MATH 2243: **Linear Algebra** & **Differential Equations**

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9.3: Ecological Models: Predators and Competitors: Review Predator/Prey Model:

In the absence of predators, prey populations grow: $\frac{dx}{dt} = ax$ where $a > 0$.
In the absence of prey, predator populations decline: $\frac{dy}{dt} = -by$ where $b > 0$.
When they "interact," prey decline at a rate $-pxy$, and predators grow at a rate qxy.
Therefore, the **predator-prey system** is:

$$\frac{dx}{dt} = ax - pxy \qquad = x(a - py),$$
$$\frac{dy}{dt} = -by + qxy \qquad = y(-b + qx).$$

Critical points: When the predator/prey are in equilibrium:
$x(a - py) = 0$ and $y(-b + qx) = 0$.

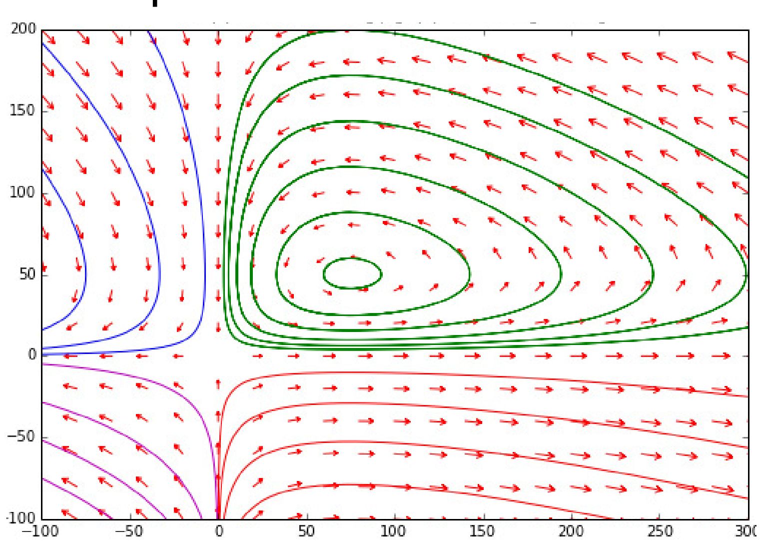
We analyze each critical point (x, y) obtained above by looking at the Jacobian…

$$\mathbf{J}(x,y) = \begin{bmatrix} \frac{\partial}{\partial x}(ax - pxy) & \frac{\partial}{\partial y}(ax - pxy) \\ \frac{\partial}{\partial x}(-by + qxy) & \frac{\partial}{\partial y}(-by + qxy) \end{bmatrix} = \begin{bmatrix} a - py & -px \\ qy & -b + qx \end{bmatrix}.$$

A common point to check is the trivial solution $(0,0)$.
$\mathbf{J}(0,0) = \begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix}$ which will tell us the behavior in the phase plane when the populations are close to extinction.

When checking points with the characteristic equation (associated with the Jacobian), we will use what we learned in the previous section to determine the shape of the phase plane around each critical point (node, saddle point, spiral, improper, closed, etc.).

Phase plane



Competing Species

The competing species model is for when different species are competing for limited resources (food/space). Without this limitation, we would merely have the following logistic equation (introduced in a previous section)…

$$\frac{dx}{dt} = a_1x - b_1x^2,$$
$$\frac{dy}{dt} = a_1y - b_2y^2.$$

However, we must now add the terms which reflect the competition between the two species for the resources.

$$\frac{dx}{dt} = a_1x - b_1x^2 - \mathbf{c_1xy} \quad = \quad x(a_1 - b_1x - c_1y),$$
$$\frac{dy}{dt} = a_1y - b_2y^2 - \mathbf{c_2xy} \quad = \quad y(a_2 - b_2y - c_2x).$$

The effect on the phase plane (and therefore the fate of each population) depends upon whether the coefficients related to “inhibiting effects" like food or space (b_1, b_2) are larger, or whether the coefficients related to the degree of competition (c_1, c_2) are larger. Specifically, whether $c_1c_2 < b_1b_2$ or $c_1c_2 > b_1b_2$.

- ♦ If $c_1c_2 < b_1b_2$, we will see that the critical point related to their coexistence is asymptotically stable, and the species coexist.
- ♦ If $c_1c_2 > b_1b_2$, the critical point is unstable, and either $x(t)$ or $y(t)$ approaches zero as one species becomes extinct.

Problem: #6 This problem deals with the competition system…

$$\frac{dx}{dt} = 60x - 4x^2 - 3xy$$
$$\frac{dy}{dt} = 42y - 2y^2 - 3xy.$$

in which $c_1c_2 = 9 > 8 = b_1b_2$, so the effect of competition should exceed that of inhibition.

The four critical points are $(0,0)$, $(0,21)$, $(15,0)$, and $(6,12)$.

Show that the linearization of the system at $(15,0)$ is: $u' = -60u - 45v, \quad v' = -3v$.

Show that the coefficient matrix of this linear system is a nodal sink for the system.

$$\mathbf{J}(x,y) = \begin{bmatrix} 60 - 8x - 3y & -3x \\ -3y & 42 - 4y - 3x \end{bmatrix}$$

$$\mathbf{J}(15,0) = \begin{bmatrix} 60 - 8(15) - 3(0) & -3(15) \\ -3(0) & 42 - 4(0) - 3(15) \end{bmatrix} = \begin{bmatrix} -60 & -45 \\ 0 & -3 \end{bmatrix}$$

Linearization?

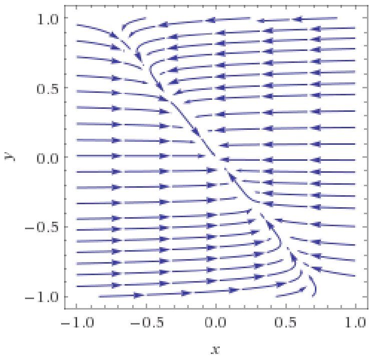
Therefore, the linearization of the system at $(15,0)$ is $u' = -60u - 45v, \quad v' = -3v$.

Behavior around the critical point?

Can you tell by looking at the matrix $\mathbf{J}(15,0)$, what the characteristic equation is... ?

$(-60-\lambda)(-3-\lambda) = 0$ and negative real eigenvalues $\lambda_1 = -60,\ \lambda_2 = -3$.

Hence $(15, 0)$ is a nodal improper sink.



Problem: #12 Consider the following predator-prey system…

$$\frac{dx}{dt} = 5x - x^2 - xy$$

$$\frac{dy}{dt} = -2y + xy$$

Which is the prey?

The prey population $x(t)$ is logistic but the predator population $y(t)$ would (in the absence of any prey) decline naturally. There are three critical points $(0,0),\ (5,0),$ and $(2,3)$ of the system.

Show that the linearization of the system at $(5,0)$ is $u' = -5u - 5v,\ v' = 3v$.

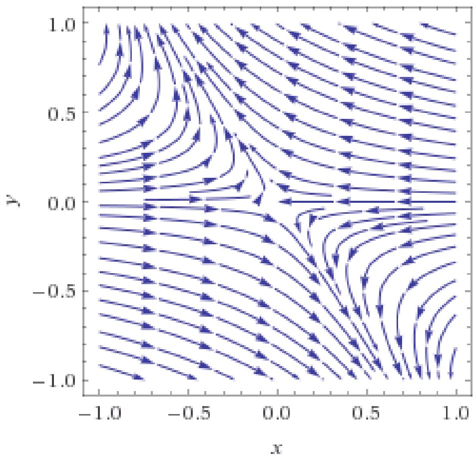
Find the eigenvalues, and classify the critical point at $(5,0)$.

$$\mathbf{J} = \begin{bmatrix} 5-2x-y & -x \\ y & -2+x \end{bmatrix}$$

At $(5,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -5 & -5 \\ 0 & 3 \end{bmatrix}$ has characteristic equation ... ?

$(-5-\lambda)(3-\lambda) = 0$ and real eigenvalues $\lambda_1 = -5,\ \lambda_2 = 3$.

Hence $(5,0)$ is a saddle point of the linearized system $u' = -5u - 5v,\ v' = 3v$.



1.0
0.5
0.0
−0.5
−1.0
y
−1.0 −0.5 0.0 0.5 1.0
x