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7.3: The Eigenvalue Method for Linear Systems

Finding the general solution to a homogeneous linear first-order system $\vec{z}' = \mathbf{A}\vec{z}$ with real coefficients, or equivalently:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n. \end{aligned}$$

Eigenvalue Solutions of $\vec{z}' = \mathbf{A}\vec{z}$ Theorem: If λ is an eigenvalue of the constant coefficient matrix $\mathbf{A}$, and if $\vec{v}$ is an eigenvector associated with λ, then the vector function $\vec{z}(t) = \vec{v}e^{\lambda t}$, is a nontrivial solution to the system. (the trivial solution is $\vec{z}(t) = \vec{0}$)

Proof: By theorem from previous section, it suffices to find n linearly independent solution vectors $\vec{z}_1, \ldots, \vec{z}_n$, and place them in a linear combination. But how to find these vectors?

Recall the educated guess when solving individual homogeneous linear DEQs: $x(t) = e^{rt}$. But this isn't a vector.

So how about $\vec{z}(t) = \vec{v}e^{\lambda t} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} e^{\lambda t} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix}$, for some constants λ, v_i.

Note that $\vec{z}' = \lambda \vec{v} e^{\lambda t}$.

To be a solution, we need $\vec{z}' = \mathbf{A}\vec{z}$, or substituting from above, $\lambda \vec{v} e^{\lambda t} = \mathbf{A}\vec{v}e^{\lambda t}$, and canceling $e^{\lambda t}$ from both sides gives us $\lambda \vec{v} = \mathbf{A}\vec{v}$.

You may recall that when $\vec{v} \neq \vec{0}$, then the above equation is the requirement for having an eigenvalue, eigenvector pair $(\lambda, \vec{v})$ of $\mathbf{A}$. ■

Steps to a Solution for $\vec{z}' = \mathbf{A}\vec{z}$.

- ♦ Solve the characteristic equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$ for the eigenvalues λ_i of $\mathbf{A}$.
- ♦ Attempt to find n linearly independent eigenvectors $\vec{v}_i$ from the λ_i, each pair $\{\lambda_i, \vec{v}_i\}$ gives you a linearly independent solution $\vec{z}_i(t) = \vec{v}_i e^{\lambda_i t}$.
- ♦ If n such vectors are found, then: $\vec{z}(t) = c_1\vec{z}_1(t) + c_2\vec{z}_2(t) + \ldots + c_n\vec{z}_n(t)$.

In general, to verify independence of solutions, check if the Wronskian $\left|\vec{v}_1 e^{\lambda_1 t} \;\ldots\; \vec{v}_n e^{\lambda_n t}\right|$ is nonzero.

Steps for Complex Eigenvalues $\lambda \in \mathbb{C}$ (which come in conjugate pairs):

- Form the complex solution $\vec{z}(t) = \vec{v}_i e^{\lambda t}$ associated either λ or $\overline{\lambda}$ (doesn't matter which).
  This involves a complex λ and complex $\vec{v}_i$.

- Notationally manipulate $\vec{z}(t)$ into the form $\vec{f}(t) + i\vec{g}(t)$ to identify the real and imaginary parts (see examples of this below)

- Once you have found these two solutions ($\vec{f}(t)$ and $\vec{g}(t)$), you are done.
  The solutions associated with the eigenvalue's conjugate $\overline{\lambda}$ are identical.

As an exercise, verify that the solutions obtained from each conjugate λ,$\overline{\lambda}$ are identical.

See the examples below to get a better feel for what this section is saying.

Exercises

Problem: **#12** Apply the eigenvalue method of this section to find a general solution of the given system:

$$x_1' = x_1 - 5x_2, \qquad x_2' = x_1 + 3x_2.$$

$$\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}$$

Characteristic Equation: $|\mathbf{A} - \lambda I| = \begin{vmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 5 = \lambda^2 - 4\lambda + 8.$

$\lambda = \frac{4 \pm \sqrt{16-32}}{2}$

Eigenvalues: $\lambda = 2 \pm 2i$.

Eigenvector Equation (**for** $2+2i$):

$$\begin{bmatrix} 1-(2+2i) & -5 \\ 1 & 3-(2+2i) \end{bmatrix} \Rightarrow \begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1-2i \\ -1-2i & -5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1-2i \\ 0 & 0 \end{bmatrix}, \qquad y = b, \; x = -b(1-2i)$$

Eigenvector: $\vec{v} = \begin{bmatrix} -b(1-2i) & b \end{bmatrix}^T = \begin{bmatrix} 1-2i & -1 \end{bmatrix}^T$, when $b = -1$.

So we have: $\vec{v}e^{(2+2i)t}$ (from above, "Notationally manipulate into the form $f(t) + ig(t)$")

$$= e^{2t}e^{2it}\begin{bmatrix} 1-2i \\ -1 \end{bmatrix}$$

$$= e^{2t}(\cos 2t + i\sin 2t)\begin{bmatrix} 1-2i \\ -1 \end{bmatrix} = e^{2t}\begin{bmatrix} (\cos 2t + i\sin 2t)(1-2i) \\ -\cos 2t - i\sin 2t \end{bmatrix}.$$

Expanding the parentheses:

$$= e^{2t}\begin{bmatrix} (\cos 2t + i\sin 2t) - 2i(\cos 2t + i\sin 2t) \\ -\cos 2t - i\sin 2t \end{bmatrix}$$

$$= e^{2t}\begin{bmatrix} \cos 2t + i\sin 2t - 2i\cos 2t + 2\sin 2t \\ -\cos 2t - i\sin 2t \end{bmatrix}.$$

Collecting the imaginary parts:

$$= e^{2t}\begin{bmatrix} \cos 2t + 2\sin 2t + i(\sin 2t - 2\cos 2t) \\ -\cos 2t - i\sin 2t \end{bmatrix}.$$

Separating the imaginary part from the real part ($f(t) + ig(t)$):

$$= e^{2t}\begin{bmatrix} \cos 2t + 2\sin 2t \\ -\cos 2t \end{bmatrix} + ie^{2t}\begin{bmatrix} \sin 2t - 2\cos 2t \\ -\sin 2t \end{bmatrix}.$$

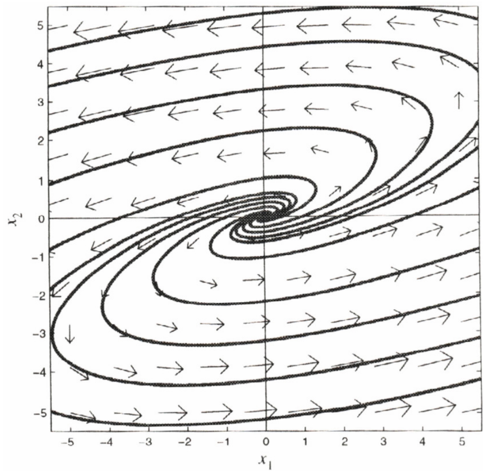
We only need **real** linearly independent solutions, so:

$$\vec{z}(t) = c_1e^{2t}\begin{bmatrix} \cos 2t + 2\sin 2t \\ -\cos 2t \end{bmatrix} + c_2e^{2t}\begin{bmatrix} \sin 2t - 2\cos 2t \\ -\sin 2t \end{bmatrix}.$$

Or, with alternate notation:

$$x_1(t) = e^{2t}[c_1(\cos 2t + 2\sin 2t) + c_2(\sin 2t - 2\cos 2t)]$$
$$= e^{2t}[(c_1 - 2c_2)\cos 2t + (2c_1 + c_2)\sin 2t]$$
$$x_2(t) = e^{2t}(-c_1\cos 2t - c_2\sin 2t).$$

The image below shows a direction field for this DEQ and some typical solution curves:



Problem: #25 Apply the eigenvalue method to find a general solution of the system.

$$x_1' = 5x_1 + 5x_2 + 2x_3, \qquad x_2' = -6x_1 - 6x_2 - 5x_3, \qquad x_3' = 6x_1 + 6x_2 + 5x_3$$

$$\mathbf{A} = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}$$

Characteristic equation: $-\lambda^3 + 4\lambda^2 - 13\lambda = 0$

Eigenvalues: $\lambda = 0$ and $2 \pm 3i$

With $\lambda = 0$ the eigenvector equation

$$\begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ gives eigenvector } \vec{v}_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T.$$

So: $\vec{z}_1(t) = \vec{v}_1 e^{0 \cdot t} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T.$

With $\lambda = 2 + 3i$ we solve the eigenvector equation

$$\begin{bmatrix} 3-3i & 5 & 2 \\ -6 & -8-3i & -5 \\ 6 & 6 & 3-3i \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To find the complex valued eigenvector $\vec{v}_2 = \begin{bmatrix} 1+i & -2 & 2 \end{bmatrix}^T.$

The corresponding complex-valued solution is

$$\vec{v}_2 e^{(2+3i)t} = e^{2t} e^{3it} \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix} = e^{2t}(\cos 3t + i \sin 3t) \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix}$$

$$= e^{2t}\begin{bmatrix} (\cos 3t + i\sin 3t) + i(\cos 3t + i\sin 3t) \\ -2\cos 3t - 2i\sin 3t \\ 2\cos 3t + 2i\sin 3t \end{bmatrix} = e^{2t}\begin{bmatrix} \cos 3t - \sin 3t + i\cos 3t - i\sin 3t \\ -2\cos 3t - 2i\sin 3t \\ 2\cos 3t + 2i\sin 3t \end{bmatrix}.$$

We are only interested in the real values, so:

$$\vec{z}_2(t) + \vec{z}_3(t) = c_2 e^{2t}\begin{bmatrix} \cos 3t - \sin 3t \\ -2\cos 3t \\ 2\cos 3t \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} \cos 3t - \sin 3t \\ -2\sin 3t \\ 2\sin 3t \end{bmatrix}.$$

Finally, we add the three solutions, with arbitrary constants.

So: $\vec{z}(t) = c_1\vec{z}_1(t) + c_2\vec{z}_2(t) + c_3\vec{z}_3(t)$

$$= c_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} \cos 3t - \sin 3t \\ -2\cos 3t \\ 2\cos 3t \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} \cos 3t - \sin 3t \\ -2\sin 3t \\ 2\sin 3t \end{bmatrix}.$$

The **scalar components** of the above general solution are:

$$x_1(t) = c_1 + e^{2t}[(c_2 + c_3)\cos 3t - (c_2 + c_3)\sin 3t],$$
$$x_2(t) = -c_1 + 2e^{2t}(-c_2\cos 3t - c_3\sin 3t),$$
$$x_3(t) = 2e^{2t}(c_2\cos 3t + c_3\sin 3t).$$

Finding the complex eigenvector from the previous problem:

$$\mathbf{A} = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}$$ **Eigenvalues**: $\lambda = 0$ and $2 \pm 3i$.

With $\lambda = 2 + 3i$ we solve the eigenvector equation…

$$\begin{bmatrix} 5-(2+3i) & 5 & 2 \\ -6 & -6-(2+3i) & -5 \\ 6 & 6 & 5-(2+3i) \end{bmatrix} = \begin{bmatrix} 3-3i & 5 & 2 \\ -6 & -8-3i & -5 \\ 6 & 6 & 3-3i \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & 6 & 3-3i \\ -6 & -8-3i & -5 \\ 3-3i & 5 & 2 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} 1 & 1 & \frac{1}{2}-\frac{1}{2}i \\ 0 & -2-3i & -2-3i \\ 3-3i & 5 & 2 \end{bmatrix}$$

Note: $-(3-3i)\left(\frac{1}{2}-\frac{1}{2}i\right) = -\left(\frac{3}{2}-\frac{3}{2}-\frac{3}{2}i-\frac{3}{2}i\right) = 3i$. So:

$$\Rightarrow \begin{bmatrix} 1 & 1 & \frac{1}{2}-\frac{1}{2}i \\ 0 & -2-3i & -2-3i \\ 0 & 2+3i & 2+3i \end{bmatrix}$$

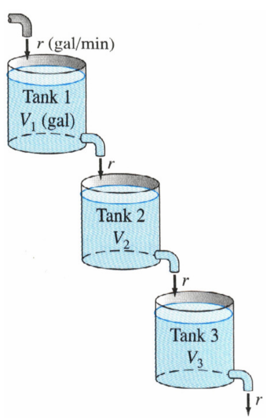
$$\Rightarrow \begin{bmatrix} 1 & 1 & \frac{1}{2}-\frac{1}{2}i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2}-\frac{1}{2}i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$z = c, \quad y = -c, \quad x = \frac{c}{2} + \frac{c}{2}i.$

$$\begin{bmatrix} \frac{c}{2}+\frac{c}{2}i \\ -c \\ c \end{bmatrix} = \begin{bmatrix} 1+1i \\ -2 \\ 2 \end{bmatrix} \text{ where } c = 2.$$

Complex valued eigenvector: $\vec{v}_2 = \begin{bmatrix} 1+i & -2 & 2 \end{bmatrix}^T.$

Problem: **#34**



This problem deals with the open three tank system. Freshwater flows into tank-1.

Mixed brine (salt water) flows from tank-1 into tank-2, from tank-2 into tank-3, and out of tank-3.

All have the flow rate $r = 60$ gallons per minute. Initial ($t = 0$) amounts of salt are:

$x_1(0) = 40\ lb$, $x_2(0) = 0$, and $x_3(0) = 0$ in the three tanks.

Initial volumes: $V_1 = 20$, $V_2 = 12$, $V_3 = 60$.

a.) **First, solve for the amounts of salt in the three tanks at time** t.

Observe that: $x_i' = [IN_i \text{ Salt}] - [OUT_i \text{ Salt}]$

$= [\text{In-Concentration}_i \times \text{In-Flow}_i] - [\text{Out-Concentration}_i \times \text{Out-Flow}_i]$

So, $x_1' = [0 \times 60] - [\frac{x_1}{20} \times 60] = -3x_1.$

Similarly: $x_2' = 3x_1 - 5x_2$, and $x_3' = 5x_2 - x_3.$

And, $\vec{z}\,' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -3x_1 \\ 3x_1 - 5x_2 \\ 5x_2 - x_3 \end{bmatrix}$.

$$= \begin{bmatrix} -3 & 0 & 0 \\ 3 & -5 & 0 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{A}\vec{z}.$$

The coefficient matrix $\mathbf{A} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -5 & 0 \\ 0 & 5 & -1 \end{bmatrix}$ has as eigenvalues, its diagonal elements:

$\lambda_1 = -3,\ \lambda_2 = -5,$ and $\lambda_3 = -1$ (as with any triangular matrix).

We find that the associated eigenvectors are:

$\vec{v}_1 = \begin{bmatrix} -4 & -6 & 15 \end{bmatrix}^T,\ \vec{v}_2 = \begin{bmatrix} 0 & -4 & 5 \end{bmatrix}^T,$ and $\vec{v}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T.$

So: $\vec{z} = \vec{v}_1 e^{-3t} + \vec{v}_2 e^{-5t} + \vec{v}_3 e^{-t}.$

Or written as a system:

$x_1(t) = -4c_1e^{-3t}$

$x_2(t) = -6c_1e^{-3t} - 4c_2e^{-5t}$

$x_3(t) = 15c_1e^{-3t} + 5c_2e^{-5t} + c_3e^{-t}.$ Now What?

The initial conditions $x_1(0) = 40,\ x_2(0) = 0,$ and $x_3(0) = 0$ give us $c_1 = -10,\ c_2 = 15,\ c_3 = 75$. So we have:

$x_1(t) = 40e^{-3t}$

$x_2(t) = 60e^{-3t} - 60e^{-5t}$

$x_3(t) = -150e^{-3t} + 75e^{-5t} + 75e^{-t}.$

b.) **Now, determine the maximal amount of salt that tank-3 ever contains**.

Remember from calculus that you can find the local maximums and minimums by taking the derivative of the function, and setting it equal to zero. For tank-3:

$x_3'(t) = 450e^{-3t} - 375e^{-5t} - 75e^{-t} = 0$

Multiplying by nonzero $\frac{1}{75e^{-t}}$:

$5e^{-4t} - 6e^{-2t} + 1 = 0$ Factoring this is the (not so) hard part.

$(5e^{-2t} - 1)(e^{-2t} - 1) = 0$

And observe that for the second factor: $e^{-2t} = 1$ when $\ln(e^{-2t}) = \ln(1)$,

or equivalently when $-2t = 0$, or $t = 0$.

Now looking at the first factor, $e^{-2t} = \frac{1}{5}$ when $\ln(e^{-2t}) = \ln(\frac{1}{5})$,

or when $-2t = -\ln 5$, or $t = \frac{1}{2}\ln 5 \sim 0.8$ min $= 48$ sec.

Since $x_3(t) = -150e^{-3t} + 75e^{-5t} + 75e^{-t}$, the maximum amount of salt ever in tank-3 is $x_3\left(\frac{1}{2}\ln 5\right) \approx 21.5$ pounds,

c.) Finally, construct a figure showing the graphs of $x_1(t)$, $x_2(t)$, **and** $x_3(t)$.

The figure below shows the graph of $x_1(t)$, $x_2(t)$, and $x_3(t)$.