Instructor: Jodin Morey moreyjc@umn.edu
Website: math.umn.edu/~moreyjc

7.2: Matrices and Linear Systems

Let's simplify a *system* of DEQs

$$\begin{aligned} x_1' &= p_{11}(t)x_2 + p_{12}(t)x_1 + f_1(t), \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_n + f_n(t) \end{aligned}$$

by expressing it as a *single* matrix equation: $\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{f}(t)$

Matrix Function

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \text{ or } \mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix}.$$

For example: $\mathbf{A}(t) = \begin{bmatrix} 0 & -e^{2t} \\ 4e^{3t} & 12t \end{bmatrix}.$

A matrix function is **continuous** or **differentiable** at a point t (or on an interval $a \leq t \leq b$) if each of its components $a_{mn}(t)$ are continuous or differentiable there.

So, $\mathbf{A}'(t) = \frac{d\mathbf{A}}{dt} = \begin{bmatrix} a_{11}'(t) & a_{12}'(t) & a_{13}'(t) \\ a_{21}'(t) & a_{22}'(t) & a_{23}'(t) \\ a_{31}'(t) & a_{32}'(t) & a_{33}'(t) \end{bmatrix}.$

Also, similar to calculus we have:
$\frac{d}{dt}(\mathbf{AB}) = \mathbf{A}'\mathbf{B} + \mathbf{AB}'$, and $\frac{d}{dt}(\mathbf{CA}) = \mathbf{CA}'$, where $\mathbf{C}$ is a constant **matrix**.

Notationally Transforming a System:

Given:

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + f_1(t), \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + f_2(t). \end{aligned}$$

Notate: $\vec{x} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\quad \mathbf{P}(t) := \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}$, $\quad \vec{f}(t) := \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$

So: $\vec{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$.

And the system becomes: $\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{f}(t)$.

A solution to this DEQ on the open interval I is a column vector function $\vec{x}(t) = (x_1(t), \ldots, x_n(t))$, such that the component functions of $\vec{x}$ satisfy the system identically on I. If the p_{ij} and f_i are continues on I, then the existence and uniqueness theorem of the previous section applies for initial conditions in I.

Similarities between Solutions to Systems and Individual DEQs:

As with our previous *individual* DEQs ($x^{(n)} = f(x^{(n-1)}, \ldots, x, t)$), with a *system* ($\vec{x}' = \mathbf{P}(t)\vec{x}$) we have:

Superposition Principal/Gen. Solutions Theorems:

If $\vec{x}_1(t),\ \vec{x}_2(t),\ \vec{x}_n(t)$ are (vector) solutions of the first order system $\vec{x}'(t) = \mathbf{P}(t)\vec{x}$, then so is the linear combination: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \ldots + c_n\vec{x}_n(t)$.

Proof: Taking the derivative of $\vec{x}$, we have: $\vec{x}' = c_1\vec{x}_1' + c_2\vec{x}_2' + \ldots + c_n\vec{x}_n'$.

And since we are given that $\vec{x}_i$ are solutions, we have $\vec{x}_i' = \mathbf{P}(t)\vec{x}_i$ for each i ($1 \le i \le n$). So:

$$\vec{x}' = c_1\mathbf{P}(t)\vec{x}_1 + c_2\mathbf{P}(t)\vec{x}_2 + \ldots + c_n\mathbf{P}(t)\vec{x}_n$$

$$= \mathbf{P}(t)(c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_n\vec{x}_n).$$

That is, $\vec{x}' = \mathbf{P}(t)\vec{x}$, as desired. ■

Independence of Vector Valued Functions: $\vec{x}_1, \ldots, \vec{x}_n$ are linearly dependent on the interval I provided that there exist constants $c_1, \ldots, c_n$, *not all zero* such that $c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \ldots + c_n\vec{x}_n(t) = \vec{0}$.

As before, they're linearly independent provided none of them is a linear combination of the others (e.g., $\vec{x}_1(t) \neq k\vec{x}_2(t)$).

Wronskian: If $\vec{x}_1(t),\ \vec{x}_2(t),\ \vec{x}_3(t)$ are solutions of $\vec{x}' = \mathbf{P}(t)\vec{x}$ on open interval I, and $\mathbf{P}(t)$ is continuous on I, then let $W(\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)) := \det\begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix}$ (note that this Wronskian doesn't involve derivatives). Then we have:

- ♦ $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$ are linearly dependent on I if and only if $W = 0$ at every point of I.
- ♦ $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$ are linearly independent on I if and only if $W \neq 0$ at every point of I.

General Solutions of Homogeneous Systems Theorem: Let $\vec{x}_1, \ldots, \vec{x}_n$ be n linearly independent solutions of the homogeneous linear DEQ $\vec{x}' = \mathbf{P}(t)\vec{x}$ on an open interval I where $\mathbf{P}(t)$ is continuous.

If $\vec{x}(t)$ is any solution whatsoever of the equation $\vec{x}' = \mathbf{P}(t)\vec{x}$ on I, then there exist numbers $c_1, \ldots, c_n$ such that:

$$\vec{x}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t), \text{ for all } t \text{ in } I.$$

As a result, general solutions can be written as $\vec{x}(t) = \mathbf{X}(t)\vec{c}$, where $\mathbf{X} := \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \ldots & \vec{x}_n \end{bmatrix}$, and $\vec{c} := \begin{bmatrix} c_1 & \ldots c_n \end{bmatrix}^T$.

And solving for initial conditions $\vec{x}(a) = \vec{b}$ with $\vec{b} := \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix}^T$ can be accomplished by substituting into the above solution as $\mathbf{X}(a)\vec{c} = \vec{b}$. In order to solve for $c_1, \ldots c_n$.

Solutions to Nonhomogeneous Systems: $\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{f}(t)$

Let $\vec{x}_p$ be a particular solution of nonhomogeneous $\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{f}(t)$ on open interval I where $\mathbf{P}(t)$, $\vec{f}(t)$ are continuous.

Let $\vec{x}_1, \ldots, \vec{x}_n$ be linearly independent solutions of the associated homogeneous DEQ $\vec{x}' = \mathbf{P}(t)\vec{x}$ on I.

If $\vec{x}(t)$ is any solution whatsoever of the nonhomogeneous DEQ on I, then there exist numbers $c_1, \ldots, c_n$ such that:

$$\vec{x}(t) = \vec{x}_p(t) + \vec{x}_c(t) = \vec{x}_p + (c_1\vec{x}_1 + \ldots + c_n\vec{x}_n).$$

Exercises

Problem: **#21** Given a system of DEQs in matrix form: $x' = \mathbf{A}\vec{x}$. Verify that the given vector functions are solutions to that system. Then, use the Wronskian to show that the solutions are linearly independent. Finally, write the general solution to the system.

System of Diff Eqs: $\vec{x}' = \begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix}\vec{x};$

Possible Solutions: $\vec{y}_1 = \begin{bmatrix} 3e^{-2t} \\ -2e^{-2t} \\ 2e^{-2t} \end{bmatrix}, \quad \vec{y}_2 = \begin{bmatrix} e^{t} \\ -e^{t} \\ e^{t} \end{bmatrix}, \quad \vec{y}_3 = \begin{bmatrix} e^{3t} \\ -e^{3t} \\ 0 \end{bmatrix}.$

Left Side: $\vec{y}_1' = -2e^{-2t}\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = e^{-2t}\begin{bmatrix} -6 \\ 4 \\ -4 \end{bmatrix},$

Right Side: $\begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix}\vec{y}_1 = \begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix} e^{-2t}\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = e^{-2t}\begin{bmatrix} -6 \\ 4 \\ -4 \end{bmatrix}.$

Simarly: $\vec{y}_2' = e^t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix} e^t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(I leave it to you to confirm the last vector is a solution).

"**Then, use the Wronskian to show that they are linearly independent**."

$$W(t) = \begin{vmatrix} 3e^{-2t} & e^t & e^{3t} \\ -2e^{-2t} & -e^t & -e^{3t} \\ 2e^{-2t} & e^t & 0 \end{vmatrix}$$

$$= e^{3t}e^t e^{-2t} \begin{vmatrix} 3 & 1 & 1 \\ -2 & -1 & -1 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= e^{2t} \begin{vmatrix} 3 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{vmatrix} = e^{2t}(-1(0-1)) = e^{2t} \neq 0, \text{ for any } t.$$

"**Finally, write the general solution to the system**."

Since we have **three** linearly independent solutions for a system of **three** first order equations, we can write the general solution:

$$\vec{x}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 = c_1 \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^t + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{3t}.$$

OR with different notation: $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 3c_1e^{-2t} + c_2e^t + c_3e^{3t} \\ -2c_1e^{-2t} - c_2e^t - c_3e^{3t} \\ 2c_1e^{-2t} + c_2e^t \end{bmatrix}$.

Problem: #29 Find a particular solution to the system in the above problem that satisfies the following initial conditions. $x_1(0) = 1,\ x_2(0) = 2,\ x_3(0) = 3$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3c_1e^{-2\cdot 0} + c_2e^0 + c_3e^{3\cdot 0} \\ -2c_1e^{-2\cdot 0} - c_2e^0 - c_3e^{3\cdot 0} \\ 2c_1e^{-2\cdot 0} + c_2e^0 \end{bmatrix} = \begin{bmatrix} 3c_1 + c_2 + c_3 \\ -2c_1 - c_2 - c_3 \\ 2c_1 + c_2 \end{bmatrix}$$

$$\mathbf{A}\vec{c} = (1,2,3) \quad \Rightarrow \left[\begin{array}{cccc} 3 & 1 & 1 & 1 \\ -2 & -1 & -1 & 2 \\ 2 & 1 & 0 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{cccc} 3 & 1 & 1 & 1 \\ 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{cccc} 1 & 0 & 0 & 3 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 0 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{cccc} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -8 \\ 0 & 1 & 0 & -3 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -8 \\ 0 & 0 & -1 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cccc} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -5 \end{array}\right] \Rightarrow \vec{c} = (3,-3,-5).$$

So, $$\left[\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \end{array}\right] = \left[\begin{array}{c} 3c_1e^{-2t} + c_2e^t + c_3e^{3t} \\ -2c_1e^{-2t} - c_2e^t - c_3e^{3t} \\ 2c_1e^{-2t} + c_2e^t \end{array}\right] = \left[\begin{array}{c} 3(3)e^{-2t} + (-3)e^t + (-5)e^{3t} \\ -2(3)e^{-2t} - (-3)e^t - (-5)e^{3t} \\ 2(3)e^{-2t} + (-3)e^t \end{array}\right]$$

$$= \left[\begin{array}{c} 9e^{-2t} - 3e^t - 5e^{3t} \\ -6e^{-2t} + 3e^t + 5e^{3t} \\ 6e^{-2t} - 3e^t \end{array}\right].$$

Problem: #34 Suppose that one of the following vector functions is a constant multiple of the other on an open interval I.

$$\vec{x}_1(t) = \left[\begin{array}{c} x_{11}(t) \\ x_{21}(t) \end{array}\right] \text{ and } \vec{x}_2(t) = \left[\begin{array}{c} x_{12}(t) \\ x_{22}(t) \end{array}\right].$$

Show that their Wronskian $W(t) = |x_{ij}(t)|$ must vanish identically on I.

$$\mathbf{W}(t) = \left|\vec{x}_1(t) \ \ \vec{x}_2(t)\right|$$

$$\vec{x}_2(t) = a_1\vec{x}_1(t) = \left[\begin{array}{c} a_1x_{11}(t) \\ a_1x_{21}(t) \end{array}\right], \text{ for some constant } a_1.$$

$$\mathbf{W}(t) = \left|\vec{x}_1(t) \ \ \vec{x}_2(t)\right| = \left|\begin{array}{cc} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{array}\right| = \left|\begin{array}{cc} x_{11}(t) & a_1x_{11}(t) \\ x_{21}(t) & a_1x_{21}(t) \end{array}\right|$$

$$\overset{C_2-a_1C_1}{=} \left|\begin{array}{cc} x_{11}(t) & 0 \\ x_{21}(t) & 0 \end{array}\right| = 0.$$

And the constant function zero is equal to zero for all choices of t, so it "vanishes identically."