MATH 2243: Linear Algebra & Differential Equations

Discussion Instructor: Jodin Morey moreyjc@umn.edu
Discussion Session Website: math.umn.edu/~moreyjc

9.2: Linear and Almost Linear Systems: Review

Behavior of the solutions of the autonomous system: $\frac{dx}{dt} = f(x,y), \qquad \frac{dy}{dt} = g(x,y).$

Isolated Critical Point: If there exists a "neighborhood" around the point containing no other critical points.

If the system in question is **almost linear**, we can approximate solutions by eliminating nonlinear terms (**linearization**).

Computation is often simplified if we translate the system so that the critical point in question is located at $(0,0)$ (see examples in the book to learn about this process).

This results are in the linear system $\vec{u}' = \mathbf{J}\vec{u}$ where $\vec{u} = \begin{bmatrix} u & v \end{bmatrix}^T$
whose coefficient matrix is the so-called **Jacobian matrix**:

$$\mathbf{J}(x_0,\ y_0) = \begin{bmatrix} f_x(x_0,\ y_0) & f_y(x_0,\ y_0) \\ g_x(x_0,\ y_0) & g_y(x_0,\ y_0) \end{bmatrix}.$$

When evaluating the system: $\vec{x}' = \mathbf{A}\vec{x}$ OR $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$,
the eigenvalues of $\mathbf{A}$, $(\lambda_1,\ \lambda_2)$ determine the type of critical points in the phase plane in the following way:

Eigenvalues of A	Type of Critical Point
Real, unequal, same sign	Improper node
Real, unequal, opposite sign	Saddle point
Real and equal	Proper or improper node (depending upon defect)
Complex conjugate	Spiral point
Pure imaginary	Center

Stability of Linear Systems: The critical point $(0,0)$ is:
- ♦ Asymptotically stable if the real parts of $\lambda_1,\ \lambda_2$ are both negative;
- ♦ Stable but not asymptotically stable if the real parts of $\lambda_1,\ \lambda_2$ are both zero ($\lambda_1, \lambda_2 = \pm qi$);
- ♦ Unstable if either $\lambda_1,\ \lambda_2$ has a positive real part.

Stability of Almost Linear Systems: When evaluating a nonlinear system, we can estimate a solution with the linearized version of the system. Let λ_1, λ_2 be the eigenvalues of this linearized system.
- ♦ If $\lambda_1 = \lambda_2$ are real numbers, then the critical point is either a node or a spiral point, and is asymptotically stable if $\lambda_1 = \lambda_2 < 0$, and unstable if $\lambda_1 = \lambda_2 > 0$.
- ♦ If λ_1 and λ_2 our pure imaginary, then $(0,0)$ is either a center or a spiral point, and may be either asymptotically stable, stable, or unstable.
- ♦ Otherwise, the critical point of the almost linear system is of the same type and stability at the critical point of the associated linear system.

Eigenvalues $\lambda_1,\ \lambda_2$ for Linearized System	Type of Critical Point for Almost Linear System
$\lambda_1 < \lambda_2 < 0$	Stable improper node
$\lambda_1 = \lambda_2 < 0$	Stable node or spiral point
$\lambda_1 < 0 < \lambda_2$	Unstable saddle point
$\lambda_1 = \lambda_2 > 0$	Unstable node or spiral point
$\lambda_1 > \lambda_2 > 0$	Unstable improper node
$\lambda_1, \lambda_2 = a \pm bi\ (a < 0)$	Stable spiral point
$\lambda_1, \lambda_2 = a \pm bi\ (a > 0)$	Unstable spiral point
$\lambda_1, \lambda_2 = \pm bi$	Stable or unstable, center or spiral point

Determining type and stability of critical point for almost linear system:

$$\frac{dx}{dt} = f(x,y),$$
$$\frac{dy}{dt} = g(x,y).$$

- First, find the critical point(s) (a,b).

- Then, calculate… $\mathbf{J}(a,b) = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix}$ for each critical point.

- To discover the associated linear system(s)…

$$\frac{du}{dt} = f_x(a,b)u + f_y(a,b)v,$$
$$\frac{dv}{dt} = g_x(a,b)u + g_y(a,b)v.$$

- Then calculate λ_1, λ_2 from each new system using our usual methods, and use the above table to determine the type and stability of the associated critical point(s).

Problem: #16 The system: $\frac{dx}{dt} = x - 2y + 1,\quad \frac{dy}{dt} = x + 3y - 9,$ has a single critical point (x_0, y_0). Apply Theorem 2 (Stability of Almost Linear Systems, page 538) to classify this critical point as to type and stability. Verify your conclusion by constructing a phase portrait for the given system.

$x - 2y + 1 = 0, \qquad x - 2y + 1 = 0$

$x = 2y - 1$ and $(2y - 1) + 3y - 9 = 5y - 10 = 0$ when $y = 2$.

Therefore, $x = 2(2) - 1 = 3$.

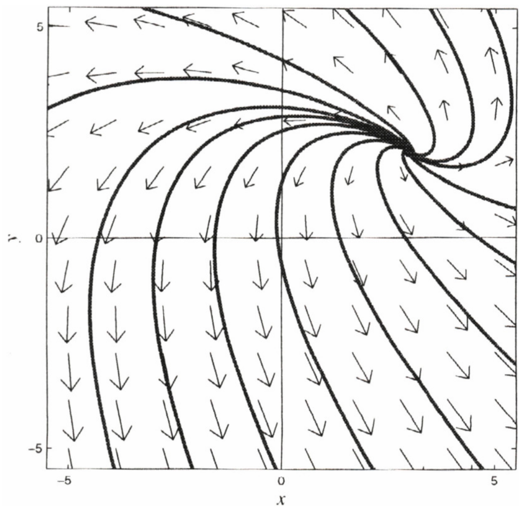
Critical point: $(3,2)$. Now What?…

The Jacobian matrix $\mathbf{J} = \begin{bmatrix} \frac{\partial}{\partial x}(x - 2y + 1) & \frac{\partial}{\partial y}(x - 2y + 1) \\ \frac{\partial}{\partial x}(x + 3y - 9) & \frac{\partial}{\partial y}(x + 3y - 9) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}.$

$$\mathbf{J}(3,2) = \mathbf{J} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}.$$

has characteristic equation $\lambda^2 - 4\lambda + 5 = 0$ and eigenvalues $\lambda_1, \lambda_2 = 2 \pm i$ that are complex conjugates with positive real part. So, ...

Hence the critical point $(3,2)$ is an unstable spiral point as shown below…



Problem: **#24** Investigate the type of the critical point $(0,0)$ of the almost linear system:

$$\frac{dx}{dt} = 5x - 3y + y(x^2 + y^2), \qquad \frac{dy}{dt} = 5x + y(x^2 + y^2).$$

Verify your conclusion by constructing a phase portrait.

We must first calculate the Jacobian matrix $\mathbf{J}$ and its eigenvalues at $(0,0)$ and at each of the other critical points we observe in our phase portrait for the given system. Then we apply Theorem 2 to determine as much as we can about the type and stability of each of these critical points of the given almost linear system.

$$\mathbf{J} = \begin{bmatrix} \frac{\partial}{\partial x}(5x - 3y + y(x^2 + y^2)) & \frac{\partial}{\partial y}(5x - 3y + y(x^2 + y^2)) \\ \frac{\partial}{\partial x}(5x + y(x^2 + y^2)) & \frac{\partial}{\partial y}(5x + y(x^2 + y^2)) \end{bmatrix} = \begin{bmatrix} 5 + 2xy & -3 + x^2 + 3y^2 \\ 5 + 2xy & x^2 + 3y^2 \end{bmatrix}$$

Now what?

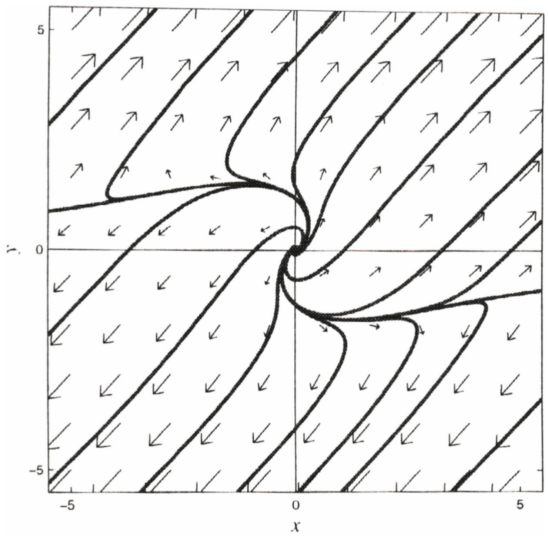
$$\mathbf{J}(0,0) = \begin{bmatrix} 5 & -3 \\ 5 & 0 \end{bmatrix} \text{ And...?}$$

Has characteristic equation $\lambda^2 - 5\lambda + 15 = 0$

and complex conjugate eigenvalues $\lambda_1 \approx 2.5 + 2.96i,$ $\quad \lambda_2 \approx 2.5 - 2.96i.$ So, ...?

Hence, $(0,0)$ is a spiral source of the given almost linear system.

The figure below shows this critical point…



Problem: #34

The term bifurcation generally refers to something "splitting apart." With regard to differential equations or systems involving a parameter, it refers to abrupt changes in the character of the solutions as the parameter is changed continuously. The problem below illustrates a sensitive case in which small perturbations (changes) in the coefficients of this almost linear system can change the type or stability (or both) of a critical point.

$$\frac{dx}{dt} = -x + \varepsilon y, \qquad \frac{dy}{dt} = x - y.$$

Show that the critical point $(0,0)$ is :

a) a stable spiral point the $\varepsilon < 0$; $\qquad$ *b*) a stable node if $0 \le \varepsilon < 1$.

$$\mathbf{J}(x,y) = \begin{bmatrix} -1 & \varepsilon \\ 1 & -1 \end{bmatrix}$$

In this case, $\mathbf{J}(0,0) = \mathbf{J}(x,y) = \begin{bmatrix} -1 & \varepsilon \\ 1 & -1 \end{bmatrix}.$

$$\begin{vmatrix} -1-\lambda & \varepsilon \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 - \varepsilon = 0.$$

$$\lambda = \frac{-2 \pm \sqrt{4-4(1-\varepsilon)}}{2} = -1 \pm \sqrt{\varepsilon}\,.$$

Recall: "Show that the critical point $(0,0)$ is :

$a)$ a stable spiral point the $\varepsilon < 0$; $b)$ a stable node if $0 \le \varepsilon < 1$."

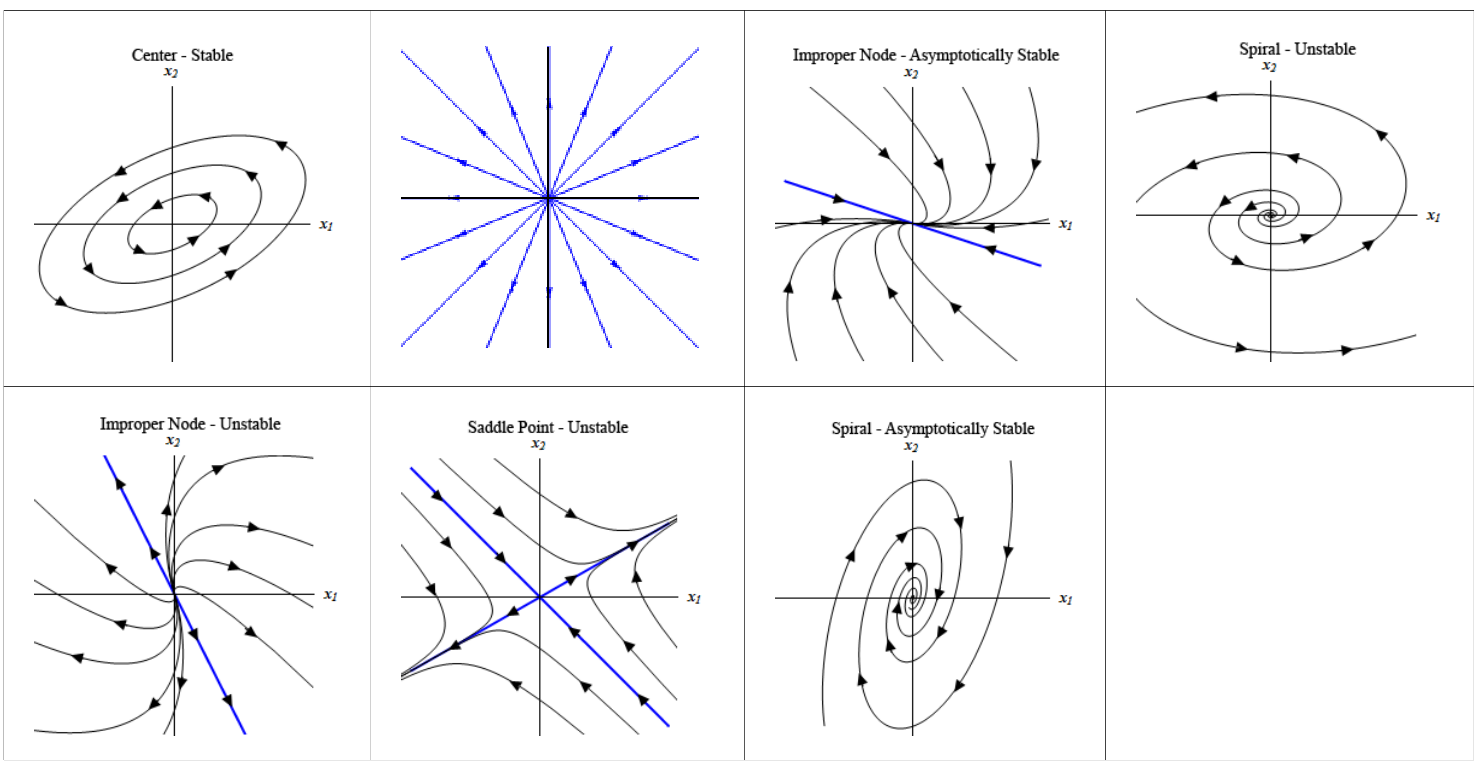
$a)$ If $\varepsilon < 0$ then $\lambda_1, \lambda_2 = -1 \pm i\sqrt{-\varepsilon}\,$. Thus the characteristic roots are complex conjugates with negative real part, so it follows that $(0,0)$ is an asymptotically stable spiral point.

$b)$ If $\varepsilon = 0$ then the characteristic roots $\lambda_1 = \lambda_2 = -1$ are equal and negative, so $(0,0)$ is an asymptotically stable node (star).

Otherwise...

If $0 < \varepsilon < 1$ then $\lambda_1, \lambda_2 = -1 \pm \sqrt{\varepsilon}$ are negative and unequal, so $(0,0)$ is an asymptotically stable improper node.

What are the eigenvalues associated with these phase portraits?



Study tips for Final Exam (in order of importance, I think):

With all of the suggestions below, don’t just read these documents, but practice working out actual problems…

- Do some problems of the type Prof. said might be on the final.
- Final exams from previous years.
- Midterms you’ve taken this semester (focus on problems you did poorly on).
- Old midterms from previous years.
- Read sections from the book that have given you difficulty.
- Quizzes and homework.
- Review my overheads.