Oral Probability Questions

These are notes made in preparation for oral exams involving the following topics in probability: Random walks, Martingales, and Markov Chains. Textbook used: "Probability: Theory and Examples," Durrett.

Chapter 4

1. **Define: Random Walk**

   Let $X_1, X_2, \ldots$ be iid taking values in $\mathbb{R}^d$

   and let $S_n = X_1 + \ldots + X_n$. S_n is a random walk.

2. **Name a <u>Random Walk</u> Theorem**
   - **RW Possibilities on R**: Four possibilities, one w/prob = 1.
     - $S_n=0$ $\forall n$, (recurrent)
     - $S_n \to \pm\infty$, (transient)
     - $-\infty=\liminf S_n<\limsup S_n=\infty$ (recurrent)
   - **RW Recurrence on R^d**:
     - S_n recurrent in d=1 if $S_n/n \to 0$ in probability. (or SSRW)
     - S_n recurrent in d=2 if $S_n/\sqrt{n}$ converges in distribution to a non-deg. norm. dist. (or SSRW)
     - S_n transient in d≥3 if is "truly three-dimensional"
   - **RW Equivalencies Theorem**: Let $\tau_0=0$ and $\tau_n=\inf\{m>\tau_{n-1}:S_m=0\}$ be time of nth return to 0. Then, $P(\tau_1<\infty)=1 \Leftrightarrow P(S_m=0 \text{ i.o.})=1 \Leftrightarrow \sum_{m=0} P(S_m=0)=\infty$.
   - **RW Convergence/Divergence Theorem**: Convergence (divergence) of $\Sigma_n P(|S_n|<\varepsilon)$, $\forall\varepsilon>0$ is sufficient to determine transience (recurrence) of S_n.
3. **Does (a version of 1) always have __________ property (related to 2)?**
   - For iid $X_1, X_2, \ldots$, is exchangeable sigma field ε trivial? Yes. By Hewitt Savage 0-1. $P(A)\in\{0,1\}$ for each $A\in\varepsilon$
   - **Types of sets for RW recurrent values (V)?** Empty set, or a closed subgroup of R^d.
   - **If V (recurrent values) is a closed subgroup, V=?** V={Possible Values}
4. **Question that leads to a Counterexample/Example.**
   - Are SSRW always recurrent? They are on d<3.
   - Are RW on R^d always recurrent w/ d<3? No, only w/ SSRW or w/ correct convergence (see above)
   - **Will Wald's theorem hold with a SSRW $S_n=X_1+\cdots+X_n$, with $X_n \in \{\pm1\}$ starting at $S_0=0$, with a stopping time T when $S_T=s\neq0$?** (Wald has X_i as iid w/E[t]<∞ and $E[X_i]<\infty$)
     Note that for any SSRW, that the time T to any position $S_T=s$ is finite, with probability one. However, the expected time is infinite. Therefore, it does not satisfy one of Wald's Theorem's assumptions.
     Proof by Contradiction: Having conditioned on $C=\{S_T=X_1+\cdots+X_T = s\}$, then the conditioned expectation $E(X_1+\cdots+X_T \mid C) = s$ is evident; furthermore, since $X_n = \pm1$ for all n with equal probability, we easily see that $\mu=E(X_n) = 0$. Under these observations, assuming Wald's Identity ($E[S_T]=\mu\cdot T$), we obtain an immediate contradiction ($s \neq 0\cdot T$).

- **If S,T are stopping times, then is it necessary that (S − T) is a stopping time?**
  S−T is not necessarily a stopping time. For a counterexample, consider the simple random walk (X_n) on {…,-1,0,1,…} starting at $X_0=0$, and let $S:=\inf\{n:X_n=1\}$ and $T:=1$. Note that $\{S-T=1\}=\{S=2\}$ which is not X_1-measurable.

Examples of stopping times

- To illustrate some examples of random times that are stopping rules and some that are not, consider a gambler playing roulette with a typical house edge, starting with $100 and betting $1 on red in each game:
- Playing exactly five games corresponds to the stopping time $\tau = 5$, and is a stopping rule.
- Playing until he either runs out of money or has played 500 games is a stopping rule.
- Playing until he is the maximum amount ahead he will ever be is not a stopping rule and does not provide a stopping time, as it requires information about the future as well as the present and past.
- Playing until he doubles his money (borrowing if necessary) is not a stopping rule, as there is a positive probability that he will never double his money.
- Playing until he either doubles his money or runs out of money is a stopping rule, even though there is potentially no limit to the number of games he plays, since the probability that he stops in a finite time is 1.

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1. **Define: Stopping Time**
   $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ a filtered prob space.
   Stopping time $T : \Omega \to \mathbb{Z}_+ \cup \{+\infty\}$ is r.v. s.t. $\{T \leq n\} \in \mathcal{F}_n$ $\forall n \geq 0$, or equivalently, $\{T = n\} \in \mathcal{F}_n$ for all $n \geq 0$.
2. **Name a Stopping Time Theorem**
   - **Wald's Identity**: Let $X_1,X_2,\dots$ be iid w/$\mu:=E[X_n]<\infty$. Set X_0 and let $S_n=X_1+\dots+X_n$, and T be stopping time w/$E[T]<\infty$. Then, $E[S_T]=\mu E[T]$.
   - If S,T,T_n are stopping times on (Ω,F,F_n,P). Then so are:
     - $S+T$, $S\wedge T:=\min(S,T)$, $S\vee T:=\max(S,T)$
     - $\liminf_n T_n$ and $\inf_n T_n$, $\limsup_n T_n$ and $\sup_n T_n$
3. **Does (a version of 1) always have __________ property (related to 2)?**
   - Are constants stopping times? Yes.
4. **Question that leads to a Counterexample/Example.**
   If stopping time T and F_T, and $X_1,X_2,\dots$ iid, is $\{X_{T+n}\}_{n>0}$ independent of F_T for all T? Yes.
   Examples of Stopping Times:
   - Constants
   - If X_n Is an adapted process, and $A\in F$, The first entry time into A is a stopping time.

Chapter 5

1. **Define: Martingale (or sub, or super)**
   X_n on (Ω,F,P,F_n), s.t.
   - X_n is adapted to F_n.
   - $E|X_n| < \infty$ for each n.
   - $E[X_{n+1}|F_n] = X_n$ a.s. $\forall$n. (or ≥, or ≤ resp.)

2. **Name a Martingale Theorem**
   - **Stopping Time (Super)Martingale Prop:** If T is a stopping time and X_n is a (super)mart, then $X_{T\wedge n}$ is a (super)mart.
   - **Submartingale Convergence:** Suppose that X_n is a sub-martingale with $\sup_n E[X_n^+]<\infty$. Then for some X, we have $X_n \to X$ a.s., where $E|X|<\infty$.
   - **Martingale Convergence:** If X_n is a martingale with $\sup_n E|X_n|<\infty$, then $X_n\to X$ a.s. and $E|X|<\infty$.
   - **Nonnegative SuperMartingale Convergence:** If X_n is a super-martingale with $X_n\geq 0$, then $X_n\to X$ a.s. and $E[X]\leq E[X_0]$
   - **Galton-Watson**: Let $\xi_i^n, i\geq 1, n\geq 0$ be iid nonnegative integer-valued r.v.s with a common $\mu:=E[\xi_i^n]\in(0,\infty)$. Define $Z_0=1$ and $Z_{n+1}=\{\xi_1^n+\ldots+\xi_{Z_n}^n$ if $Z_n>0$; and 0 if $Z_n=0$.
     Then, $(Z_n/\mu^n)_{n\geq 0}$ is a martingale with respect to $F_n=\sigma(\xi_1^m : i\geq 1, 0\leq m<n)$.
3. **Does (a version of 1) always have __________ property (related to 2)?**
   - *Do supermartingales always converge a.s.?* Not necessarily, it's guaranteed when X_n nonnegative.
   - *If $\mu<1$, Then P(extinction) = ?* P(extinction) = 1.
4. **Question that leads to a Counterexample/Example.**
   - **When $\mu=1$, is P(extinction) equal to 1?** Only when $P(\xi_i=1) <1$.
   - **From Durrett Exmpl. 5.2.3: Do nonnegative martingales converge in L^1?**
     Not always. Let S_n be a symmetric simple random walk with $S_0 = 1$, i.e., $S_n = S_{n-1} +\xi_n$ where $\xi_1, \xi_2, \ldots$ are i.i.d. with $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. Let $N = \inf\{n : S_n = 0\}$ and let $X_n = S_{N\wedge n}$. Since the martingale property is closed under stopping times, X_n is a nonnegative martingale. The Nonnegative SuperMartingale Convergence Theorem implies X_n converges a.s. to a limit $X_\infty < \infty$ that must be $\equiv 0$, since convergence to $k > 0$ is impossible. (If $X_n = k > 0$ then $X_{n+1} = k \pm 1$.) Since $EX_n = EX_0 = 1$ for all n and $X_\infty = 0$, convergence cannot occur in L^1. $E|X_n-X_\infty| = E[X_n] \to 1 \neq 0$.
   - **Consider the random walk $S_n=X_1+\cdots+X_n$ starting at zero with X's having $P(X_i=1) = P(X_i=-1) = ½$, a martingale. Now if $T=\inf\{n\geq 0:S_n=1\}$. Can we bound T?**
     No. For any $n \in \{1,2,...\}$ we have $P(S_k\leq 0 \text{ for all } k\leq n)\geq P(X_1=\ldots=X_n=-1)=1/2^n$ since $\{S_k\leq 0 \text{ for all } k\leq n\}\subseteq\{T>n\}$, this implies $P(T>n)\geq P(S_k\leq 0 \text{ for all } k\leq n)\geq 1/2^n>0$. As $n\in N$ is arbitrary, this proves that T is unbounded.
   - **Do all Martingales which converge in probability, also do so in L^1?**
     No. Any martingale which converges almost surely but not in $\mathbf{L^1}$ does the job (since a.s. conv. implies conv. in prob.); see example 5.2.3 above.

- **If $E(X_{n+1}|X_n)=X_n$ for all n, must X_n be a martingale (instead of $E(X_{n+1}|F_n)=X_n$)?**
  No. Let $(Y_j)_{j\in N}$ be a sequence of iid r.v. such that $EY_j=0$. Fix $N \in \{1,2,...\}$ and define: $X_n:=\sum_{j=1}^{n} Y_j$ for all $n\le N$, and $X_n:=\sum_{j=1}^{N} Y_j+Y_1-Y_2=X_N+Y_1-Y_2$ for all $n>N$.
  For $n\le N$ and $n>N+1$, the condition $E(Xn|X_{n-1})=X_{n-1}$ is obviously satisfied.
  For $n=N+1$, we have $E(X_{N+1}|X_N)=X_N+E(Y_1|X_N)-E(Y_2|X_N)$. Since $(Yj)_{j\in N}$ is identically distributed and independent, we have $E(Y_1|X_N)=E(Y2|XN)$ and therefore $E(X_{N+1}|X_N)=X_N$. On the other hand,

$$\mathbb{E}(X_{N+1} \mid \mathcal{F}_N) = X_N + 2\underbrace{\mathbb{E}(Y_1 \mid \mathcal{F}_N)}_{\mathbb{E}(X_1|\mathcal{F}_N)=X_1} - \underbrace{\mathbb{E}(Y_1+Y_2 \mid \mathcal{F}_N)}_{\mathbb{E}(X_2|\mathcal{F}_N)=X_2} = X_N + 2Y_1 - (Y_1+Y_2)$$
$$= X_{N+1} \neq X_N.$$

  So, X_n is not a martingale.

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1. **Define: <u>Optional Stopping Sigma-Field</u>**
   Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\ge 0}, \mathbb{P})$ and T be stopping time.
   Denote by $\mathcal{F}_T$, the σ-field of "events which occur prior to time T."
   In symbols: $\mathcal{F}_T := \{A \in \mathcal{F} : A \cap \{T \le n\} \in \mathcal{F}_n, \ \forall n \ge 0\}$.
2. **Name an <u>Optional Stopping Time Theorem</u>**

Optional Stopping Thm for SubMarts (or mart)	If S,T are stopping times w/$\mathbb{P}(S \le T < \infty) = 1$, and $(X_{T\wedge n})_{n\ge 0}$ is UI submart, then $\mathbb{E}[X_T|\mathcal{F}_S] \ge X_S$ a.s. Consequently, $\mathbb{E}[X_S] \le \mathbb{E}[X_T]$. (switch to ='s for mart)

3. **Does (a version of 1) always have __________ property (related to 2)?**
   - If T is a stopping time, then is F_T a Sigma field? Yes
   - If X_n is UI sub-martingale and T a stopping time, is $X_{T\wedge n}$ UI? Yes
   - If $S\le T$ are stopping times, then is $F_T\subseteq F_S$? No, but $F_S\subseteq F_T$.

4. **Question that leads to a Counterexample/Example.**
   - **If T is a stopping time, and X_n adapted, then is $X_T \in F_T$?** Not necessarily, this is only guaranteed when $P(T<\infty)=1$.

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1. **Define: Conditional Expectation**
   $(\Omega, \mathcal{F}, P)$ w/$X \in L^1$, $G \subseteq \mathcal{F}$, Y:=$\mathbb{E}[X|G]$ is unique s.t.
   Y is G-measurable and $\mathbb{E}|Y| < \infty$.
   $\mathbb{E}[\mathbb{E}[X|G]1_A] = \mathbb{E}[Y1_A] = \mathbb{E}[X1_A]$, $A \in G$

2. **Name a Conditional Expectation Theorem**
   - **Conditional MCT**: Let G ⊆ $\mathcal{F}$.

     Let $X, X_n \geq 0$ be integrable r.v.s and $X_n \uparrow X$.

     Then $\mathbb{E}[X_n|G] \uparrow \mathbb{E}[X|G]$ a.s.
   - **Conditional DCT:** Let G ⊆ $\mathcal{F}$.

     If $X_n \to X$ a.s. and $|X_n| \leq Y$ for some integrable r.v. Y.

     Then $\mathbb{E}[X_n|G] \to \mathbb{E}[X|G]$ a.s.
   - **Conditional Jensen's:** Let G ⊆ $\mathcal{F}$.

     If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, $\mathbb{E}|X| < \infty$ and $\mathbb{E}|\varphi(X)| < \infty$,

     then $\mathbb{E}[\varphi(X)|G] \geq \varphi(\mathbb{E}[X|G])$ a.s.
3. **Does (a version of 1) always have __________ property (related to 2)**

4. **Question that leads to a Counterexample/Example.**
   - **If X,Y are two random variables and E(X|Y)=E(X), are X and Y independent?**
     Not necessarily. Let X∈{−1,0,1}, each with probability ⅓. Let Y=X^2. Note that X and Y are not independent. However, observe that E(X|Y=0)=0 and E(X|Y=1) = ⅓ (-1) + ⅓(1) = 0, so E(X|Y)=0=E(X) with probability 1.

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1. **Define: Uniform Integrability**

   Family of r.v.s $(X_\alpha)_{\alpha\in\Lambda}$ is uniformly integrable (UI) if

   $\sup_{\alpha\in\Lambda}\mathbb{E}[|X_\alpha|1_{\{|X_\alpha|>M\}}] \to 0$ as $M \to \infty$.

   Remrk: Since $\mathbb{E}|X_\alpha| \leq M + \mathbb{E}[|X_\alpha|1_{\{|X_\alpha|>M\}}]$, then $UI \Rightarrow L^1$-bounded **uniformly for $(X_\alpha)_{\alpha\in\Lambda}$.**
2. **Name a UI Theorem**
   - **Sub σ-field UI Lemma**: Let X∈L^1(Ω,F,P). Then, {E[X|G]:G a σ-field ⊂F} is UI. Used in Levy's Fwd Law.
   - **If X_n→X in probability, then TFAE**:
     - {X_n} is UI.
     - X_n→X in L^1.   E|X_n - X| → 0.
     - E|X_n|→E|X|<∞.
   - **Convergence in Prob Corollary:**
     - If X_n→X in prob. and {X_n} is UI ⇔ X_n → X in L^1.
     - If X_n→X in prob and |X_n|≤Y for some Y∈L^1 (L^1 bounded), then X_n→X in L^1.
   - **Submartingale Equivalencies Thm**: For a submart X_n, TFAE:
     - {X_n} is UI.
     - X_n converges a.s. and in L^1.
     - X_n converges in L^1.
     - If X_n is a martingale, then ∃ integrable r.v. X so that X_n = E[X|F_n].

3. **Does (a version of 1) always have __________ property (related to 2)?**
   - Do UI sub martingales converge almost surely? Yes.

4. **Question that leads to a Counterexample/Example.**
   - **For a reverse martingale $(X_{-n})_n$, clearly, $E[X_0]=X_{-n}$, for each $n\in\{1,2,\ldots\}$. Is $E[X_0 \mid F_{-n}]$ UI?**
     Yes. Proof: Since $(X_{-n})_n$ is a martingale, we have: $E[X_0]<\infty$. So by the Subsigma Field UI Lemma, we have $E[X_0 \mid F_{-n}]$ is UI.
   - **Durrett Example 5.5.1. Suppose $X_1,X_2,\ldots$ are UI and $X_n\rightarrow X$ a.s. Need $E(X_n|F)$ converge a.s.?**
     No. Let $Y_1, Y_2, \ldots$ and $Z_1,Z_2,\ldots$ be independent r.v.'s with $P(Y_n = 1) = 1/n$, $P(Y_n = 0) = 1 - 1/n$, $P(Z_n = n) = 1/n$, $P(Z_n = 0) = 1 - 1/n$. So our counterexample uses $X_n := Y_nZ_n$. Observe that $E(X_n : |X_n|\geq 1) = n/n^2$, so X_n is UI. Also, $P(X_n > 0) = 1/n^2$ so $\Sigma P(X_n > 0)<\infty$, $P(\{X_n > 0\}$ i.o.$)=0$, and the Borel-Cantelli lemma implies $X_n \rightarrow 0$ a.s. Let $F = \sigma(Y_1, Y_2, \ldots)$. Then, $E(X_n|F) = Y_nE(Z_n|F) = Y_nE[Z_n] = Y_n$. Since $Y_n \rightarrow 0$ in L^1 but not a.s., the same is true for $E(X_n|F)$. Since $\sum P\{Y_n>½\} = \sum 1/n = \infty$. Then, apply Borel-Cantelli.
   - **Does every sequence X_n which converges almost surely, also converge in L^1?**
     No, take the sequence $n \cdot 1_{[0,1/n]}$, and note that it converges almost surely to zero. Also note that $E[n \cdot 1_{[0,1/n]}] = 1$ for all n. So, $\text{Lim}\, E[n \cdot 1_{[0,1/n]}-X] = \text{Lim}E[n \cdot 1_{[0,1/n]}] = 1 \neq 0$.
   - **For a martingale X_n does UI imply integrability of $\sup|X_n|$?**
     No, but the counterexamples are not trivial.
   - **Non-trivial martingale which converges almost surely to 0**
     Let $Y_1,Y_2,\ldots$ be nonnegative i.i.d. random variables with $E[Y_m]=1$ and $P(Y_m=1)<1$.
     (i) Show that $X_n=\prod_{m\leq n}Y_m$ defines a martingale. (ii) Use an argument by contradiction to show $X_n\rightarrow 0$ a.s.
     (i) is easy to check.
     (ii) Let $X: = \lim X_n$. The Hewitt-Savage zero one law says (since $X \in$ {exchangeable sigma field}) that X is almost surely a constant. Also, $X=Y_1\cdot\prod_{i=2}^{\infty}Y_i$ has the same distribution as $Y_1\cdot X$. Since Y_1 is not constant a.s., this forces $X\in\{0,\infty\}$, but $X\neq\infty$ since by Fatou and Y_n independence we have: $E(X)=E(\lim X_n)=E(\lim\prod_{m\leq n}Y_m)\leq \lim E(\prod_{m\leq n}Y_m)=\lim\prod_{m\leq n}E(Y_m)=1$. So $X=0$, and $X_n\rightarrow 0$ a.s.

Chapter 6

1. **Define: Markov Chain**
   An $\{F_n\}$-adapted stochastic process X_n taking values in $(S,\mathcal{S})$ is called a Markov chain if it has the **Markov Property**: $P(X_{n+1}\in B|F_n)=P(X_{n+1}\in B|X_n)$ a.s. for each $B\in S$, $n\geq 0$.
2. **Name a Markov Chain Theorem**
   - **Decomposition Theorem**: Let $R=\{x : \rho_{xx}=1\}$ be the recurrent states of a Markov chain. R can be written as $\cup_i R_i$, where each R_i is closed and irreducible. [This results shows that for the study of recurrent states we can, without loss of generality, consider a single irreducible closed set.]
   - **For an irreducible and recurrent chain (Corolary 6.46):**
     - The stat/inv measures are unique up to constant multiples.
     - If μ is a stat/inv measure, then $\mu(x)>0$ for all x.
   - **If p is irreducible and has a stationary distribution π.**
     - **Calculating Stat/Inv Distribution**: $\pi(x)=1/E_x[T_x]$.
     - **Theorem D6.5.7**: Any other stationary measure is a multiple of π.

- **Theorem 6.70** (Markov Chain Convergence Theorem): Consider an irreducible, aperiodic Markov chain with stationary distribution π. Then, $p^n(x,y) \to \pi(y)$ as $n\to\infty$, for all $x,y \in S$.
- **Theorem 6.62** (Asymptotic Density of Returns): Let $y\in S$ be recurrent, and $N_n(y) = \sum^n_i 1_{\{Xi=y\}}$, then $\lim N_n(y)/n=(1/E_y[T_y])1_{\{Ty<\infty\}}$, P_x- a.s.

3. **Does (a version of 1) always have __________ property (related to 2)?**

4. **Question that leads to a Counterexample/Example.**
   - **Multivalued Markov Chain**: If $\xi_0,\xi_1,\ldots$ are iid$\in\{H,T\}$, each with p=½, then $X_n:=\{\xi_n,\xi_{n+1}\}$ is a Markov chain.
   - **(HW 3): If $\xi_0,\xi_1,\ldots$ are iid$\in\{-1,1\}$ with p=½, and $S_0=0$, $S_n:=\xi_1+\xi_2+\ldots+\xi_n$, and $X_n=\max\{S_m : 0\le m\le n\}$. Then is X_n is a Markov chain?**
     No. Observe the sequence $(X_1,X_2,X_3)=(1,1,1)$. This can occur if $(S_1,S_2,S_3)=(1,0,1)$, or if $(S_1,S_2,S_3)=(1,0,-1)$. Therefore, we have: $P(X_4=2|X_1=1,X_2=1,X_3=1)=(1/2)\cdot(1/2)=1/4$. Alternatively, take the sequence $(X_1,X_2,X_3)=(0,0,1)$, and observe that this only occurs in only one way, namely if $(S_1,S_2,S_3)=(-1,0,1)$. Therefore, $P(X_4=2|X_1=0,X_2=0,X_3=1)=1\cdot(1/2)=1/2$. Therefore, since the dependence includes more than just the previous value, X_n is not a Markov chain.

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1. **Define: Stationary Distribution**
   It's a stationary/invariant measure that is also a probability measure: $\pi p=\pi$ such that $\pi(y)=\Sigma_{x\in S}\pi(x)p(x,y)$, and $\Sigma_{x\in S}\pi(x)=1$. It represents a possible equilibrium for the chain.
2. **Name a Stationary DistributionTheorem**
   - **If p is irreducible and has a stationary distribution π.**
     - **Calculating Stat/Inv Distribution**: $\pi(x)=1/E_x[T_x]$.
     - **Theorem D6.5.7**: Any other stationary measure is a multiple of π.
   - **Recurrence from Positive Stat/Inv Distributions**: If π is a stationary/invariant distribution of a Markov chain and $\pi(x)>0$ for some x, then that x is recurrent.
   - **Theorem 6.70** (Markov Chain Convergence Theorem): Consider an irreducible, aperiodic Markov chain with stationary distribution π. Then, $p^n(x,y)\to\pi(y)$ as $n\to\infty$, for all $x,y\in S$.
3. **Does (a version of 1) always have __________ property (related to 2)?**
   - **What are sufficient conditions for a Markov chain's stat/inv measures to be unique up to constant multiples?** That it be irreducible and recurrent.
   - **What are sufficient conditions for a Markov chain's stat/inv measure, If it exists, to have the property $\mu(x)>0$ for all x?** That it be irreducible and recurrent.
   - **What are sufficient conditions for a Markov chain's stat/inv distribution, if it exists, to be unique?** That it be irreducible and recurrent.
   - **Assume a Markov chain is irreducible and recurrent, what are sufficient conditions to allow us to conclude that the stat/inv distribution cannot exist?** The stat/inv measure has infinite mass.
   - **If π is a stat/inv distribution and π(x)>0, what we know about x?** It is recurrent.
   - **If you have an irreducible Markov chain, and there is a positive recurrent value, does this imply the existence of a stationary distribution?** Yes.
   - **If you have an irreducible Markov chain, and every state is positive recurrent, does this imply the existence of a stationary distribution?** Yes.
   - **If you have an irreducible Markov chain that has a stationary distribution, does this imply the existence of a positive recurrent value?** Yes.

4. **Question that leads to a Counterexample/Example.**
    - **Let X_n be a Markov chain, where S is the state space and P is the transition matrix. Is every closed class recurrent?** No, for example a biased random walk on the integers is transient. *Finite* closed classes, on the other hand, are always recurrent.

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1. **Define: Markov Chain Recurrence**
   A state y∈S is called recurrent if $\rho_{yy}=1$, and is called transient if $\rho_{yy}<1$.
2. **Name a Recurrence Theorem**
    - **Decomposition Theorem**: Let $R=\{x : \rho_{xx}=1\}$ be the recurrent states of a Markov chain. R can be written as $\cup_i R_i$, where each R_i is closed and irreducible. [This results shows that for the study of recurrent states we can, without loss of generality, consider a single irreducible closed set.]
    - **Theorem 6.62** (Asymptotic Density of Returns): Let y∈S be recurrent. Then $\lim N_n(y)/n=(1/E_y[T_y])1_{\{Ty<\infty\}}$, P_x- a.s.
3. **Does (a version of 1) always have __________ property (related to 2)?**
4. **Question that leads to an Counterexample/Example.**

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1. **Define: Markov Chain Irreducibility**
   Markov chain is irreducible if it is possible to get to any state from any state. Formally, if its state space is a single communicating class, i.e., x↔y for all x,y∈S.
2. **Name an Irreducibility Theorem**
    - **Decomposition Theorem**: Let $R=\{x : \rho_{xx}=1\}$ be the recurrent states of a Markov chain. R can be written as $\cup_i R_i$, where each R_i is closed and irreducible. [This results shows that for the study of recurrent states we can, without loss of generality, consider a single irreducible closed set.]
  - **For an irreducible and recurrent chain (Corolary 6.46):**
    - The stat/inv measures are unique up to constant multiples.
    - If μ is a stat/inv measure, then μ(x)>0 for all x.
  - **If p is irreducible and has a stationary distribution π.**
    - **Calculating Stat/Inv Distribution**: $\pi(x)=1/E_x[T_x]$.
    - **Theorem D6.5.7**: Any other stationary measure is a multiple of π.
    - **Theorem 6.70** (Markov Chain Convergence Theorem): Consider an irreducible, aperiodic Markov chain with stationary distribution π. Then, $p^n(x,y)\to\pi(y)$ as $n\to\infty$, for all x,y∈S.
3. **Does (a version of 1) always have __________ property (related to 2)?**
4. **Question that leads to a Counterexample/Example.**
    - **If an irreducible Markov chain has period 2, then for every state i∈S do we have $(P_{ii})^2 > 0$?** No, consider P=

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

Note that $P^2=Id$, so period=2 and x↔y. So it is irreducible. But, $P_{ii}=0$, so $(P_{ii})^2=0$.

Other Counterexamples/Examples

- **Are Martingales always Markov processes?**
  No, assume that $(Z_t)_{t\geqslant 2}$ are independent, integrable, nonconstant (say, standard normal distributions), μ=0, and Z_t independent of some X_0, where $X_0:=X_1:=1$ and $X_t:=X_{t-1}+Z_tX_{t-2}$ for every t⩾2. $F_n=\sigma\{X_1, \dots, X_n\}$. Then $E[X_t \mid F_{t-1}]= E[X_{t-1}\mid F_{t-1}] + E[Z_tX_{t-2}\mid F_{t-1}] = X_{t-1} + X_{t-2}E[Z_t\mid F_{t-1}] = X_{t-1}$ for every t⩾1 (hence, if X_0 is integrable, $(X_t)_{t\geqslant 0}$ is a martingale) but $(X_t)_{t\geqslant 0}$ is not a Markov process since the conditional distribution of X_t on F_{t-1} does not depend on X_{t-1} only, but on (X_{t-1},X_{t-2}).
- **If X_n is a homogeneous Markov chain, is it true that $X_{n\wedge 2}$ is also a homogeneous Markov chain?**
  No. Consider the random walk on {…,-1,0,1,…} that with probability ⅓ each either: stays at its position, goes to the right, or to the left. We consider the particular transition probability: $p^n(0,2):=P(X_{n^2}= 2 \mid X_{(n-1)^2}= 0)$, which if X_n is a homogeneous Markov chain, should not depend on n. But guess what? It depends on n. We have $p^1(0,2)=P(X_1=2 \mid X_0=0)=0$, while $p^2(0,2)=P(X_4=2 \mid X_1=0)>0$.

- **If $X_n\in\{-1,1\}$, $S_0=0$, and $S_n:=X_1+\dots+X_n$. Then is $(|S_n|)_{n\geq 0}$ a Markov-chain?**
  Not necessarily. Let $F_n=\boldsymbol{\sigma}\{\mathbf{=X_1,\dots,X_n}\}$. It is not a markov chain unless p=½ (probability of a step to the left), and a counterexample is to take n=1; then $|S_1|=1$ but $P(|S_2|=2)=p \neq ½$ if the first step was to $S_1=-1$, but is $P(|S_2|=2)=1-p \neq ½$ if the first step was to $S_1=+1$. So, $P(|S_2|=2 : F_1)\in\{p, 1-p\}$ is not equal to $P(|S_2|=2 : |S_1|) = ½(1-p) + ½p = ½ \notin \{p, 1-p\}$, and $(|S_n|)_{n\geq 0}$ is not a Markov-chain

- **Does every chain that has a stationary distribution have a limiting distribution?** No.
  Recall that a Markov chain has a limiting distribution if $\pi_j=\lim_{n\to\infty}p^n_{ij}$, ∀i∈S, exists. In particular, if the limit does not depend on the starting state (and hence distribution) of the chain.
  We know a Markov Chain $\{X_n\}$ with a stat. distrib. μ as its initial distribution is a stationary process, because if $X_0\sim\mu$ is a stationary distribution, then for each n, $X_n\sim\mu p_{n-1}=\mu$. So, $(X_0,X_1,\cdots,X_n)\sim(X_m,X_{m+1},\cdots,X_{m+n})$.
  Durrett said a special case to keep in mind for counterexamples is the Markov chain: $X_n:\Omega \to S=\{0,1\}$ with transition probability p(0,1)=p(1,0)=1, and stationary distribution μ(0)=μ(1)=½. Now let $X_0\in\{0,1\}$ w/probability ½ (so not starting with the stat. dist.), so $(X_0,X_1,\cdots)=(0,1,0,\cdots)$ or $(1,0,1,\cdots)$ with probability ½. Note that it does not have a limiting distribution. Durrett is demonstrating that this chain satisfies stationarity, and that it is useful to keep this Markov chain in mind when *picturing* what stationarity means. In particular this is a commonly used counterexample to distinguish between stationary distributions, and limiting distributions.
  Regarding the limiting distribution, note that in this case $\lim_{n\to\infty}p^n_{01}=1$ and $\lim_{n\to\infty}p^n_{11}=0$, so the limit does not exist. Any chain that has a limiting distribution necessarily is stationary (since π can be shown to satisfy the stationarity property). The converse however is not true: and this is what the counterexample shows, since the limit above only exists if the chain is started from μ(0)=μ(1)=1/2, and not from an arbitrary distribution. In general for finite, irreducible Markov chains
  - A stationary distribution always exists.
  - Existence of a limiting distribution implies stationarity.
  - If, in addition to being finite and irreducible, the chain is also aperiodic, then a limiting distribution is guaranteed to exist.