1 Differential Calculus

The derivative of a function, \( f(x) \), is another function, \( f'(x) \), which tells you the way the original function changes. Specifically, \( f'(a) \), is the instantaneous rate of change of the function \( f(x) \) at the value \( x = a \). The number \( f'(a) \) is also the slope of the tangent line to the function \( f(x) \) at \( x = a \).

We can define the derivative using limits as

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

or equivalently,

\[
f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}.
\]

Note that the derivative will only be defined when this limit exists. In other words we can’t take a derivative at cusps, jumps and vertical asymptotes.

1.1 Power Rule

\[
\frac{d}{dx} x^n = nx^{n-1}
\]

1.2 Product Rule

\[
\frac{d}{dx} [f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)
\]

1.3 Quotient Rule

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}
\]

1.4 Chain Rule

\[
\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)
\]

1.5 Properties

The first derivative of a function \( f(x) \) determines where the function \( f(x) \) is increasing and decreasing. Specifically we have

| \( f'(a) > 0 \) | \( f(x) \) is increasing at \( x = a \) |
| \( f'(a) < 0 \) | \( f(x) \) is decreasing at \( x = a \) |
| \( f'(a) = 0 \) | \( f(x) \) is neither increasing or decreasing \( x = a \) |

Definition 1. A critical point to the function \( f(x) \) is a point where the function is neither increasing or decreasing. The \( x \)-coordinate \( x = a \) is a critical point if \( f'(a) = 0 \).
Given that \( f'(a) = 0 \), the function \( f(x) \) has a horizontal tangent line at the point \((a, f(a))\). For nice functions, \( f(x) \), this leads to one of three possibilities which can be tested by the second derivative. Typically, \( a \) will be an extrema, either a local maximum or local minimum. Otherwise, \( a \) is a “flattening out point” or saddle which is neither a max or min.

**Theorem 1.** Given that \( f'(a) = 0 \), we can classify the critical point \( x = a \) for the function \( f(x) \) as follows.

\[
\begin{array}{|c|c|}
\hline
f''(a) < 0 & \text{The function } f(x) \text{ has a local maximum at the point } x = a \\
f''(a) > 0 & \text{The function } f(x) \text{ has a local minimum at the point } x = a \\
f''(a) = 0 & f(x) \text{ does not have an extrema at } x = a \\
\hline
\end{array}
\]

**Remark 1.** This defines local extrema, a global max or min may occur where the first derivative is not zero. However, this may only occur for closed and bounded domains.

The **second derivative** of a function \( f(x) \), which is related to curvature, determines the concavity of the function \( f(x) \). Specifically we have

\[
\begin{array}{|c|c|}
\hline
f''(a) > 0 & f(x) \text{ is concave up at } x = a \\
f''(a) < 0 & f(x) \text{ is concave down at } x = a \\
\hline
\end{array}
\]

**Definition 2.** An inflection point of the function \( f(x) \) is an x-coordinate \( x = a \) such that \( f''(a) = 0 \) and \( f'' \) has opposite signs just to the left and right of \( x = a \) e.g. \( f''(a + \epsilon) < 0 \) and \( f''(a - \epsilon) > 0 \).

Loosely speaking the **concavity** of a function tells you if the function looks like a parabola opening up or down. If you were to pick two nearby points on the graph of \( f(x) \) and draw the line between them then that line segment will be above or below \( f(x) \). If the line segment is above then \( f(x) \) is concave up in this region and if the line segment is below then \( f(x) \) is concave down in this region.

### 1.6 Partial Derivatives

In many cases one variable may depend on more than one thing that is changing. If we consider a variable \( z \) that depends on two variables \( x \) and \( y \), then \( z \) will be a function of \( x \) and \( y \) i.e.

\[ z = f(x, y). \]

In order to determine the ways in which the variable \( z \) changes we will need to consider how it changes with respect to a single variable first. To do this we use **partial derivatives** which we define using limits as

the **partial derivative** of \( f \) with respect to \( x \)

\[ f_x = \frac{\partial}{\partial x} f(x) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}. \]
the partial derivative of $f$ with respect to $y$

$$f_y = \frac{\partial}{\partial y} f(x) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

The partial derivative of $f$ with respect to $x$ is the *instantaneous rate of change in $z = f(x, y)$ with respect to $x$ while holding $y$ fixed.* In other words $f_x = \frac{\partial f}{\partial x}$ tells you how $f$ changes as you change just $x$. The computation of partial derivatives is quite simple given that we already know how to compute derivatives in 1 dimension. For example, to compute $f_x$ we take a derivative just as we normally would but we treat the variable $y$ like a parameter i.e. like a constant fixed value.

**Example 1.** Given $f(x, y) = x^2 y^3 + e^{xy} + \ln(xy)$, calculate $f_x$ and $f_y$

$$f_x = \frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} x^2 y^3 + \frac{\partial}{\partial x} e^{xy} + \frac{\partial}{\partial x} \ln(xy) = y^3 \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial x} e^{xy} + \frac{\partial}{\partial x} \ln(xy) = y^3 (2x) + e^{xy} \frac{\partial}{\partial x} (xy) + \frac{1}{xy} \frac{\partial}{\partial x} (xy) = 2xy^3 + ye^{xy} + \frac{1}{x}$$

$$f_y = \frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} x^2 y^3 + \frac{\partial}{\partial y} e^{xy} + \frac{\partial}{\partial y} \ln(xy) = x^2 \frac{\partial}{\partial y} y^3 + \frac{\partial}{\partial y} e^{xy} + \frac{\partial}{\partial y} \ln(xy) = x^2 (3y^2) + e^{xy} \frac{\partial}{\partial y} (xy) + \frac{1}{xy} \frac{\partial}{\partial y} (xy) = 3x^2 y^2 + xe^{xy} + \frac{1}{y}$$

## 2 Discrete Dynamical System on $\mathbb{R}$

\[
\begin{align*}
\begin{cases}
  x_{n+1} &= f(x_n; a) \\
  x_0 &= x_0 
\end{cases}
\end{align*}
\]

### 2.1 Variables, parameters

In ?? above the state variable $x_n$ is the quantity which is updated according to the function $f(x)$ at each time step or iteration. The important concept is that the way the state variable changes only depends on the current value of the state variable (state of the system).

The concept of an algebraic variable, or unknown quantity, should be familiar. A parameter is similar to a variable in that it is represented by a symbol representing the
parameter, but we make the distinction that a parameter is a fixed number. In both the starting value or initial condition, $x_0$, and the arbitrary variable $a$ are parameters. For example, consider discrete exponential growth or decay:

$$
\begin{align*}
    x_{n+1} &= a \times x_n \\
    x_0 &= x_0
\end{align*}
$$

with solution

$$
x_N = a^N x_0.
$$

The parameters $a$ and $x_0$ are fixed while the variable $x_n$ changes at each iteration. In fact, the solution depends only on the parameters and the time step. Different values of the parameter $a$ create different results. Specifically, if $a > 1$ the system exhibits growth while if $a < 1$ the system exhibits decay.

### 2.2 Orbits, periodic orbit

**Definition 3.** An orbit is a solution to a dynamical system.

In particular, given an initial condition, $x_0$ an orbit to a discrete dynamical system, as in (1), is a list of values $\{x_0, x_1, x_2, \ldots\}$ which satisfy the condition $x_{n+1} = f(x_n; a)$.

### 2.3 Equilibrium

Finding equilibria in dynamical systems is a crucial first step to understanding behavior of any system in question.

**Definition 4.** An equilibrium is a constant solution to a dynamical system.

In other words an equilibrium solution is a numerical value which when entered as an initial condition creates an orbit in which every entry is identical to the initial condition i.e. the solution will be $\{x_0, x_0, x_0, \ldots\}$.

**Example 2.** The discrete dynamical system

$$
\begin{align*}
    x_{n+1} &= 1 - x_n \\
    x_0 &= \frac{1}{2}
\end{align*}
$$

has an orbit $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\}$ since $f(\frac{1}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$, thus $\frac{1}{2}$ is an equilibrium point to the discrete dynamical system with updating function $f(x_n) = 1 - x_n$.

The method used to calculate equilibrium points is quite simple. Given the dynamical system (1) an equilibrium $E$, is a fixed point of the updating function so it must be true that $E = f(E; a)$ since $x_{n+1} = x_n = E$. In fact, the general method for calculating equilibria will be to replace both $x_{n+1}$ and $x_n$ by the new variable $E$ and solve. If the dynamical system contains arbitrary parameters then your equilibrium point will almost certainly also contain arbitrary parameters.

In the example above we have $E = 1 - E$ so that $2E = 1$ and $E = \frac{1}{2}$ as we have seen.
2.4 Stability of Equilibria

Stability is an incredibly important topic within dynamical systems. If a solution is not stable then it will likely not be observed in nature. Roughly speaking a solution is stable if other solutions nearby the stable solution stay nearby. For the moment we can only consider stability of equilibrium solutions. An equilibrium is stable if initial conditions close to equilibrium give rise to orbits whose entries $x_n$ are also close to the equilibrium value.

**Definition 5. An equilibrium, $E$, is a stable equilibrium point** if given a small positive constant $\epsilon$ there exist a small positive constant $\delta$ such that all initial conditions in the interval $I_\delta$: $E - \delta < x_0 < E + \delta$ have solutions in the interval $I_\epsilon$: $E - \epsilon < x_n < E + \epsilon$.

This mathematical definition is perhaps a bit hard to read but is really just a thorough way of stating that solutions which start close (in $I_\epsilon$) stays close (in $I_\delta$). Specifically, if an equilibrium is stable and you are looking for solutions which stay in the interval $I_\epsilon$ (the given interval) then, in theory, you should be able to supply the interval $I_\delta$.

Figure 1: The green orbit is stable where as the orange orbit is unstable. The red lines bound $I_\epsilon$ and the blue lines bound $I_\delta$.

### 2.4.1 first derivative test

For a discrete dynamical system of the form (1), the first derivative of the updating function $f'(x; a)$ is a nice way to check stability. The simple example (2) has stable equilibrium $E = 0$ for growth problems $a > 1$ and unstable equilibrium $E = 0$ for decay problems $a < 1$. In this example the parameter $a$ is the constant value of the derivative of the updating function $f'(x_n; a)$. As it turns out this same idea extends to the stability of an equilibrium point.

**Theorem 2.** Let $E$ be an equilibrium point to the system (1) then the stability of the equilibrium point $E$ is determined as follows

- $|f'(E; a)| > 1 \quad \Rightarrow \quad E$ is an unstable equilibrium
- $|f'(E; a)| < 1 \quad \Rightarrow \quad E$ is a stable equilibrium

We neglect classifying stability if $f'(E; a) = 1$. The first derivative is not enough information to determine stability in this case.
2.5 Stability of homogeneous second order discrete system

In class we have seen solutions and behavior for a one dimensional linear system. The next simplest model would have a quadratic updating function. In particular, we consider a \textit{homogeneous} updating function, which means \( f(0) = 0 \)

\[
\begin{align*}
x_{n+1} &= ax_n + bx_n^2 \\
x_0 &= x_0
\end{align*}
\]

Following the method for calculating equilibria above we get the equilibrium equation

\[ E = aE + bE^2 \]

which reduces to

\[ 0 = E[(a - 1) + b]. \]

This gives us the two equilibria \( E_0 = 0 \) and \( E_1 = \frac{1-a}{b} \). Next we will determine for what values of the parameters \( a \) and \( b \) the equilibria \( E_0 \) and \( E_1 \) are stable or unstable. This reduces to checking \( |f'(E; a, b)| \) where

\[
\begin{align*}
f(x; a, b) &= ax + bx^2 \\
f'(x; a, b) &= a + 2bx.
\end{align*}
\]

So we find that

\[ f'(E_0; a, b) = a \quad \text{and} \quad f'(E_1; a, b) = a + 2b\left(\frac{1-a}{b}\right) = 2 - a. \]

From Theorem 2 this tells us that the equilibrium \( E_0 \) will be stable for \( |a| < 1 \) and \( E_1 \) will be stable for \( |2 - a| < 1 \) or \( 1 < a < 3 \).

\textbf{Remark 2.} The stability of the equilibria \( E_0 \) and \( E_1 \) does not depend on the parameter in the quadratic term. Furthermore, for linear growth rates in the range \( 1 < a < 3 \) \( x_n = 0 \) is unstable and \( x_n = \frac{1-a}{b} \) is an asymptotically stable equilibrium so that if \( b < 0 \) all positive values tend to the equilibrium \( E = \frac{1-a}{b} \).

3 Modeling a Single Population

...

4 Integral Calculus

area under the curve, Riemann Sum, definite integral, indefinite Integral , antiderivative

4.1 Power Rule

\[
\int x^n dx = \frac{x^{n+1}}{n+1}
\]
4.2 substitution

4.3 Fundamental Theorem of Calculus

5 Continuous Dynamical System on $\mathbb{R}$

$$\begin{cases}
  x' &= f(x, t; a) \\
  x(0) &= x_0
\end{cases}$$  \hspace{1cm} (3)

This type of continuous dynamical system is known as an ordinary differential equation or ODE. A solution to is a function $x(t)$, or curve, such that $\frac{d}{dt}x(t) = f(x(t), t; a)$. Furthermore, $x(t)$ satisfies the initial condition $x(0) = x_0$ (where $x(0)$ is the function value at zero and $x_0$ is prescribed data or a parameter.) Similar to discrete systems an ODE may contain parameters $a$.

Example 3. Let $f(x, t; a) = ax$ and $x(0) = 2$, then $x(t) = 2e^{at}$ is the solution to

$$\begin{cases}
  x' &= ax \\
  x(0) &= 2
\end{cases}$$

We can check this since $\frac{d}{dt}x(t) = \frac{d}{dt}2e^{at} = a2e^{at} = ax$ and $x(0) = 2e^0 = 2$.

The “full” solution to an ODE, ordinary differential equation, is called the General Solution. The General Solution is a family of curves parametrized by the arbitrary constants that show up from integration. In some way we can visualize the General Solution as the flow seen in a direction field or slope field. A Particular Solution to an ODE is a specific curve. Any particular solution will have the same form as the general solution although the arbitrary constants will be eliminated by the Initial Conditions.

For example, the general solution to $y'' + y' - 2y = 0$ is

$$y(x) = C_1e^{-2x} + C_2e^x$$

where $C_1$ and $C_2$ are arbitrary constants (numbers yet to be determined by data). If in addition we know that $y(0) = 0$ and $y'(0) = -3$ then we can calculate the particular solution as

$$y(x) = e^{-2x} - e^x.$$  

5.1 Integrable ODE

The simplest types of ODEs are ones which can be solved directly by integration. In this case the ODE can be written in the form

$$\begin{cases}
  x' &= f(t) \\
  x(0) &= x_0
\end{cases}$$

If this is the case then we can integrate the differential equation from the starting time $t_0$ up to an arbitrary time $t$, but first we replace the independent variable $t$ by the “dummy variable” $\tau$ to avoid silly things like $\int_1^t t dt$. So,

$$\int_{t_0}^t \frac{dx}{d\tau} d\tau = \int_{t_0}^t f(\tau) d\tau$$
which after using the Fundamental Theorem of Calculus reduces to

\[ x(t) - x(t_0) = \int_{t_0}^{t} f(\tau) d\tau \]

so that the solution \( x(t) \) to (??) is given by

\[ x(t) = x_0 + \int_{t_0}^{t} f(\tau) d\tau. \] (4)

**Example 4.** Compute the solution to

\[
\begin{align*}
  x' &= 3t^2 - t \\
  x(0) &= 3
\end{align*}
\]

Using equation (??) we first need to compute the integral \( \int_{t_0}^{t} 3\tau^2 - \tau d\tau \) where \( t_0 = 0 \)

\[
\int_{t_0}^{t} 3\tau^2 - \tau d\tau = \tau^3 - \frac{\tau^2}{2} \bigg|_{t_0}^{t} = t^3 - \frac{t^2}{2} - 0 + 0
\]

So, we find from formula (3) that

\[ x(t) = 3 + t^3 - \frac{t^2}{2}. \]

It is a good idea to check the solution is correct by plugging in \( t = 0 \) to see that \( x(0) = 3 \) and checking \( \frac{d}{dt}[t^3 - \frac{t^2}{2}] = 3t^2 - t \)

### 5.2 Fixed Points of Autonomous Differential Equations

**Definition 6.** An autonomous differential equation is a differential equation where \( x' = f(x; a) \). In other words the differential equation does not depend on the independent variable \( t \)

The example (??) was an autonomous differential equation.

An **Autonomous Differential Equation** is simply a differential equation in which only the dependent variable shows up in the equation. We can write any first order Autonomous Differential Equation as

\[
\frac{dx}{dt} = f(x).
\]

Equations of this type have special solutions called **Equilibrium Solutions** which are in fact **Equilibrium Points**. Pausing to consider the meaning of equilibrium we think about various situations of balance and a loss of dynamics. The derivative measures change, so we say that equilibrium occurs when \( \frac{dx}{dt} = 0 \). From this idea we can define **Equilibrium points** of autonomous equations as the numerical values, \( x_e \), which make the function \( f(x) \) vanish. In other words the values \( x_e \) are the zeros of \( f(x) \)

\[ f(x_e) = 0 \]
5.3 Stability of Equilibrium Points

The equilibrium points, $x_e$, or fixed points will show up as horizontal asymptotes in the direction fields and are the only points plotted in Phase Diagrams or Phase-line diagrams. The phase diagram gives us a practical way to check the Stability of Equilibrium Points.

**Definition 7.** An equilibrium point, $x_e$, is asymptotically stable if solutions to the corresponding ODE with initial condition $x(0) = x_0$ near the equilibrium point have solutions such that $\lim_{t \to \infty} x(t) = x_e$.

In other words solutions which start near the equilibrium point aproach the equilibrium point as time increases.

We will characterize the stability of an equilibrium point $x_e$ for an ODE as follows:

<table>
<thead>
<tr>
<th>$f'(x_e)$</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 0$</td>
<td>Stable</td>
</tr>
<tr>
<td>$&gt; 0$</td>
<td>Unstable</td>
</tr>
<tr>
<td>$= 0$</td>
<td>Quasi-Stable</td>
</tr>
</tbody>
</table>

The question mark is a little ambiguous but its purpose is to point out that this characterization can become complicated. When checking the stability of an equilibrium point you should just sketch a phase diagram instead of memorizing what appear to be arbitrary rules. The case $f'(x_e) = 0$ becomes complicated because three things can happen. In Calculus we learned how to find local maxima and minima of functions. These values occur precisely when $f'(x_e) = 0$. If you have that $f'(x_e) = 0$ then $x_e$ could be either a local max, local min or a Saddle point which is really an inflection point. The local max and local min correspond to the equilibrium point being Quasi-Stable. The saddle could correspond to either a stable point or an unstable point depending on if $f(x)$ is increasing or decreasing near $x_e$ i.e. if $f(x) < 0$ or $f(x) > 0$ near $x_e$. Consider the two cases $\frac{dx}{dt} = \pm x^3$. The pragmatic perspective here is if $f'(x_e) = 0$ draw a phase diagram and save yourself a headache.

**Remark 3.** The real difficulty in determining stability of equilibrium solutions to autonomous differential equations is in curve sketching of the right hand side. From the graph of $f(x)$ it is straigtforward to determine stability by a phase line diagram.

![Phase diagrams for the stable case $x' = -x^3$ and the unstable case $x' = x^3$.](image)

Figure 2: Phase diagrams for the stable case $x' = -x^3$ and the unstable case $x' = x^3$.

Both cases have the equilibrium point $x_e = 0$ with $f'(x_e) = 0$.

**Remark 4.** The arrow in any interval of a phase line diagram indicates the sign of the derivative of the solution. In other words the arrow indicates if the solution is increasing or decreasing in the given interval. This information is sufficient to determine stability of the equilibrium points, where the solution is constant.
5.4 Bifurcation and Behaviour of Solutions

We now know how to find fixed points of autonomous differential equations

\[ x' = f(x) \]

and determine their stability by looking at either a phase line diagram or \( f'(x) \). We now want to look a little further into how a parameter can change both the number of fixed points and their stability. This is the idea of a bifurcation. For all the following examples a bifurcation occurs when the parameter \( a \) is zero. This is because we will only consider certain normal forms, which are simplified or model cases for which most problems can be reduced to. When considering bifurcations we will look at autonomous equations with parameter \( a \) i.e. equations of the form

\[ x' = f(x; a). \]

To try and understand the change in behavior from \( a < 0 \) to \( a > 0 \) we will look at bifurcation diagrams which are plots of stable and unstable equilibria in the \( a - x \) plane. Solid lines represent stable equilibria and dashed lines represent unstable equilibria. The vertical line for any fixed parameter value \( a = \)constant is exactly the phase line diagram for that specific parameter value.

Example 5. Determine the equilibria and their stability for the equation

\[ x' = ax - x^3. \]

The equilibrium equation factors as \( x(a - x^2) = 0 \), which has 1 or 3 solutions depending if \( a \) is positive or negative so we must consider the two case separately. Note that \( f'(x; a) = a - 3x^2 \) which may be used to determine stability.

\( a \leq 0 \)

In this case \( x = 0 \) is the only equilibrium and the phase line diagram can be drawn to check that \( x = 0 \) is a stable equilibrium solution. Otherwise, we can check that \( f'(0; a) = a < 0 \) so that \( x = 0 \) is a stable equilibrium.

\[ x = 0 \]

\( a > 0 \)

In this case \( x = 0, \sqrt{a}, -\sqrt{a} \) are all equilibrium solutions. The phase line diagram now looks like so that the equilibrium solution \( x = 0 \) is unstable and the equilibrium solutions \( x = \pm \sqrt{a} \) are stable, this agrees with the derivative test since

\[
\begin{align*}
  f'(x = 0; a) &= a \\
  f'(x = \pm \sqrt{a}; a) &= -2a.
\end{align*}
\]
6 Modeling with Dynamical Systems

- Temperature
- Trajectories
- Chemical decay
- ...

7 Linear Algebra

7.1 Matrices, Vectors, Trace
7.2 Dimension
7.3 Basis, Span
7.4 Determinant
7.5 Eigenvalues, Eigenvectors

8 Modeling with Linear Algebra

- Economics
- Nutrition
- ...

9 Dynamical Systems on $\mathbb{R}^2$

\[
\begin{align*}
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} &=
\begin{bmatrix}
  f(x, y) \\
  g(x, y)
\end{bmatrix} \\
\begin{bmatrix}
  x(0) \\
  y(0)
\end{bmatrix} &=
\begin{bmatrix}
  x_0 \\
  y_0
\end{bmatrix}
\end{align*}
\]
9.1 second order ODE
9.2 Phase Plane
9.3 null clines

10 Population Models: two species
10.1 predator prey
10.2 competing species

11 Infinite dimensional dynamical systems
PDE

11.1 Laplace equation
Steady State, minimization

11.2 Heat equation
diffusion

11.3 Wave equation
Transport