering equation (2.10) we find

$$\Theta_x \Theta^{-1} + \mathrm{Ad}\ \Theta(\Psi_x \Psi^{-1}) + \beta(u_m) + \epsilon(\lambda) = 0 \ . \tag{3.3}$$

This can be separated using the direct sum decomposition (3.2) of $\mathfrak{g}$. Assuming $\Theta \in K_o$, the image of $\exp : \mathfrak{K} \to K$, so $\Theta = e^{\theta}$, the linear transformation $\mathrm{Ad}\ \Theta$ can be written in block matrix format

$$\mathrm{Ad}\ \Theta = \begin{pmatrix} R(\theta) & S(\theta) \\ 0 & T(\theta) \end{pmatrix} , \tag{3.4}$$

where $R(\theta) = \mathrm{Ad}\ \Theta \mid \mathfrak{K}$, $S(\theta) : \mathfrak{H} \to \mathfrak{K}$ and $T(\theta) : \mathfrak{H} \to \mathfrak{H}$ are linear maps. Using lemma 2.2, (3.3) separates as

$$\varphi(\mathrm{ad}\ \theta)\theta_x + S(\theta)\psi + \beta(u_m) = 0 \ ,$$

$$T(\theta)\psi + \epsilon(\lambda) = 0 \ ,$$

where $\psi = \Psi_x \Psi^{-1} \in \mathfrak{H}$. Eliminating ψ leads to the transformation

$$\varphi(\mathrm{ad}\ \theta)\theta_x - S(\theta)T(-\theta)\epsilon(\lambda) + \beta(u_m) = 0 \ , \tag{3.5}$$

between u and θ . . Equivalently.

$$\varphi(\mathrm{ad}\ \theta)\theta_x + R(\theta)S(-\theta)\epsilon(\lambda) + \beta(u_m) = 0 \ , \tag{3.5'}$$

since

$$\mathrm{Ad}\ \Theta^{-1} = \begin{pmatrix} R(-\theta) & -R(-\theta)S(\theta)T(-\theta) \\ 0 & T(-\theta) \end{pmatrix} .$$

Since β is a linear isomorphism by assumption, (3.5) can be solved for u_m :

$$u_m = F(\theta, \theta_x, \lambda) \quad . \tag{3.6}$$

Similarly, writing the operator M as

$$M = D_t + \gamma(u^{(k-1)}, \lambda) + \delta(u^{(k-1)}, \lambda) \ , \tag{3.7}$$

where $\gamma : J_{k-1}\mathcal{Z} \times \Lambda \to \mathfrak{K}$, $\delta : J_{k-1}\mathcal{Z} \times \Lambda \to \mathfrak{H}$, (2.11) reduces to the temporal counterpart to (3.5):

$$\varphi(\text{ad } \theta)\theta_t + R(\theta)S(-\theta)\delta(u^{(k-1)}, \lambda) + \gamma(u^{(k-1)}, \lambda) = 0 \quad . \tag{3.8}$$

We will refer to (3.5,8) as the <u>deformation equations</u>. In the special case $m = 0$, the derivatives of u in (3.8) can be substituted for according to (3.6), leading to the <u>modified equation</u>

$$\varphi(\text{ad } \theta)\theta_t + R(\theta)S(-\theta)\delta(F^{(k-1)}(\theta, \theta_x, \lambda), \lambda) + \gamma(F^{(k-1)}(\theta, \theta_x, \lambda), \lambda) = 0 \tag{3.9}$$

More generally, the modified equation for θ will be integro-differential. If the subgroup $\mathfrak{K}$ is abelian, $R(\theta)$ and $\varphi(\text{ad } \theta)$ are both identity maps, so the modified equation (3.9) is a genuine evolution equation for $\theta(x,t)$. More generally, singularities will occur when $\varphi(\text{ad } \theta)$ is no longer invertible. The key point is that one evolution equation for u has been modified by the nontrivial deformation into a different evolution equation in θ . Solutions of the modified equation then yield solutions of the original equations. Discrete symmetries of the modified equations, as emphasized by Chen, lead to Bäcklund transformations for the original equation. Examples will bear out the efficacy and generality of this generalization of Chen's approach to Bäcklund and Miura transformations.

4. Remarks on "True" Deformations and Bäcklund Transformations

The deformation equations (3.5,8) are not true deformations in the sense of Kupershmidt because they do not contain a parameter μ such that the deformation reduces to the identity and the modified equation reduces to the original equation as $\mu \to 0$. In the simplest cases, meaning when the subgroup K is abelian and can up to covering be identified with its Lie algebra, and the spectral parameter λ enters so that $\epsilon(0) = 0$, λ itself enters as the true deformation parameter. At $\lambda = 0$, the transformation (5.5) reduces to

$$\theta_x + \beta(u_m) = 0 ,$$

which, since β is an isomorphism, relates θ_x linearly to u_m. Differentiating the modified equation with respect to x and setting $\lambda = 0$ yields the original evolution equation, as can readily be checked. More generally, when K is abelian, but $\epsilon(0) \neq 0$, one must scale θ and look at $\lambda \to \infty$ or some other singularity of ϵ. Thus, in the case of the KdV equation, $\lambda^{-1} = \mu$ becomes the true deformation parameter. More generally, when K is no longer abelian, one must look at the infinitesimal versions of the modified equations. These, under an appropriate scaling of θ, should yield the original evolution equations, but the details become very messy and impractical in general. It is often easier to treat each case individually.

As is well known, [3], discrete symmetries of the modified equations yield Bäcklund transformations for the original evolutionary system. Fordy, [5], has pointed out that these are in reality automorphisms of the underlying Lie algebra. As yet there is no general method for finding which automorphism, if any, is appropriate. Since the constructions are well-known in the standard examples, we leave aside the general question for future investigation.

It is of interest to put these special types of Bäcklund transformations into the jet-bundle theoretic formulations of Kosmann-Schwarzbach, [14], and Pirani, Robinson and Shadwick, [23]. In particular, the question arises as to the relation of the flat connection arising from the prolongation structure of the evolution equation with the flat connection constructed directly from the Backlund transformation in the latter reference. Again, this will not be treated here.

5. Transformations Associated with $sl(2,\mathbb{R})$

As originally proposed by Zakharov and Shabat, [30], and re-emphasized by Ablowitz, Kaup, Newell and Segur (AKNS) [1], many of the fundamental examples of soliton equations can be integrated by scattering problems associated with the Lie group $SL(2,\mathbb{R})$. Here we derive the most general deformations for such evolution equations, and show how the standard examples (KdV, modified KdV, sine-Gordon) fit into our theory. These calculations are for the most part reconstructions of the original method of Chen, [3], in our group-theoretic context.

The first step is to classify the possible decompositions (3.2) of $sl(2,\mathbb{R})$. Assuming $\mathfrak{K}$ to be one-dimensional, $\mathfrak{H}$ must clearly be a Borel (maximal solvable) subalgebra, [9], all of which are conjugate. Let n, a be a basis for $\mathfrak{H}$ with commutation relation

$$[n,a] = 2n \quad . \tag{5.1}$$

In the usual 2×2 matrix representation, of $sl(2,\mathbb{R})$, $\mathfrak{H}$ can be taken as the subalgebra of lower triangular matrices, so that

$$n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \qquad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

Using the fact that $\mathfrak{H}$ is its own normalizer, it is easy to show that there are precisely three non-conjugate one-dimensional complements $\mathfrak{K}$ to $\mathfrak{H}$: $\mathfrak{K}_-$, $\mathfrak{K}_+$, $\mathfrak{K}_0$ with basis element e_-, e_+, e_0, respectively, having the following commutation relations:

$$[e_\nu, n] = a \ , \ [e_\nu, a] = -2e_\nu + 4\nu n \quad , \tag{5.2}$$

where $\nu = -1$, $+1$ or 0, so $e_{-1} = e_-$, etc. In the above matrix representation,

$$e_\nu = \begin{pmatrix} 0 & 1 \\ \nu & 0 \end{pmatrix} \quad , \quad \nu = \pm 1, 0 \quad .$$

No interesting evolution equations arise where the subalgebra $\mathfrak{K}$ in (3.4) is two-dimensional, hence up to conjugacy there are three distinct types of scattering problems associated with $SL(2,\mathbb{R})$. The connections on the principal bundle are spanned by

$$\begin{aligned} L_\nu &= D_x + u\, e_\nu + \xi(\lambda)a + \eta(\lambda)n \;, \\ M &= D_t + A\, a + B\, e_\nu + C\, n \;, \end{aligned} \tag{5.3}$$

where ξ, η are fixed functions of the spectral parameter $\lambda \in \mathbb{R}$, and A,B,C are real-valued functions on $J_{k-1}Z \times \Lambda$, i.e. functions of λ , u and the derivatives of u of orders $\leq k-1$. The integrability conditions (2.7) and commutation relations (5.1,2) lead to generalized AKNS relations:

$$\begin{aligned} & D_x A - \eta B + uC = 0 \;, \\ & D_x B - 2uA + 2\xi B = u_t \;, \\ & D_x C + (2\eta + 4\nu u)A - 4\nu\xi B - 2\xi C = 0 \quad . \end{aligned} \tag{5.4}$$

Certain choices of $\xi(\lambda)$, $\eta(\lambda)$ lead to interesting, nontrivial solutions of the integrability equations (5.4), but it is an open question as to the most general form of these functions allowable.

Corresponding to each such solution, we get a deformation associated with the given decomposition of $sl(2,\mathbb{R})$.

<u>Theorem</u> 5.1 Let A,B,C be nontrivial solutions of the integrability equations (5.4) leading to an evolution equation $u_t = K$, so $K = [D_x B - 2uA + 2\xi B]|_{\lambda=0}$. The deformation equations for this evolution equation assume the forms:

$$
\begin{aligned}
\text{i)} \quad \mathfrak{K}_- : \quad & \theta_x + u + \xi \sin 2\theta - \eta \sin^2\theta = 0 , \\
& \theta_t + B + A \sin 2\theta - C \sin^2\theta = 0 . \\
\text{ii)} \quad \mathfrak{K}_+ : \quad & \theta_x + u + \sinh 2\theta - \sinh^2\theta = 0 , \\
& \theta_t + B + A \sinh 2\theta - C \sinh^2\theta = 0 . \\
\text{iii)} \quad \mathfrak{K}_0 : \quad & \theta_x + u + 2\xi\theta - \eta\theta^2 = 0 , \\
& \theta_t + B + 2A\theta - C\theta^2 = 0 .
\end{aligned}
\tag{5.5}
$$

(Of course, the actual modified equation is obtained by substituting for u according to the first equation into the second equation.)

We do the case of $\mathfrak{K}_0$, the others being similar. Let $\Theta = \exp(\theta\, e_0)$. Then from (5.2),

$$
\mathrm{Ad}\, \Theta (Aa + Be_0 + Cn) = (A + \theta C)a + (B - 2\theta A - \theta^2 C)e_0 + Cn
$$

Therefore, the linear map $S(\theta): \mathfrak{H} \to \mathfrak{K}$ in (3.4) assumes the form

$$
S(\theta)a = -2\theta e_0 , \quad S(\theta)n = -\theta^2 e_0 .
$$

Since $\mathfrak{K}$ is abelian, the deformation equations (3.5,8) are then easily seen to assume the form listed in the statement of the theorem.

The main point of interest is that the deformation equations (5.5) arise naturally from the underlying group theory. Chen's somewhat ad hoc original method of introducing a projective coordinate and then, in the $\mathfrak{K}_\pm$ cases, making an inspired change of coordinates to introduce the trigonometric or hyperbolic functions now has the proper interpretation. The change of coordinates is precisely that one that converts the projective coordinate into the additive subgroup coordinate for the abelian subgroup $K \subset SL(2,\mathbb{R})$ generated by the subalgebra $\mathfrak{K}$. We now illustrate the use of theorem 5.1 for some of the more classical examples.

Example 5.2 The Korteweg-de Vries Equation.

The appropriate decomposition is that using $\mathfrak{R}_0$, with

$$\xi(\lambda) = \lambda \quad , \quad \eta(\lambda) = 1 \quad .$$

The spatial half of the deformation equations (5.5) is

$$\theta_x + 2\lambda\theta - \theta^2 + u = 0 \ ,$$

which is nothing but Gardner's generalization, [22], of the Miura transformation. The KdV equation

$$u_t = u_{xxx} - 6uu_x \tag{5.6}$$

arises from the solution

$$A = 4\lambda^3 - 2\lambda u + u_x \ ,$$
$$B = 4\lambda^2 u - 2\lambda u_x + u_{xx} - 2u^2 \ ,$$
$$C = 4\lambda^2 - 2u \ ,$$

of (5.5). The temporal half of the deformation is then

$$\theta_t + 4\lambda^2 u - 2\lambda u_x + u_{xx} - 2u^2 + 2\theta(4\lambda^3 - 2\lambda u + u_x) - \theta^2(4\lambda^2 - 2u) = 0 \ ,$$

which, on substitution, simplifies to the Modified KdV equation

$$\theta_t = \theta_{xxx} - 6\theta^2\theta_x + 4\lambda\theta\theta_x \ . \tag{5.7}$$

Invariance under the symmetry $\theta \to -\theta$, $\lambda \to -\lambda$ leads to the Bäcklund transformation for the KdV equation, [3], [27].

Example 5.3. Modified KdV and sine-Gordon Equations.

Here the $\mathfrak{R}_-$ decomposition is appropriate with $\xi(\lambda) = \lambda$, $\eta(\lambda) = 0$.
The deformation equations have spatial half

$$\theta_x + u + \lambda \sin 2\theta = 0 \quad .$$

For the modified KdV equation,

$$A = 4\lambda^3 + 2\lambda u^2 ,$$
$$B = 4\lambda^2 u - 2\lambda u_x + u_{xx} + 2u^3 ,$$
$$C = -2\lambda u_x ,$$

so that

$$u_t = u_{xxx} + 6u^2 u_x \quad . \tag{5.8}$$

The "modified" modified KdV equation then assumes the form, [3],

$$\theta_t = \theta_{xxx} + 2\theta_x^3 + 6\lambda(\sin^2 2\theta)\theta_x \quad . \tag{5.9}$$

Again, the discrete symmetry $\theta \to -\theta$, $\lambda \to \lambda$ leads to the Bäcklund transformation. For the sine-Gordon equation

$$w_{xt} = \sin w , \tag{5.10}$$

let $u = \frac{1}{2} w_x$, so

$$A = 1/4\lambda \cos w , \quad B = 1/4\lambda \sin w , \quad C = 1/2\lambda \sin w \quad .$$

The deformation is then

$$\theta_x + w_x + 2\lambda \sin \theta = 0$$
$$\theta_t + \frac{1}{2\lambda} \sin(w + \theta) = 0 ,$$

where, for simplicity, we have replaced θ by $\frac{1}{2}\theta$. These can be written as

$$\theta_{xt} = \sqrt{1 - (2\lambda\theta_t)^2} \sin \theta , \tag{5.11}$$

[3], or in integro-differential form

$$\theta_t = \frac{1}{2\lambda}\sin(2\lambda \int \sin\theta\, dx) \quad . \qquad (5.11')$$

In either case the same discrete symmetry $\theta \mapsto -\theta$, $\lambda \mapsto \lambda$ yields the Bäcklund transformation.

Example 5.4 Harry Dym Equation.

Here we use the $\mathfrak{R}_0$ decomposition with $\xi(\lambda) = 0$, $\eta(\lambda) = \lambda$. The solutions

$$A = \lambda(u^{-1/2})_x ,$$
$$B = -2\lambda u^{1/2} + (u^{-1/2})_{xx} ,$$
$$C = -2\lambda^2 u^{-1/2}$$

of (5.4) yields the evolution equation

$$u_t = (u^{-1/2})_{xxx} , \qquad (5.12)$$

whose integrability was first noticed by Dym, cf [17], [20]. The deformation equations

$$\theta_x + u - \lambda\theta^2 = 0,$$
$$\theta_t - 2\lambda u^{1/2} + (u^{-1/2})_{xx} + 2\lambda(u^{-1/2})_x\theta + 2\lambda^2 u^{-1/2}\theta^2 = 0,$$

yield the modified Dym equation:

$$\theta_t + 2\lambda(\theta(\lambda\theta^2 - \theta_x)^{-1/2})_x + ((\lambda\theta^2 - \theta_x)^{-1/2})_{xx} = 0 \quad . \qquad (5.13)$$

In this case, no relevant nontrivial discrete symmetry appears to be available, so we cannot find a Bäcklund transformation.

Example 5.5 Some Novel Equations.

Different choices of the way the spectral parameter λ enters into the scattering operator give different evolution equations. Here we present two new evolution equations arising from $SL(2,\mathbb{R})$ prolongations, together with their deformations. Both of these example use the decomposition corresponding to $\nu = 0$.

For $\xi(\lambda) = 1$, $\eta(\lambda) = \lambda$, the functions

$$A = (-2u^{-1/2} + (u^{-1/2})_x)\lambda \ ,$$
$$B = -2u^{1/2}\lambda - 2(u^{-1/2})_x + (u^{-1/2})_{xx} \ ,$$
$$C = -2u^{-1/2}\lambda^2$$

satisfy the integrability conditions (5.4) for the evolution equation

$$u_t = (u^{-1/2})_{xxx} - 4(u^{-1/2})_x \ ,$$

a variant of the Harry Dym equation. The deformation equations (5.5) read

$$\theta_x + u + 2\theta - \lambda\theta^2 = 0 \ ,$$
$$\theta_t - 2u^{1/2}\lambda - 2(u^{-1/2})_x + (u^{-1/2})_{xx} +$$
$$+(-4u^{-1/2} + 2(u^{-1/2})_x)\lambda\theta + 2u^{-1/2}\lambda^2\theta^2 = 0 \ ,$$

which can be combined into a somewhat complicated modified equation. As for the Dym equation, no Bäcklund transformation is apparent.

The choice $\xi(\lambda) = \lambda$, $\eta(\lambda) = \lambda$ has solutions

$$A = -e^{2w}\lambda^2 + w_x e^{2w}\lambda$$
$$B = -2w_x e^{2w}\lambda + (w_{xx} + 3w^2)e^{2w} \ ,$$
$$C = -e^{2w}\lambda^2 \ ,$$

leading to the evolution equation

$$w_{xt} = (w_{xxx} + 8w_x w_{xx} + 6w_x^3)e^{2w} \ ,$$

where $w_x = u$. The deformation equations are

$$\theta_x + w_x + \lambda\theta - \lambda\theta^2 = 0 \ ,$$

$$\theta_t - (-2w_x\lambda + (w_{xx} + 3w^2) + (-2\lambda^2 + w_x\lambda)\theta + \lambda^2\theta^2)e^{2w} = 0 \, .$$

The modified equation is now of integro-differential kind, but with no Bäcklund transformation.

6. Three-Wave Interaction Equations.

In this section we discuss the deformation theory for a real version of the three-wave interaction equations

$$
\begin{aligned}
u_t - \rho u_x &= \xi vw , \\
v_t - \sigma v_x &= -\eta \; uw , \\
w_t - \tau \; w_x &= \zeta \; uv ,
\end{aligned}
\tag{6.1}
$$

where $\rho, \sigma, \tau, \xi, \eta, \zeta$ are real constants, with $\xi, \eta, \zeta, \rho-\sigma, \rho-\tau, \sigma-\tau$ all positive (other signs can be treated analogously.) As originally noticed by Zakhavov and Manakov, [29], and developed by Kaup, [12], this system can be integrated via a scattering problem associated with the Lie group $SL(3,\mathbb{R})$. To describe this prolongation structure, we first rescale u,v,w so that (6.1) is in the "normalized" form

$$
\begin{aligned}
\upsilon_t - \rho u_x &= (\sigma - \tau) vw , \\
v_t - \sigma v_x &= (\tau - \rho) uw , \\
w_t - \tau w_x &= (\rho - \sigma) vw \; .
\end{aligned}
\tag{6.2}
$$

Consider the Iwasawa decomposition, [9],

$$
sl(3,\mathbb{R}) = \mathfrak{K} + \mathfrak{H} ,
$$

where $\mathfrak{K}$ is a maximal compact and $\mathfrak{H}$ a Borel subalgebra. Since all such decompositions are conjugate, we may use the standard 3×3 matrix representation of $sl(3,\mathbb{R})$ with $\mathfrak{K}$ being the subalgebra $so(3,\mathbb{R})$ of skew-symmetric matrices and $\mathfrak{H}$ the subalgebra of lower triangular matrices. In this representation, the connection leading to (6.2) is spanned by

$$L = D_x + \begin{pmatrix} 0 & u & v \\ -u & 0 & w \\ -v & -w & 0 \end{pmatrix} + \lambda \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \epsilon \end{pmatrix} ,$$

$$M = D_t + \begin{pmatrix} 0 & \rho u & \sigma v \\ -\rho u & 0 & \tau w \\ -\sigma v & -\tau w & 0 \end{pmatrix} + \lambda \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

where

$$\gamma = 2\tau - \sigma - \rho \ , \ \delta = 2\sigma - \tau - \rho \ , \ \epsilon = 2\rho - \sigma - \tau \ ,$$

and

$$a = (\rho + \sigma)\tau - 2\sigma\rho \ , \ b = (\rho + \tau)\sigma - 2\rho\tau \ , \ c = (\sigma + \tau)\rho - 2\sigma\tau \ .$$

Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

be a basis of $\mathfrak{K}$, so that $\theta = \theta_1\sigma_1 + \theta_2\sigma_2 + \theta_3\sigma_3 \in \mathfrak{K}$ can be represented by the column vector $\underset{\sim}{\theta} = (\theta_1 , \theta_2 , \theta_3)^T$. For this basis of $\mathfrak{K}$ it is easy to check that $\mathrm{Ad}\, e^{\theta}|\mathfrak{K}$ has the matrix representation

$$R(\underset{\sim}{\theta}) = \begin{pmatrix} \cos\theta - c\theta_1^2 & c\theta_1\theta_2 + s\theta_3 & -c\theta_1\theta_3 + s\theta_2 \\ c\theta_1\theta_2 - s\theta_3 & \cos\theta - c\theta_2^2 & c\theta_2\theta_3 + s\theta_1 \\ -c\theta_1\theta_3 - s\theta_2 & c\theta_2\theta_3 - s\theta_1 & \cos\theta - c\theta_3^2 \end{pmatrix}$$

where $c = (\cos\theta - 1)/\theta^2$, $s = \sin\theta/\theta$ and $\theta = \|\underset{\sim}{\theta}\| = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{1/2}$.

Similarly, $\varphi(\mathrm{ad}\,\underset{\sim}{\theta})|\mathfrak{K}$ has matrix representation

$$\Phi(\underset{\sim}{\theta}) = \begin{pmatrix} 1 + \hat{s}(\theta_2^2 + \theta_3^2) & \hat{s}\theta_1\theta_2 - c\theta_3 & -\hat{s}\theta_1\theta_3 - c\theta_2 \\ \hat{s}\theta_1\theta_2 + c\theta_3 & 1 + \hat{s}(\theta_1^2 + \theta_3^2) & \hat{s}\theta_2\theta_3 - c\theta_1 \\ -\hat{s}\theta_1\theta_3 + c\theta_2 & 3\theta_2\theta_3 + c\theta_1 & 1 + \hat{s}(\theta_1^2 + \theta_2^2) \end{pmatrix}$$

where $\hat{s} = (\sin\theta - \theta)/\theta^3$. Finally we must compute the transformation

$S(\underset{\sim}{\theta}) : \mathfrak{H} \to \mathfrak{K}$, but it clearly suffices to compute the restriction of $S(\underset{\sim}{\theta})$ to the diagonal elements in $\mathfrak{H}$, since these are the only ones appearing in L , M. Note first that in the 3×3 matrix representation of $sl(3, \mathbb{R})$,

$$\exp(\underset{\sim}{\theta}) = R(\hat{\underset{\sim}{\theta}})$$

for $\underset{\sim}{\theta} = (\theta_1, \theta_2, \theta_3) \in \mathfrak{K}$ and $\hat{\underset{\sim}{\theta}} = (\theta_3, \theta_2, \theta_1)$. Therefore, if $\Delta = \mathrm{diag}(\alpha, \beta, \gamma)$, $\alpha + \beta + \gamma = 0$, to find $S(\underset{\sim}{\theta})\Delta$ we need merely compute the supra-diagonal elements of $R(\hat{\underset{\sim}{\theta}})\Delta R(-\hat{\underset{\sim}{\theta}})$. Therefore

$$S(\theta)\Delta = \begin{pmatrix} (\beta-\alpha)s \cos\theta\ \theta_1 + (\alpha-\gamma)cs\theta_3^2\theta_1 + (\gamma-\beta)cs\theta_2^2\theta_1 - (\alpha\theta_3^2 + \beta\theta_2^2 + \gamma(2\theta^2+\theta_2^2))c^2 - 3\gamma c \\ (\alpha-\beta)cs\theta_3^2\theta_2 + (\gamma-\alpha)s \cos\theta\ \theta_2 + (\beta-\gamma)cs\theta_1^2\theta_2 - (\alpha\theta_3^2 + \beta(2\theta^2+\theta_2^2) + \gamma\theta_1^3)c^2 - 3\beta c \\ (\beta-\alpha)cs\theta_2^2\theta_3 + (\alpha-\gamma)cs\theta_1^2\theta_3 + (\gamma-\beta)s \cos\theta\ \theta_3 - (\alpha(2\theta^2+\theta_3^2) + \beta\theta_2^2 + \gamma\theta_1^2)c^2 - 3\alpha c \end{pmatrix}$$

$$\equiv \underset{\sim}{s}(\alpha, \beta, \gamma, \theta)$$

Therefore we have the following deformation equations:

$$\Phi(\theta)\ \theta_x + u + R(\theta)\ \underset{\sim}{s}(\gamma, \delta, \epsilon, -\theta) = 0 ,$$

$$\Phi(\theta)\theta_t + \Delta u + R(\theta)\underset{\sim}{s}(a, b, c, -\theta) = 0 ,$$

where $\Delta = \mathrm{diag}(\rho, \sigma, \tau)$. Thus the modified three-wave interaction equations assume the form

$$\Phi(\theta)\theta_t - \Delta\Phi(\theta)\theta_x - \Delta\ R(\theta)\underset{\sim}{s}(\gamma, \delta, \epsilon, -\theta) + R(\theta)\underset{\sim}{s}(a, b, c, -\theta) = 0$$

which, if written out in full detail, is extremely complicated. No discrete symmetries are readily apparent, so we do not seem to get a Bäcklund transformation.

7. Some Open Problems

Several important gaps remain to be filled in the above scheme.

i) How can the construction be generalized to equations with prolongation structures where the Lie algebra and scattering operators do not decompose into two subalgebras where the variables u_m lie in one and the spectral parameter λ in the other subalgebra? Clearly some such generalization must work as the examples of deformations and Bäcklund transformations for higher order Lax pairs, [19], and generalized Klein-Gordon equations modelled on Toda systems associated with semi-simple Lie algebras, [6], [7], demonstrate. Fordy, in a recent preprint, [5], shows how generalizing Chen's construction to "projective representation" might be used in these cases, but the underlying group-theoretical structure is not apparent. More work is needed to fully comprehend these systems.

ii) Since the modified equations arise from completely integrable evolution equations, these equations must also be integrable. In particular they must have a prolongation structure, and the question is how this is related to the prolongation structure of the original equations. Unfortunately it is not always true that the underlying Lie algebras of the prolongation structures of the two systems are the same; Pirani and Soliani [24], have shown this for the "modified-modified KdV equation," (5.9). This could lead to some very interesting Lie-algebraic constructions.

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