Quantization of bi-Hamiltonian systems

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One of the distinguishing features of soliton equations is the fact that they can be written in Hamiltonian form in more than one way. Here we compare the different quantized versions of the soliton equations arising in the AKNS inverse scattering scheme. It is found that, when expressed in terms of the scattering data, both quantized versions are essentially identical.

In 1975, one of the present authors[1] showed how to obtain the quantized levels of the nonlinear Schrödinger equation using the action-angle variables (canonical coordinates) of the AKNS scattering data. The symplectic form used to effect the reduction to canonical coordinates was based on the standard Hamiltonian structure for the nonlinear Schrödinger equation. The method used was a nonlinear generalization of one of the standard methods for the second quantization of the electromagnetic field. As presented in the textbook by Schiff,[2] one takes the classical electromagnetic field and decomposes it into normal modes (Fourier components). The key idea in this approach is that the classical electromagnetic Hamiltonian will decompose into a sum of noninteracting classical Hamiltonians, each of which has just two degrees of freedom and is easily quantized by itself. This method of quantization bypasses all the inherent difficulties of fully quantizing the system, including the factor-ordering problem, defining the quantum field operators for the fundamental fields, etc.[3] It is fundamentally based on the symmetries of the classical system, and reduces the problem to one of quantizing noninteracting particles.[4] In this way, the original difficult second quantization problem is reduced to a simpler set of noninteracting problems. The advantage of this simpler solution is tremendous when one considers the information that one can glean from it. First, one can obtain the spacings of the energy levels. One also discovers which quantum variables will commute, and which modes will have a particle-like behavior. Of course, for a full quantum theory, one still has to deal with a number of remaining difficult problems, including finding a consistent factor-ordering for the quantum operators, evaluating matrix elements, etc. Unfortunately, the solution to this larger quantization problem may well be multivalued.[3] However, in the meantime, one has been able to immediately isolate the above mentioned important features of second quantization, and, very importantly, those quantities which would have the same common solution for every possible consistent second quantization. Thus, any difficulty which would be found at this level would also be present in *any* quantum field theory. And a study by this method can provide valuable insight into the structure of the more thorny parts of the second-quantization problem.

The symplectic form used in Ref. 1 to effect the reduction to canonical coordinates was based on the first Hamiltonian structure for the nonlinear Schrödinger equation. In 1978, Magri[5] showed how many soliton equations, including the nonlinear Schrödinger equation, could be written as bi-Hamiltonian systems, meaning that they have two distinct, but compatible, Hamiltonian structures. Indeed, his fundamental result showed that, subject to some technical hypotheses[5,6] any bi-Hamiltonian system is completely integrable in the sense that it has infinitely many conservation laws in involution and corresponding commuting Hamiltonian flows.

From the viewpoint of quantum mechanics, the existence of more than one Hamiltonian structure for a given classical mechanical system raises the possibility of there existing more than one quantized version of this system, even at the level of quantization considered in Ref. 1. The resulting ambiguity in the quantization procedure raises serious physical doubts as to the mathematical framework of quantization. However, the main result to be proven here is that, for AKNS soliton equations,[7] both quantized versions are essentially the same. We demonstrate that, in terms of the respective canonical coordinates on the scattering data, the two Hamiltonians have identical expressions, and hence identical quantum versions. Indeed, we conjecture that this phenomenon is true in general: *quantization does not depend on the underlying Hamiltonian structure*. (The results of Dodonov *et al.*,[8] in which an ambiguity in the quantization procedure for certain finite-dimensional bi-Hamiltonian systems is supposedly demonstrated, are erroneous, since they fail to incorporate the important topological properties of phase space properly in their picture. Indeed, their ambiguity is just a version of the ambiguity inherent in the quantization of two-dimensional Hamiltonian systems, which we discuss in detail below.) Moreover, we will see that for the other members of the associated hierarchy of soliton equations the only difference in the quantum versions is in the choice of weighting factor for the quantum operators corresponding to the continuous spectrum, the weight being determined by the classical dispersion relation, and the replacement of the bound state Hamiltonians. Thus, the effect of quantizing different members of the soliton hierarchy will only be significant for the bound states/solitons.

Our presentation relies heavily on the notation and results in earlier papers by Kaup and Newell[1,9,10] on the closure of the squared eigenfunctions for the AKNS scattering

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problem. The key to our result is the well-known fact that the recursion operator, which is built out of the two Hamiltonian operators for the system[5,6] is essentially the squared eigenfunction operator. Since variations in the potential for the AKNS scattering problem are expressed in terms of the squared eigenfunctions, the second symplectic form can be simply written down in explicit form. In terms of the scattering data, it differs from the first symplectic form only by a weighting factor in the continuous spectrum, and a change in the discrete components. However, the corresponding difference in weighting factors for the two Hamiltonians exactly cancels out the weighting factor for the two symplectic forms, while the discrete components reduce simply to the quantization of a two-dimensional Hamiltonian system, based on different symplectic structures. Thus, the entire quantum ambiguity reduces to the simple matter of an ambiguity in the quantization of two-dimensional Hamiltonian systems, a problem that is easily handled.

Our notation is as follows. Hamilton's equations are

$$\partial_t Q^\alpha = J^{\alpha\beta}\partial_\beta H, \tag{1}$$

where $Q=\{Q^\alpha\}$ are the dynamical variables (the p's and the q's), $J=[J^{\alpha\beta}]$ is the Hamiltonian operator, which determines the underlying Hamiltonian structure of the phase space, and H is the Hamiltonian function or density. For instance, for a harmonic oscillator, one would take

$$Q=\begin{pmatrix} q\\ p\end{pmatrix},\quad J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\quad \text{and}\quad H=\tfrac{1}{2}(p^2+q^2).$$

When Q is a function of a continuous variable, the sum over the dummy indices in (1) is understood to include the appropriate integration, and the partial derivative is understood to be a functional derivative instead. The Poisson bracket determined by such a Hamiltonian operator has the form

$$\{F,G\}=(\partial_\alpha F)J^{\alpha\beta}\partial_\beta G, \tag{2}$$

which requires the symplectic two-form to be

$$\Omega=\tfrac{1}{2}\,dQ^\alpha\wedge J^{-1}_{\alpha\beta}\,dQ^\beta. \tag{3}$$

For the harmonic oscillator, this reduces to the familiar canonical form

$$\Omega=dp\wedge dq. \tag{4}$$

Therefore, the operator J needs to be skew adjoint, and satisfy the additional condition that the Poisson bracket (2) satisfy the Jacobi identity, which is equivalent to the requirement that the two-form Ω can be closed.[6]

Before presenting the main results, we discuss a simple but crucial fact that any two-dimensional Hamiltonian system has a unique quantized version, even though it has many different Hamiltonian structures. In terms of the standard Hamiltonian structure prescribed by the canonical two-form (4), Hamilton's equations take the classical form[11]

$$p_t=-\frac{\partial H}{\partial q},\quad q_t=\frac{\partial H}{\partial p}. \tag{5}$$

In $\mathbb{R}^2$, any nonzero two-form $\lambda(p,q)dp\wedge dq$ is always closed, and hence determines a Hamiltonian operator

$$\tilde{J}=\begin{pmatrix} 0 & -\dfrac{1}{\lambda}\\ \dfrac{1}{\lambda} & 0\end{pmatrix}.$$

It is easy to see that (5) can be written in Hamiltonian form using this second Hamiltonian structure if and only if λ is a function of the Hamiltonian H. In this case, the new Hamiltonian function is

$$H_2(p,q)=\Phi[H(p,q)],$$

where $\Phi(\xi)$ is any nonvanishing scalar function, and

$$\Omega_2=\Phi'[H(p,q)]dp\wedge dq \tag{6}$$

is the second symplectic form. Re-expressing Ω_2 in canonical form will lead to new canonical variables $\tilde{p}$, $\tilde{q}$, and an ostensibly different quantized version. However, provided this transformation does not affect the phase space topology, it is not hard to see that these two quantized versions will end up being identical, at least in the semi-classical limit, and so there is no ambiguity in the (semi-classical) quantization of two-dimensional Hamiltonian systems.

We now turn to our problem at hand. For simplicity, we will consider the general nonlinear Schrödinger equation

$$iq_t=-q_{xx}+2rq^2, \tag{7a}$$

$$ir_t=r_{xx}+2qr^2, \tag{7b}$$

in detail. However, our arguments will work equally well for any other soliton equation associated with the AKNS spectral problem[7]; see the remarks at the end of the paper. For $r=\pm q^*$, (7) reduces to the single equation

$$iq_t=-q_{xx}\pm 2(q^*q)\,q, \tag{8}$$

which is the form of the nonlinear Schrödinger equation in which all physical constants, e.g., $\hbar$, m, etc., have been set equal to 1. According to Magri,[5] the nonlinear Schrödinger equation can be written as a bi-Hamiltonian system

$$\Psi_t=J_1\,\partial H_1=J_2\,\partial H_2. \tag{9}$$

The first Hamiltonian can be identified with the (signed) energy

$$H_1=\pm E=\int_{-\infty}^{\infty}(q_xr_x+q^2r^2)dx, \tag{10}$$

while the second Hamiltonian is the field momentum

$$H_2=P=i\int_{-\infty}^{\infty}(rq_x-qr_x)dx. \tag{11}$$

The two Hamiltonian operators are given by

$$J_1=\sigma_2=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}, \tag{12}$$

$$J_2=\frac{1}{2}\sigma_1\partial_x+\begin{pmatrix} q\displaystyle\int_{-\infty}^{x} q & -q\displaystyle\int_{-\infty}^{x} r\\ -r\displaystyle\int_{-\infty}^{x} q & r\displaystyle\int_{-\infty}^{x} r\end{pmatrix},\quad \sigma_1=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}. \tag{13}$$

(In our notation,[6] we have omitted the delta functions used by some authors.) Moreover, these Hamiltonian structures are compatible, in the sense that any linear combination $c_1J_1+c_2J_2$ is also Hamiltonian. Therefore, according to the theorem of Magri the operator

$$R = J_2 \cdot J_1^{-1} \tag{14}$$

is a recursion operator for the general nonlinear Schrödinger equation, leading to an infinite hierarchy of mutually commuting bi-Hamiltonian flows.

To determine the two quantized versions of the nonlinear Schrödinger equation, we need to introduce canonical coordinates and momenta, which will be found among the scattering data for the associated eigenvalue problem. We begin by recalling how this was done in Ref. 1 for the first symplectic form. The general nonlinear Schrödinger equation can be solved using the AKNS eigenvalue problem[7]

$$v_{1,x} + i\zeta v_1 = q v_2, \quad v_{2,x} - i\zeta v_2 = r v_1 . \tag{15}$$

We let

$$\phi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

be the solution to (15) satisfying the boundary conditions

$$\phi \to e^{-i\zeta x}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x \to -\infty, \quad \phi \to \begin{pmatrix} a(\zeta)e^{-i\zeta x} \\ b(\zeta)e^{i\zeta x} \end{pmatrix}, \quad x \to \infty,$$

for $\operatorname{Im}\zeta > 0$. Similarly, let

$$\bar\phi = \begin{pmatrix} \bar v_1 \\ \bar v_2 \end{pmatrix}$$

be the solution to (15) satisfying the boundary conditions

$$\bar\phi \to e^{-i\zeta x}\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad x \to -\infty,$$

$$\bar\phi \to \begin{pmatrix} \bar b(\zeta)e^{-i\zeta x} \\ -\bar a(\zeta)e^{i\zeta x} \end{pmatrix}, \quad x \to \infty ,$$

for $\operatorname{Im}\zeta < 0$. This serves to define the scattering coefficients a, b, $\bar a$, $\bar b$, which also satisfy

$$\bar a(\zeta)a(\zeta) + \bar b(\zeta)b(\zeta) = 1 . \tag{16}$$

The ratio $\rho(\xi) = b(\xi)/a(\xi)$, ξ real, serves to define the continuous spectrum of the scattering data for (16). The zeros of $a(\zeta)$ in the upper half plane correspond to the bound states, and are denoted as $\zeta_j = \xi_j + i\eta_j$, $j = 1,\dots,N$. Finally let b_j denote the value of b at ζ_j, and let ρ_j denote the residue of ρ at the pole ζ_j. Similar quantities are defined for the eigenvalues $\bar\zeta_j$.

In Ref. 1 it was shown how to express the first symplectic two-form in terms of the scattering data in the case $r = \pm q^*$. Tracing through the calculation there in the more general case, we find that

$$\begin{aligned}\Omega_1 &= i\int_{-\infty}^{\infty} \{\delta q \wedge \delta r\} dx \\ &= \frac{i}{\pi}\int_{-\infty}^{\infty} \{\delta \log b(\xi) \wedge \delta \log[\bar a(\xi)a(\xi)]\} d\xi \\ &\quad - 2\sum_{j=1}^{N} (\delta\zeta_j \wedge \delta \log b_j + \delta\bar\zeta_j \wedge \delta \log \bar b_j) ,\end{aligned} \tag{17}$$

where the last sum is absent if $r = +q^*$, since there are no bound states. When $r = \pm q^*$, then $\bar a(\xi) = a(\xi)^*$, and $\bar b(\xi) = \mp b(\xi)^*$. In this case one can choose canonically conjugate variables by letting

$$A_j = 4\eta_j, \quad p_j = -4\xi_j, \quad p(\xi) = -(i/\pi)\log|a(\xi)| ,$$

represent the momenta (p's), and letting

$$B_j = \arg b_j, \quad q_j = \log|b_j|, \quad q(\xi) = \arg b(\xi)$$

represent the conjugate coordinates (q's) for the system. The first Hamiltonian functional is then expressed as

$$\begin{aligned}H_1 = \pm E = \frac{4}{\pi}\int_{-\infty}^{\infty} \xi^2 \log(|a(\xi)|) d\xi \\ - \frac{8i}{3}\sum_{j=1}^{N} (\bar\zeta_j^3 - \zeta_j^3) .\end{aligned} \tag{18}$$

From this expression, the quantized form follows directly as in Ref. 1.

For the second symplectic form, we first recognize that by (12), (13) and Ref. 7,

$$J_2 = L^A J_1 = L^A \sigma_2 , \tag{19}$$

where L^A is the recursion operator for the squared eigenfunctions. Recall that the *squared eigenfunctions* corresponding to (15) are the functions

$$\Psi(\zeta,x) = \begin{pmatrix} v_1(\zeta,x)^2 \\ v_2(\zeta,x)^2 \end{pmatrix} .$$

We define the corresponding quantities Ψ_j for the bound states ζ_j similarly. The key result[10] is that the recursion operator L^A, given in (19), has the squared eigenfunctions as eigenstates:

$$L^A \Psi = \zeta\Psi, \quad L^A \Psi_j = \zeta_j \Psi_j . \tag{20}$$

Thus we can compute the second symplectic form

$$\Omega_2 = \tfrac{1}{2}\langle \delta V^A | \wedge \sigma_2 (L^A)^{-1} | \delta V\rangle .$$

Now, according to (B3) of Ref. 10,

$$\begin{aligned}\delta V = \frac{1}{\pi}\int_{-\infty}^{\infty} [\delta\rho(\xi)\Psi(\xi) - \delta\bar\rho(\xi)\bar\Psi(\xi)] d\xi \\ - 2i\sum_{j=1}^{N} (\delta\rho_j \Psi_j + \rho_j \delta\zeta_j \chi_j + \delta\bar\rho_j \bar\Psi_j + \bar\rho_j \delta\bar\zeta_j \bar\chi_j).\end{aligned}$$

Therefore, using (20),

$$\begin{aligned}(L^A)^{-1}\delta V &= \frac{1}{\pi}\int_{-\infty}^{\infty} [\delta\rho(\xi)(L^A)^{-1}\Psi(\xi) - \delta\bar\rho(\xi)(L^A)^{-1}\bar\Psi(\xi)] d\xi \\ &\quad - 2i\sum_{j=1}^{N} (\delta\rho_j (L^A)^{-1}\Psi_j + \rho_j\delta\zeta_j (L^A)^{-1}\chi_j + \delta\bar\rho_j (L^A)^{-1}\bar\Psi_j + \bar\rho_j\delta\bar\zeta_j (L^A)^{-1}\bar\chi_j) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\delta\rho(\xi)\Psi(\xi)}{\xi + i\epsilon} - \frac{\delta\bar\rho(\xi)\bar\Psi(\xi)}{\xi - i\epsilon}\right) d\xi - 2i\sum_{j=1}^{N} \left(\delta\left(\frac{\rho_j}{\zeta_j}\right)\Psi_j + \frac{\rho_j}{\zeta_j}\delta\zeta_j\chi_j + \delta\left(\frac{\bar\rho_j}{\bar\zeta_j}\right)\bar\Psi_j + \frac{\bar\rho_j}{\bar\zeta_j}\delta\bar\zeta_j\bar\chi_j\right),\end{aligned}$$

where we have moved the integral over the continuous spectrum off the real axis to avoid the singularity at $\zeta = 0$. Therefore the only difference between the computation of Ω_1 and the new symplectic form Ω_2 are the weighting factors $1/\xi$ in the continuous spectrum, and $1/\zeta_j$ in the discrete spectrum. A similar calculation as was used to produce (17) now gives

$$\Omega_2 = \frac{i}{\pi} \fint_{-\infty}^{\infty} \{\delta \log[\bar{a}(\xi)a(\xi)] \wedge \delta \arg b(\xi)\} \frac{d\xi}{\xi} + \frac{1}{2} \bar{b}(0) b(0) \delta \log \frac{\bar{a}(0)}{a(0)} \wedge \delta \log \frac{\bar{b}(0)}{b(0)} - 2 \sum_{j=1}^{N} \{\delta \log \zeta_j \wedge \delta \log b_j + \delta \log \bar{\zeta}_j \wedge \delta \log \bar{b}_j\}, \tag{21}$$

where the two complex integrals have combined to give the principal value in the leading term, and extra discrete term comes from the associated residues at the pole $\zeta = 0$. When $r = \pm q^*$, canonically conjugate variables are provided by the momenta

$$\hat{A}_j = 4 \arg \zeta_j, \quad \hat{p}_j = -4 \log|\zeta_j|,$$

$$\hat{p}(\xi) = -(i/\pi\xi) \log|a(\xi)|,$$

and the conjugate coordinates

$$\hat{B}_j = \arg b_j, \quad \hat{q}_j = \log|b_j|, \quad \hat{q}(\xi) = \arg b(\xi),$$

provided $\xi \neq 0$. In addition, the point $\xi = 0$ appears separately as the extra residue term in the expression for Ω_2, so this particular mode survives the principal value cancellation in a new discrete form. However, there is no simple formula for the relevant canonical variables there. Also, in the case $r = \pm q^*$, this term vanishes because $\bar{a}(0) = a(0)$, and so this extra complication does not arise. All the other modes for the continuous spectrum are related according to the simple reweighting

$$p(\xi) = \xi \hat{p}(\xi). \tag{22}$$

For the second Hamiltonian structure, the Hamiltonian functional giving the nonlinear Schrödinger equation is the momentum (11). According to the calculations in Ref. 1, it can be expressed in terms of the scattering data as

$$H_2 = P = \frac{4}{\pi} \int_{-\infty}^{\infty} \xi \log|a(\xi)| d\xi - 4i \sum_{j=1}^{N} (\bar{\zeta}_j^2 - \zeta_j^2). \tag{23}$$

Comparing with (18), we see that, in terms of the respective canonical variables, the continuous spectrum contribution is exactly the same weighted sum of the continuous canonical momentum variable associated with the respective symplectic two forms:

$$H_1: \quad \frac{4}{\pi} \int_{-\infty}^{\infty} \xi^2 p(\xi) d\xi \quad \text{versus}$$

$$H_2: \quad \frac{4}{\pi} \int_{-\infty}^{\infty} \xi p(\xi) d\xi = \frac{4}{\pi} \int_{-\infty}^{\infty} \xi^2 \hat{p}(\xi) d\xi.$$

Therefore, the continuous modes have identical quantizations. (The singular point $\xi = 0$ plays no role as both Hamiltonians make no contribution to this mode.) As for the bound states, we are reduced to the case of a collection of integrable two-dimensional Hamiltonian systems with different Hamiltonian structures. For the original symplectic form Ω_1, the Hamiltonian system corresponding to the discrete eigenvalue ζ_j has the form

$$(\log b_j)_t = -\frac{1}{2} \frac{\partial H_1}{\partial \zeta_j} = 4i\zeta_j^2, \quad (\zeta_j)_t = \frac{1}{2} \frac{\partial H_1}{\partial \log b_j} = 0,$$

and similarly for the eigenvalues $\bar{\zeta}_j$. (We are just reproducing the classical calculation of the evolution of the discrete scattering data for soliton equations.) For the second symplectic form Ω_2, the Hamiltonian system corresponding to the discrete eigenvalue ζ_j now takes the form

$$(\log b_j)_t = -\frac{1}{2} \frac{\partial H_1}{\partial \log \zeta_j} = 4i\zeta_j^2,$$

$$(\log \zeta_j)_t = \frac{1}{2} \frac{\partial H_1}{\partial \log b_j} = 0,$$

and similarly for the eigenvalues $\bar{\zeta}_j$. Thus, these two dimensional Hamiltonian systems are identical, even though they use two different Hamiltonian structures:

$$-2\delta\zeta_j \wedge \delta \log b_j \quad \text{versus} \quad -2\delta \log \zeta_j \wedge \delta \log b_j.$$

However, as we remarked above, we take as fundamental the fact that a two-dimensional Hamiltonian system has a unique quantization, even though it has many different Hamiltonian structures. Therefore the bound states for the nonlinear Schrödinger equation also have identical quantizations. We conclude that both Hamiltonians lead to the same quantized verison of the nonlinear Schrödinger equation.

As a final remark, we recall that the other soliton equations appearing in the AKNS scheme can be written in the form

$$\binom{q}{r}_t = \Omega(L^A)\binom{q}{r}_x,$$

where $\Omega(\xi)$ determines the linear dispersion relation.[7] These can all be written in bi-Hamiltonian form using the same two Hamiltonian structures as above. An identical calculation, which we omit for the sake of brevity, will show that the two quantized versions of any member of these AKNS hierarchies will lead to the same quantum version. Moreover, it is not hard to see that the only difference between the quantized versions of two different members of the same soliton hierarchy is in the weighting factor $\Omega(\xi)$ for the modes corresponding to the continuous spectrum [with appropriate discrete contributions at the points where $\Omega(\xi) = 0$] and replacement of the discrete Hamiltonians by $\Omega(\zeta_j)$ and $\Omega(\bar{\zeta}_j)$, respectively. Thus the only distinction between the various quantized versions of a soliton hierarchy is in the weighting assigned to the continuous modes, and the replacement of the Hamiltonian governing the evolution of the bound states. Finally, we note that the same considerations will apply to other soliton equations, such as the Korteweg–de Vries equation, as the key fact that the recursion operator is the squared eigenfunction operator remains valid.

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