MATH 2243: Linear Algebra & Differential Equations

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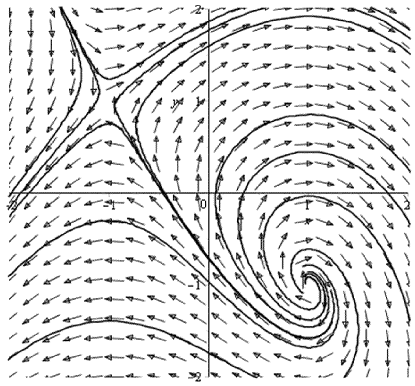
7.4: Solution Curves of Linear Systems



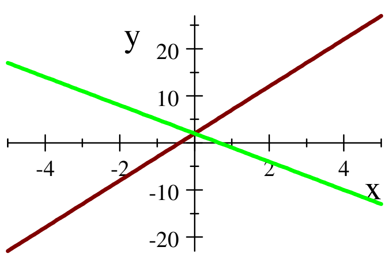
Nonlinear DEQ w/two critical points

In subsequent courses, you can learn how to linearize a nonlinear DEQ around critical points (the light blue dots in image above) to discover the local behavior of the solution curves. By doing this with each of the critical points, you can patch them together to discover the behavior of solutions in a complicated system.

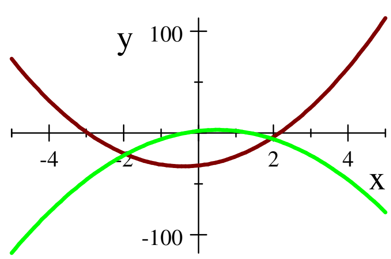
For now, assume a linear first-order homogeneous constant-coefficient system: $\vec{x}' = \mathbf{A}\vec{x}$, where $\mathbf{A}$ is 2×2. Let's call λ_1, λ_2 the eigenvalues of $\mathbf{A}$. This section explores how one can predict the geometry of the solutions to our system by simply knowing λ_1, λ_2.

This is similar to how you've learned the graphs of:

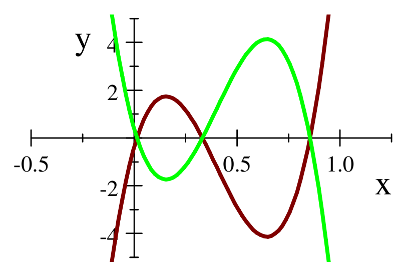
$f_1 = cx + d$, $f_2 = cx^2 + dx + e$,   and $f_3 = cx^3 + dx^2 + ex + f$.



Straight line. Red with $c > 0$.



Curve with one extrema. Red with $c > 0$.



Curve with zero or two extrema

The sign of c tells you immediately from the equation about the shape of the graph.

Recall that:

- If $\lambda_1, \lambda_2 \in \mathbb{R}$, solutions take the form: $\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$.
- If $\lambda, \overline{\lambda} = p \pm iq$, then the normal process using λ gives a tentative solution $\vec{f}(t)e^{\lambda t} + i\vec{g}(t)e^{\lambda t}$,

  from which we get the real valued solution(s): $\vec{x}(t) = c_1\vec{f}(t)e^{\lambda t} + c_2\vec{g}(t)e^{\lambda t}$.
- If $\lambda_1 = \lambda_2 \in \mathbb{R}$, $\mathbf{A}$ may not have two linearly independent eigenvectors.

  If Independent $\vec{v}_1, \vec{v}_2$, then $x(t) = c_1\vec{v}_1 e^{\lambda t} + c_2\vec{v}_2 e^{\lambda t}$.

Otherwise: $x(t) = c_1\vec{v}_1 e^{\lambda t} + c_2(\vec{v}_1 t + \vec{v}_g)e^{\lambda t}$, where $\vec{v}_g$ is a "**generalized**" **eigenvector**.

So perhaps in analogy to above, we find qualitatively different solutions depending upon the signs of constants λ_1, λ_2, and also from c_1, c_2.

Eigenvalue Scenarios

For *real* eigenvalues, observe we have the following possibilities:

Distinct eigenvalues

- A. Nonzero and opposite sign ($\lambda_1 < 0 < \lambda_2$)
- B. Both negative ($\lambda_1 < \lambda_2 < 0$)
- C. Both positive ($0 < \lambda_2 < \lambda_1$)
- D. One zero and one negative ($\lambda_1 < \lambda_2 = 0$)
- E. One zero and one positive ($0 = \lambda_2 < \lambda_1$)

Repeated eigenvalue

- F. Positive ($\lambda_1 = \lambda_2 > 0$)
- G. Negative ($\lambda_1 = \lambda_2 < 0$)
- H. Zero ($\lambda_1 = \lambda_2 = 0$)

Observe that the origin ($\vec{x} \equiv \vec{0}$) is always an equilibrium solution of $\vec{x}' = \mathbf{A}\vec{x}$. However, in the cases below we explore the different behaviors that can occur *around* the origin.

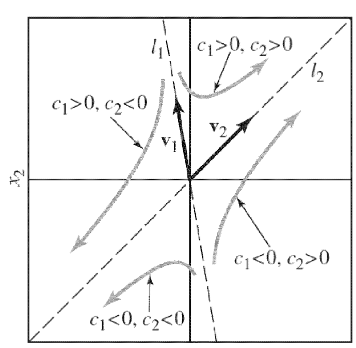
In what follows, it's helpful to remember that a positive eigenvalue represents pushing solutions away from the origin along the eigenvector, and a negative eigenvalue represents pulling solutions toward the origin.

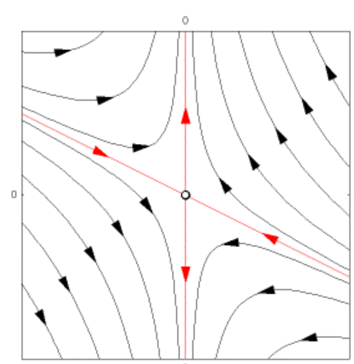
Saddle Points

A. Nonzero Distinct Eigenvalues of Opposite Sign: $\lambda_1 < 0 < \lambda_2, \qquad x(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$

If c_1 and c_2 are nonzero, we have two asymptotes, namely lines l_1 and l_2 passing through the origin (see image below), and parallel to the eigenvectors $\vec{v}_1, \vec{v}_2$. On one of these lines, solutions are moving toward the origin as $t \to \infty$, and on the other line solutions move away from the origin. The origin itself is an equilibrium solution, and is referred to as a **saddle point**.

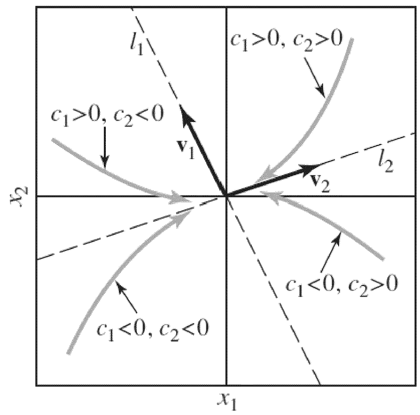
Saddle Point: Two trajectories approach the critical point, but all others are unbounded as $t \to \infty$.





B. Distinct Negative Eigenvalues: $\lambda_1 < \lambda_2 < 0, \qquad x(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$

Every trajectory approaches the origin as $t \to \infty$.



Now that we have some examples as a frame of reference, let's learn some terms.

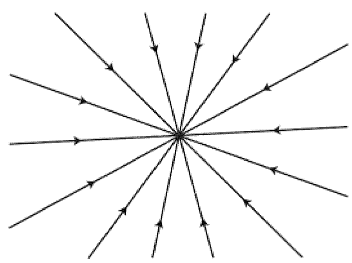
The preceding is an example of a *node*:

Node: Every trajectory approaches (or recedes) from the origin as $t \to \infty$, AND

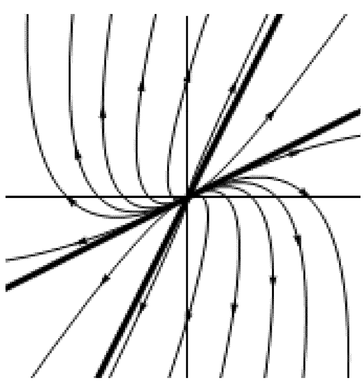
every trajectory is tangent, at the origin, to some straight line through the origin.

A **star point**, also called a **proper node** (seen below) is another example of a node, where each trajectory is a straight line thru the origin. And, approaching the origin, each trajectory is tangent at the origin to a straight line (namely, itself; since it is a straight line!).

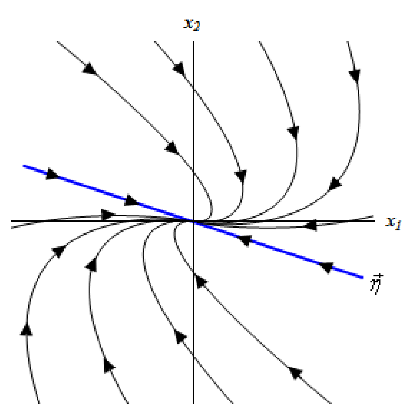
Proper Node: Trajectories approach or recede in all directions.



Improper Node: All trajectories become tangent to just one or two vectors as they approach the origin.



Improper Nodal Source, independent



Improper Nodal Sink, dependent $\vec{v}_i$

Sink: All trajectories approach the critical point (the origin) as $t \to \infty$.

Source: All trajectories flow away from the critical point (the origin) as $t \to \infty$.

Principle: Time Reversal

Let $\vec{x}(t)$ be a solution of the 2D linear system $\vec{x}' = \mathbf{A}\vec{x}$, then the function $\tilde{x}(t) = \vec{x}(-t)$ is a solution of the system $\tilde{x}' = -\mathbf{A}\tilde{x}$.

Proof: Note that $\tilde{x}' = \frac{d}{dt}\vec{x}(-t) = \frac{d}{d(-t)}\frac{d(-t)}{dt}\vec{x}(-t) = -\vec{x}'(-t)$.

Therefore, the left-hand side of $\tilde{x}' = -\mathbf{A}\tilde{x}$ becomes $-\vec{x}'(-t)$, and the right side becomes

$-\mathbf{A}\vec{x}(-t)$, giving us: $\vec{x}'(-t) = \mathbf{A}\vec{x}(-t)$.

And since we know $\vec{x}'(t) = \mathbf{A}\vec{x}(t)$ for all t, the equality is satisfied and
$\tilde{x}(t)$ is a solution of the system $\tilde{x}' = -\mathbf{A}\tilde{x}$. ■

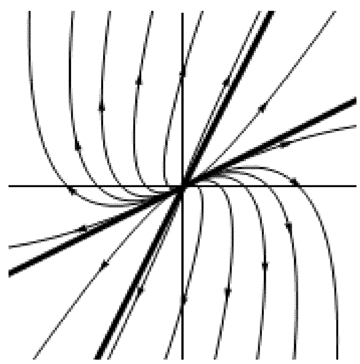
Relatedly, if a matrix $\mathbf{A}$ has eigenvalues λ_1, λ_2, then $-\mathbf{A}$ has the negative of those eigenvalues.

Proof: $\mathbf{A}\vec{v} = \lambda_1\vec{v}$, then multiplying by -1 gives us: $-\mathbf{A}\vec{v} = -\lambda_1\vec{v}$. And similarly for λ_2. ■

Continuing on with our cases:

Improper Nodal Source

C. **Distinct Positive Eigenvalues**: $0 < \lambda_2 < \lambda_1$, $x(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$

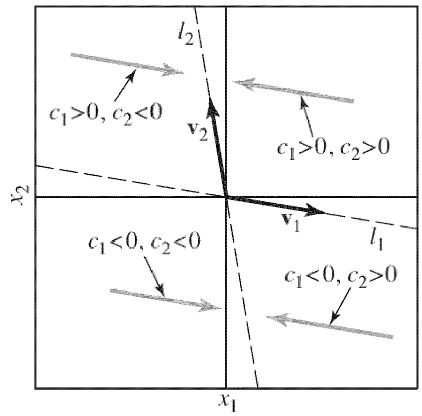


Zero Eigenvalues and Straight-Line Solutions

D. **One Zero and One Negative Eigenvalue**: $\lambda_1 < \lambda_2 = 0$, $\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2$.

$\vec{x}' = c_1\lambda_1\vec{v}_1 e^{\lambda_1 t}$, all solutions are moving tangent to $\vec{v}_1$.

Observe that each point lying on the line l_2 represents a constant solution (equilibrium solution) to the system (instead of the origin being the only constant solution).



E. One Zero and One Positive Eigenvalue: $0 = \lambda_2 < \lambda_1,\qquad \vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2$

By the principle of time reversal, the trajectories in this system ($\vec{x}' = \mathbf{A}\vec{x}$) are identical to those in $\vec{x}' = -\mathbf{A}\vec{x}$, except that they flow in the opposite direction. And notice that the eigenvalues you get from $-\mathbf{A}$ are $-\lambda_1 < -\lambda_2 = 0$, and puts us in the previous case of "One Zero and One Negative Eigenvalue." So just reverse the direction of the flow lines in the previous graph.

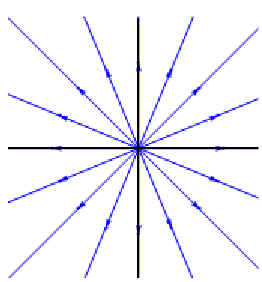
Repeated Eigenvalues - Proper and Improper Nodes

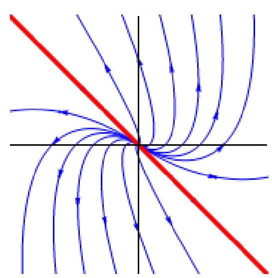
F. Repeated Positive Eigenvalue: $\lambda > 0$

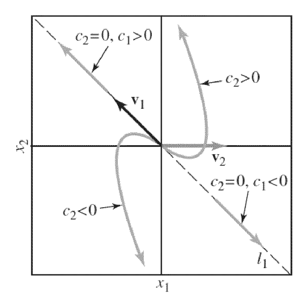
We have two cases:

- Two Independent Eigenvectors: $x(t) = c_1\vec{v}_1 e^{\lambda t} + c_2\vec{v}_2 e^{\lambda t}$
- One Independent Eigenvector: $x(t) = c_1\vec{v}_1 e^{\lambda t} + c_2(\vec{v}_1 t + \vec{v}_2)e^{\lambda t}$

These generate sources which are either a star or an improper node, respectively.

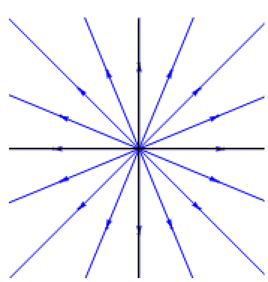






Case: **Two Independent Eigenvectors**

We will learn that solutions for this case look like those in the graph below.



First, we must learn some lemmas.

Lemma: If **A** has a repeated eigenvalue λ with two linearly independent eigenvectors, then *every* nonzero vector is an eigenvector of **A** with eigenvalue λ.

Proof: Let $\vec{v}_1, \vec{v}_2$, be linearly independent eigenvectors associated with λ. Therefore, we have $\mathbf{A}\vec{v}_i = \lambda\vec{v}_i$.

Now choose *any* $\vec{v}_0$. We must show $\mathbf{A}\vec{v}_0 = \lambda\vec{v}_0$.

Observe that since $\vec{v}_1, \vec{v}_2$ are linearly independent in $\mathbb{R}^2$, we have $\vec{v}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$, for some $c_1, c_2 \in \mathbb{R}$.

Therefore, $\mathbf{A}\vec{v}_0 = \mathbf{A}(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\mathbf{A}\vec{v}_1 + c_2\mathbf{A}\vec{v}_2$

$= c_1\boldsymbol{\lambda}\vec{v}_1 + c_2\boldsymbol{\lambda}\vec{v}_2 = \lambda(c_1\vec{v}_1 + c_2\vec{v}_2) = \lambda\vec{v}_0.$

So we have $\mathbf{A}\vec{v}_0 = \lambda\vec{v}_0$ and $\vec{v}_0$ is an eigenvector of $\mathbf{A}$. ■

Lemma: If $\mathbf{A}$ has a repeated eigenvalue λ with two linearly independent eigenvectors, then $\mathbf{A} = \lambda\mathbf{I}$.

Proof: Without loss of generality, let's choose the independent eigenvectors to be $\vec{v}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

By the previous Lemma we have $\mathbf{A}\vec{v} = \lambda\vec{v}$ for every $\vec{v}$. Using $\vec{v}_1$, we have $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$,

and expanding the left-hand side we have $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$.

Similarly, using $\vec{v}_2$, we calculate $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}$.

Therefore, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \lambda\mathbf{I}.$ ■

Therefore, the system $\vec{x}' = \mathbf{A}\vec{x}$ becomes:

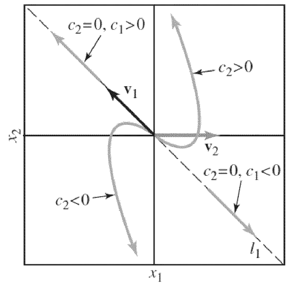
$$x_1' = \lambda x_1$$
$$x_2' = \lambda x_2.$$

Solving these separable DEQs individually, we get:

$$x_1 = c_1e^{\lambda t}$$
$$x_2 = c_2e^{\lambda t}$$

Or $\vec{x}(t) = e^{\lambda t}\vec{c}$, where $\vec{c}$ is solved for from initial conditions. In other words, the solution just pushes the initial condition out from the origin.

Case: **Only One Independent Eigenvector** $x(t) = c_1\vec{v}_1e^{\lambda t} + c_2(\vec{v}_1t + \vec{v}_2)e^{\lambda t}$, where $\vec{v}_2$ is a "generalized" eigenvector.



For the improper node, if $c_2 \neq 0$ (i.e., the initial condition is not on $\vec{v}_1$) then as $t \to -\infty$, each trajectory approaches the origin along a solution curve which is tangent there to the vector $\vec{v}_1$. Alternatively, as $t \to +\infty$, the trajectory moves far from the origin and all trajectories become essentially parallel to the vector $\vec{v}_1$, but pointing in the opposite direction to when the solution was near the origin.

G: **Repeated Negative Eigenvalue**: $\lambda < 0$

By the time reversal property, we conclude the same properties as the positive case, except that the flow arrows reverse. Therefore we find a proper nodal (star) or improper nodal sink.

H: **Repeated Zero Eigenvalue**: $\lambda = 0$

We have 2 cases:

- Two Independent Eigenvectors,
- One Independent Eigenvector.

Case - **Two Independent Eigenvectors**:

Lemma: If **A** has a repeated zero eigenvalue λ, and two linearly independent eigenvectors $\vec{v}_1, \vec{v}_2$, then $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Proof: Without loss of generality, let's choose the independent eigenvectors to be $\vec{v}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Now observe $\mathbf{A}\vec{v}_1 = \lambda\vec{v}_1$. The left-hand side becomes $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$,

and the right hand side becomes $0 \cdot \vec{v} = \vec{0}$.

Therefore, $a_{12} = a_{22} = 0$. A similar calculation for $\vec{v}_2$ reveals that $a_{11} = a_{21} = 0$.

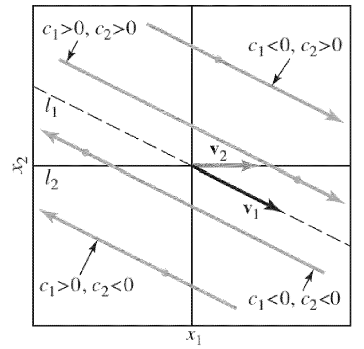
The conclusion follows. ■

Observe that we can choose *any* vector $\vec{v}$, and $\mathbf{A}\vec{v} = \vec{0} = 0\vec{v}$, since **A** is the zero matrix, and therefore every vector is an eigenvector of **A**.

Therefore, the system $\vec{x}' = \mathbf{A}\vec{x}$ becomes: $x_1' = 0$ and $x_2' = 0$, and integrating we find that the solutions are constants $x_1(t) = c_1$ and $x_2(t) = c_2$, where (c_1, c_2) are simply the initial values.

In other words, each point (c_1, c_2) in the plane becomes an equilibrium solution where the "trajectories" are the fixed points $\vec{x}(t) \equiv \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T$.

Case - One Independent Eigenvector: $x(t) = c_1\vec{v}_1 + c_2(\vec{v}_1 t + \vec{v}_2)$, where $\vec{v}_1$ is the eigenvector, and $\vec{v}_2$ is what is called a "generalized eigenvector," which is explained in the next section of the book. In the phase plane below, note that l_1 is thought of as a median strip dividing two opposite “lanes of traffic."

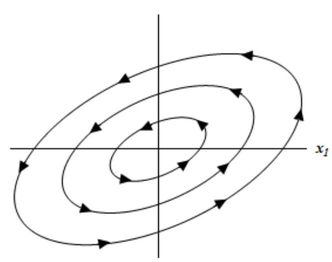


Complex Conjugate Eigenvalues and Eigenvectors

For complex eigenvalues, there are two basic situations:

- A. **Pure imaginary eigenvalues**: $\lambda_1, \lambda_2 = \pm iq$
- B. **Complex Eigenvalues with Nonzero Real Parts**: $\lambda_1, \lambda_2 = p \pm iq$, with $p \neq 0$.

A. **Pure imaginary eigenvalues**: $\lambda_1, \lambda_2 = \pm iq$

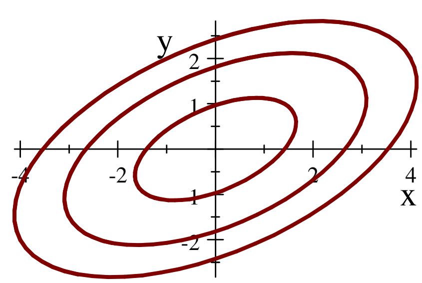


center

Because of the concentric nature of the solutions around the origin, the origin is referred to as a **center equilibrium**.

Knowledge of the eigenvalues and eigenvectors of **A** are insufficient to predict the direction of flow of the trajectories.

For example, if we examine $\vec{x}' = \mathbf{A}\vec{x}$ with $\mathbf{A} = \begin{bmatrix} 6 & -17 \\ 8 & -6 \end{bmatrix}$ using our usual methods, we calculate trajectories which look like:



with the solution moving counterclockwise as t increases.

However, if we instead examine $\vec{x}' = (-\mathbf{A})\vec{x}$, we notice it has the same eigenvalues and eigenvectors. And therefore the same solution curves as above, but with solutions moving clockwise as t increases (principle of time reversal).

So how can we know direction of flow in advance of calculating the solution?

Lemma: For $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if $c > 0$ the flow will be counterclockwise. If $c < 0$ the flow will be clockwise.

Proof: We will monitor the direction in which the solution curve flows as it crosses the positive x-axis.

If s is any positive number, then any trajectory for the system $\vec{x}' = \mathbf{A}\vec{x}$ passing through $(s, 0)$ (which sits on the positive x-axis) satisfies:

$$\vec{x}' = \mathbf{A}\vec{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} s \\ 0 \end{bmatrix} = \begin{bmatrix} as \\ cs \end{bmatrix} = s\begin{bmatrix} a \\ c \end{bmatrix}, \text{ at the point } (s, 0).$$

Therefore, at $(s, 0)$ the direction of flow of the solution curve is a positive scalar multiple of the vector $\begin{bmatrix} a & c \end{bmatrix}^T$.

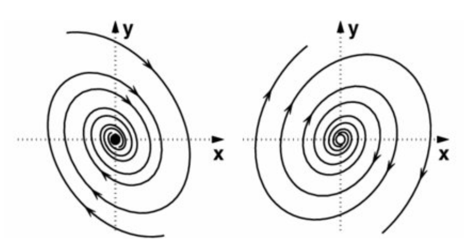
Since c cannot be zero when $\lambda_{1,2}$ imaginary (see problem 39 in the book), this vector either points "upward" into the 1st quadrant (if $c > 0$), or "downward" into the 4th quadrant (if $c < 0$). ■

Complex Eigenvalues: Spiral

B. **Complex Eigenvalues with Nonzero Real Parts**: $\lambda_1, \lambda_2 = p \pm iq$, with $p \neq 0$.

As before, positive real part is associated with a source, and negative real part is associated with a sink.

General solution: $x(t) = c_1 e^{pt}\left(\vec{a}\cos qt - \vec{b}\sin qt\right) + c_2 e^{pt}\left(\vec{b}\cos qt + \vec{a}\sin qt\right)$, where $\vec{a}$ and $\vec{b}$ are the real and imaginary parts of a complex valued eigenvector $\vec{v}$ of $\mathbf{A}$.



Sprial sink, Sprial source

Exercises

Problem: #4 For Problem 4 in Section 7.3, categorize the eigenvalues and eigenvectors of the coefficient matrix **A** according to Fig. 7.4.16 and sketch the phase portrait of the system by hand. Then use a computer system or graphing calculator to check your answer.

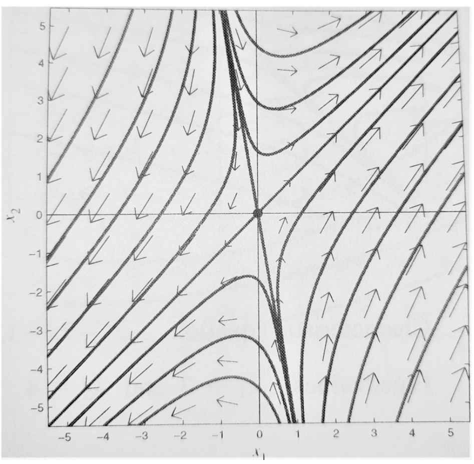
In section 7.3, the system was: $x_1' = 4x_1 + x_2, \qquad x_2' = 6x_1 - x_2.$

Where $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}$, with eigenvalues: $\lambda_{1,2} = -2, 5$

Eigenvectors: $\vec{v}_1 = \begin{bmatrix} 1 & -6 \end{bmatrix}^T$ and $\vec{v}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

With real eigenvalues on either side of zero, we expect a saddle point with trajectories moving outward along the eigenvector $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ with the positive eigenvalue $\lambda = 5$, and trajectories moving inward along the eigenvector $\begin{bmatrix} 1 & -6 \end{bmatrix}^T$ with the negative eigenvalue $\lambda = -2$.

And indeed, upon graphing we find:



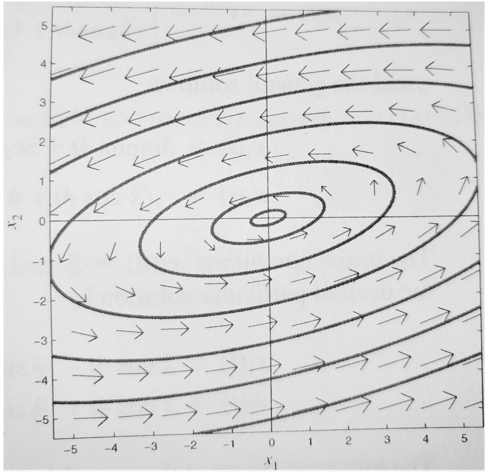
Problem: #8 For Problem 8 in Section 7.3, categorize the eigenvalues and eigenvectors of the coefficient matrix **A** according to Fig. 7.4.16 and sketch the phase portrait of the system by hand. Then use a computer system or graphing calculator to check your answer.

For section 7.3, the system was: $x_1' = x_1 - 5x_2, \qquad x_2' = x_1 - x_2.$

Where $\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$, with eigenvalues: $\lambda_{1,2} = \pm 2i$.

Eigenvectors: $\vec{v}_{1,2} = \begin{bmatrix} 1 \pm 2i & 1 \end{bmatrix}^T$.

With purely imaginary eigenvalues, we expect a center. And with $c = 1 > 0$, we expect the trajectories to be counterclockwise. And indeed, upon graphing we find:



Problem: #12 For Problem 12 in Section 7.3, categorize the eigenvalues and eigenvectors of the coefficient matrix **A** according to Fig. 7.4.16 and sketch the phase portrait of the system by hand. Then use a computer system or graphing calculator to check your answer.

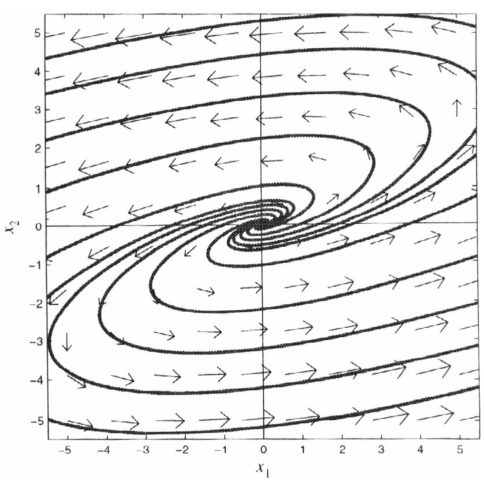
Recall that in the class notes for section 7.3, the system was: $x_1' = x_1 - 5x_2, \qquad x_2' = x_1 + 3x_2.$

We determined there, that the eigenvalues and eigenvectors were

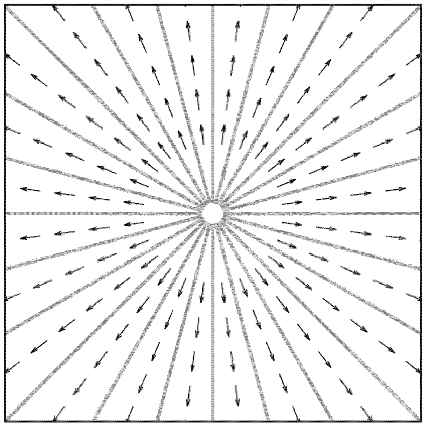
Eigenvalues: $\lambda_{1,2} = 2 \pm 2i.$

Eigenvectors: $\vec{v} = \begin{bmatrix} 1 \pm 2i & -1 \end{bmatrix}^T.$

Observe that the real part of λ_i is greater than zero, so we expect a source. Also, since the imaginary parts of λ_i are not zero, we expect a spiral. And from $\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}$, we have $c = 1 > 0$, so we expect the spiral to be counterclockwise. And indeed, upon graphing we find:



Problem: #21 The following phase portrait corresponds to a linear system of the form $\vec{x}' = \mathbf{A}\vec{x}$ in which the matrix A has two linearly independent eigenvectors. Determine the nature of the eigenvalues and eigenvectors. (For example, you might discern that the system has pure imaginary eigenvalues, or that it has real eigenvalues of opposite sign; that an eigenvector associated with the positive eigenvalue is roughly $\begin{bmatrix} 2 & -1 \end{bmatrix}^T$, etc.)



Observe that there is no circular motion, so it appears that the eigenvalues are real. It also appears we have a source (arrows pointing out from the origin), and there are no trajectories that head towards the origin, so these real eigenvalues appear to be positive. Furthermore, each trajectory near the origin is tangent to a straight line, notably itself (since all of the trajectories are straight lines). Therefore, not only is it a node, it is a star point, or proper node. There are no restrictions on the eigenvectors, other than those given that the eigenvectors be linearly independent.