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5.5 - Non-Homogeneous DEQs & Undetermined Coefficients

Non-homogeneous DEQs are of the form: $Ly = f(x)$, where L is a differential operator explained in 5.3.

For example: $y^{(7)} + 2y^{(3)} + 3y'' + 4y' + 5y = f(x)$. How to solve them?

Earlier we learned that general solutions have the form $y = y_c + y_p$, where y_c is the complementary solution we get from the characteristic equation, and y_p is any particular solution. But how do we find a y_p?

Undetermined Coefficients, the Justification

To solve the DEQ above, we need some function y such that when we take various derivatives $(y^{(7)}, y^{(3)}, y'', y', y)$, multiply them by some constants $(y^{(7)}, 2y^{(3)}, 3y'', 4y', 5y)$, and add them together, we get $f(x)$. So if we limit the type of expression $f(x)$ can be, we might come up with a good guess for y.

Polynomial: If we assume $f(x)$ is a polynomial, note that the derivative of a polynomial is a polynomial. So a reasonable guess y_i for a particular solution that we could substitute into the left-hand side of the DEQ would be a polynomial $y_i := A_1 + A_2x + \ldots + A_{n-1}x^n$. The A_j are yet-to-be-determined coefficients and n is the highest power of x in $f(x)$. Substituting y_i into the left-hand side and taking derivatives as necessary, we could then compare the two sides of the equation to determine the undetermined coefficients A_i.

Exponential: Similarly, if $f(x)$ is an exponential functions (e.g.,$7e^{5x}$), note that the derivative of an exponential is also an exponential $((7e^{5x})' = 35e^{5x})$. So a reasonable guess y_i for a particular solution that we could substitute into the left-hand side of the DEQ would be an exponential Ae^{5x}. The A is a yet-to-be-determined coefficients. Substituting y_i into the left-hand side and taking derivatives as necessary, we could then compare the two sides of the equation to determine the undetermined coefficient A.

Trigonometric: Similarly, if $f(x)$ is a sine or cosine function (e.g., $7\cos 3x$), note that the derivative of a $\sin/\cos$ is also a $\sin/\cos$. So a reasonable guess y_i for a particular solution that we could substitute into the left-hand side of the DEQ would be an $A\sin 3x + B\cos 3x$. The A, B are yet-to-be-determined coefficients. Substituting y_i into the left-hand side and taking derivatives as necessary, we could then compare the two sides of the equation to determine the undetermined coefficients A, B.

Even better, we can merge these three facts into a procedure (seen below) which allows for $f(x)$ to combine these types of functions.

Linear Independence

But before we write down the procedure, there is still one difficulty to deal with. The processes laid out above may result in a y_i which has terms that are linearly dependent with terms in y_c.

For example from $y''' - 3ry'' + 3r^2y' - r^3y = (2x-3)e^{rx}$ you would calculate $y_c = c_1e^{rx} + c_2xe^{rx} + c_3x^2e^{rx}$, and

$y_i = Ae^{rx} + Bxe^{rx}$. So what is wrong with this?

First observe that the terms in y_i are linearly dependent with terms in y_c

$Ae^{rx} = kc_1e^{rx}$ where $k = \frac{A}{c_1}$, and $Bxe^{rx} = kc_2xe^{rx}$ where $k = \frac{B}{c_2}$.

In other words, y_i just consists of solutions from our y_c.

But being solutions to the homogeneous version of our DEQ means that substituting them into our nonhomogeneous DEQ will just give us $0 = (2x-3)e^{rx}$. So it is not a solution to the nonhomogeneous DEQ.

We certainly don't get the opportunity to solve for the undetermined coefficients A, B in y_i.

So how do we amend y_0 to produce the solutions we are looking for?

The trick is to first rewrite our DEQ above as: $(D-r)^3y = 2xe^{rx} - 3e^{rx}$.

Then we recall something from section 5.3, that is $(D-r)^k[u(x)e^{rx}] = D^k(u(x))e^{rx}$.

So if we multiply both sides of our DEQ by $(D-r)^2$, we have:

$$(D-r)^5y = (D-r)^2(2xe^{rx} - 3e^{rx}) = (D-r)^2(2xe^{rx}) - (D-r)^2(3e^{rx})$$
$$= D^2(2x)e^{r_1x} - D^2(3)e^{r_1x} = 0 \bullet e^{r_1x} - 0 \bullet e^{r_1x} = 0.$$

In other words, any solution y which satisfies our original nonhomogeneous DEQ, $((D-r)^3y = (2x-3)e^{rx})$ also satisfies $(D-r)^5y = 0$.

Observe from our previous analysis that solutions to this DEQ can take the form:

$y(x) = c_1e^{rx} + c_2xe^{rx} + c_3x^2e^{rx} + Ax^3e^{rx} + Bx^4e^{rx}$, where I have suggestively chosen notation for the constant coefficients.

In other words, if I multiply y_i by x^3, these are likely to produce solutions to my nonhomogeneous DEQ.

Undetermined Coefficients, the Method

The method of **Undetermined Coefficients** assumes f is of the form:

$f(x) = Ax^ke^{rx}\cos(tx)$ or $Ax^ke^{rx}\sin(tx)$, where $k,\ r, t \geq 0$

(or $f(x)$ can consist of several terms of this form added together)

Steps to solving…

♦ Determine **complementary solution** $y_c = c_1y_1 + c_2y_2$. (where $f(x) = 0$)

Example, for $y'' + 2y = f(x)$, we calculate: $y_c = c_1x + c_2e^{-2x}$.

♦ Define a **pre-trial solution**: $y_i := p_1(x) + \ldots + p_n(x)$.

Example, if $f(x) = \sin x + 7xe^{-2x}$, then $y_i = A\sin x + B\cos x + (C + Dx)e^{-2x}$. $(*)$

Find the pre-trial solution in three steps. For each term in $f(x)$:

◊ **Trig-Step**: If there is $\sin tx$ or $\cos tx$ in the term, write: $\sin tx + \cos tx$.

◊ **Exponential-Step**: Next, if there is an exponential e^{rx}, multiply what you have by e^{rx}.

Example: $e^{rx}\sin tx + e^{rx}\cos tx$.

◊ **Power-Step**: Finally, determine k (the power of x). Note that you may have $k = 0$.

Then, multiply each term of what you have by $(A + Bx + Cx^2 + \ldots + Lx^k)$,

with different constants for each term. If $k = 0$, then you just multiply by A.

Example: $Ae^{rx}\sin tx + Bxe^{rx}\sin tx + Ce^{rx}\cos tx + Dxe^{rx}\cos tx$, when $k = 1$.

♦ Next, we need for the terms of our **trial solution** to be linearly independent from our **complimentary solution**.

So, for each term p_i of our **pre-trial solution**: $y_i = p_1(x) + \ldots + p_n(x)$, determine the smallest power s_i of x, such that $x^{s_i}p_i$ isn't a constant multiple of any term in our complementary solution: $y_c = c_1y_1 + c_2y_2$.

(we're removing duplicates to acheive independence of the two sets of solutions).

Putting it together we have a **trial solution**: $y_{trial} = x^{s_1}p_1 + \ldots + x^{s_n}p_n$.

(continuing with our example $(*)$ above: $y_{trial} = A\sin x + B\cos x + Cxe^{-2x} + Dx^2e^{-2x}$).

♦ Substitute y_{trial} into $Ly = f(x)$ (taking derivatives as necessary), and determine the coefficients $(A, B, \ldots)$ by comparing the two sides of the equation.

The result we label y_p (our **particular solution**).

♦ **General Solution**: $y = y_c + y_p$. (combination of complementary & particular solution)

Here's a video explanation from Khan Academy:
https://www.khanacademy.org/math/differential-equations/second-order-differential-equations#undetermined-coefficients

But what if $f(x)$ isn't in the form required by Undetermined Coefficients?

Variation of Parameters, the Justification

So how do we form y_p? We saw in undetermined coefficients that sometimes the solutions to our DEQ are similar to the solutions (y_1, y_2, etc.) in $y_c = c_1y_1 + c_2y_2 + \ldots$, but multiplied by some power of x.

What if we made the assumption that something similar happens for more complicated $f(x)$; that our particular solution takes the form: $y_p = u_1y_1 + u_2y_2 + \ldots$, where u_i are functions of x.

Below we work with an 2nd order DEQ, but the process works for nth order DEQs.

If: $y'' + P(x)y' + Q(x)y = f(x)$, with $y_c = c_1y_1 + c_2y_2$,

we write a particular solution guess as $y_p := u_1y_1 + u_2y_2$.

If we were to substitute this into our DEQ, there would be two unknown functions u_1, u_2, but only one equation (constraint) in the form of our DEQ. Generally, one would need two equations to pin down both u_1, u_2.

We could write down a new constraint, but how would we know if it was correct? That's easy, if the resulting constraint results in u_1, u_2 which solve our DEQ, then it was correct. And although our textbook doesn't tell us why it works, the constraint $u_1'y_1 + u_2'y_2 = 0$ leads us to solutions of the DEQ.

So let's substitute y_p into our DEQ, using our 2nd constraint along the way to simplify things in order to derive an algorithm for solving this type of DEQ.

Note: $y_p' = (u_1y_1' + u_2y_2') + (u_1'y_1 + u_2'y_2)$.

Applying our second constraint, this becomes $y_p' = u_1y_1' + u_2y_2'$.

Taking another derivative: $y_p'' = (u_1'y_1' + u_2'y_2') + (u_1y_1'' + u_2y_2'')$.

Recall that both y_1, y_2 satisfy the homogeneous DEQ: $y_i'' + Py_i' + Qy_i = 0$. Rearranging: $y_i'' = -Py_i' - Qy_i$.

So substituting this into our second derivative:

$$\begin{aligned} y_p'' &= (u_1'y_1' + u_2'y_2') + (u_1(-Py_1' - Qy_1) + u_2(-Py_2' - Qy)) \\ &= (u_1'y_1' + u_2'y_2') - P \cdot (u_1y_1' + u_2y_2') - Q \cdot (u_1y_1 + u_2y_2) \\ &= (u_1'y_1' + u_2'y_2') - Py_p' - Qy_p. \end{aligned}$$

Substituting these into our DEQ:

$$[(u_1'y_1' + u_2'y_2') - Py_p' - Qy_p] + Py_p' + Qy_p = u_1'y_1' + u_2'y_2' = f(x).$$

So our two constraints become: $u_1'y_1 + u_2'y_2 = 0$ and $u_1'y_1' + u_2'y_2' = f(x)$.

Variation of Parameters, the Method

This is for non-homogeneous DEQs $Ly = f(x)$ not in the form necessary for Undetermined Coefficients.

Steps to solving...

1. Determine the complementary solution: $y_c = c_1y_1(x) + c_2y_2(x)$.
2. Differentiate: y_1, y_2 to get y_1', y_2'.
3. Write down: $u_1'y_1 + u_2'y_2 = 0$, and $u_1'y_1' + u_2'y_2' = f(x)$; where u_1', u_2' are unknown.
4. Solve for u_1' and u_2' (two equations, two unknowns).
5. Integrate u_1' and u_2', (using zeros as the constants of integration).
6. Particular Solution is: $y_p = u_1y_1 + u_2y_2$.
7. As above, the Gen. Solution is: $y_g = y_c + y_p$.

There is another way to characterize this.

Observe that if we solve for u_1' in the 1st equation, we have $u_1' = -\frac{u_2'y_2}{y_1}$.

And substituting this in the 2nd equation: $\left(-\frac{u_2' y_2}{y_1}\right) y_1' + u_2' y_2' = u_2'(-\frac{y_2}{y_1} y_1' + y_2') = f(x)$.

And solving for $u_2' = \frac{y_1 \cdot f(x)}{y_1 y_2' - y_1' y_2}$, giving us: $u_1' = -\frac{\left(\frac{y_1 \cdot f(x)}{y_1 y_2' - y_1' y_2}\right) y_2}{y_1} = -\frac{y_2 \cdot f(x)}{y_1 y_2' - y_1' y_2}$.

Observe that $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$.

Therefore, we have $u_2' = \frac{y_1 \cdot f(x)}{W}$, and $u_1' = -\frac{y_2 \cdot f(x)}{W}$.

So, alternatively to steps 2-6 above we have:

$y_p = u_1 y_1 + u_2 y_2 = -y_1 \int \frac{y_2 \cdot f(x)}{W} dx + y_2 \int \frac{y_1 \cdot f(x)}{W} dx$, where $W(y_1, y_2)$ is the Wronskian.

Exercises

Problem: **#26** The roots of the equation $r^2 - 6r + 13 = 0$ are $r = 3 \pm 2i$.
Using the undetermined coefficients method, write down the **general form of a particular solution** for:
$y'' - 6y' + 13y = xe^{3x} \sin 2x$ (this means you don't solve for the coefficients).

$e^{(3+2i)x} = e^{3x}(\cos 2x + i \sin 2x)$

$y_c = c_1 e^{3x} \cos 2x + c_2 e^{3x} \sin 2x$.

$y_i = (A + Bx)e^{3x} \sin 2x + (C + Dx)e^{3x} \cos 2x$ (pre-trial solution)

Clearing up any linear dependence between y_i and y_c, we get:

$y_{trial} = (Ax + Bx^2)e^{3x} \sin 2x + (Cx + Dx^2)e^{3x} \cos 2x$

General form of a Particular Solution:

$y_g = y_c + y_{trial} = (c_1 e^{3x} \cos 2x + c_2 e^{3x} \sin 2x) + (Ax + Bx^2)e^{3x} \sin 2x + (Cx + Dx^2)e^{3x} \cos 2x$.

Problem: **#18** Find a particular solution y_p of $y^{(4)} - 5y'' + 4y = e^x - xe^{2x}$.

$r^4 - 5r^2 + 4 = 0$,

$(r^2 - 4)(r^2 - 1)$ $r^2 = \{1, 4\}$.

$r = \pm 1, \pm 2$.

$y_c = c_1e^{-x} + c_2e^{x} + c_3e^{-2x} + c_4e^{2x}.$

$y_i = Ae^{x} + (B + Cx)e^{2x}$ (pre-trial solution)

$y_{trial} = Axe^{x} + Bxe^{2x} + Cx^2e^{2x}$

We need to determine the fourth derivative so that we can plug this into our original equation.

$$\begin{aligned} y'_{trial} &= (Axe^{x} + Ae^{x}) + (Be^{2x} + 2Bxe^{2x}) + (2Cxe^{2x} + 2Cx^2e^{2x}), \\ &= (Ax + A)e^{x} + (2Bx + B)e^{2x} + (2Cx + 2C)xe^{2x}. \\ y''_{trial} &= (Ax + 2A)e^{x} + (4Bx + 4B)e^{2x} + (4Cx^2 + 8Cx + 2C)e^{2x}. \\ y'''_{trial} &= (Ax + 3A)e^{x} + (8Bx + 12B)e^{2x} + (8Cx^2 + 24Cx + 12C)e^{2x}. \\ y^{(4)}_{trial} &= (Ax + 4A)e^{x} + (16Bx + 32B)e^{2x} + (16Cx^2 + 64Cx + 48C)e^{2x}. \end{aligned}$$

Recall the original equation: $y^{(4)} - 5y'' + 4y = e^{x} - xe^{2x}$.

So the LHS trial version is: $(Ax + 4A)e^{x} + (16Bx + 32B)e^{2x} + (16Cx^2 + 64Cx + 48C)e^{2x}$

$- 5[(Ax + 2A)e^{x} + (4Bx + 4B)e^{2x} + (4Cx^2 + 8Cx + 2C)e^{2x}] + 4[Axe^{x} + Bxe^{2x} + Cx^2e^{2x}]$

$= (Ax + 4A)e^{x} + (16Bx + 32B)e^{2x} + (16Cx^2 + 64Cx + 48C)e^{2x}$

$- [(5Ax + 10A)e^{x} + (20Bx + 20B)e^{2x} + (20Cx^2 + 40Cx + 10C)e^{2x}] + [4Axe^{x} + 4Bxe^{2x} + 4Cx^2e^{2x}]$

$= -6Ae^{x} + (12B + 38C)e^{2x} + 24Cxe^{2x}.$

Now what?

$-6Ae^{x} + (12B + 38C)e^{2x} + 24Cxe^{2x} = e^{x} - xe^{2x}$

$-6A = 1, \qquad 12B + 38C = 0, \qquad 24C = -1$

$A = -\frac{1}{6}, \qquad C = -\frac{1}{24}, \qquad 12B + 38\left(-\frac{1}{24}\right) = 0, \qquad 12B = \frac{19}{12}, \qquad B = \frac{19}{144}.$

Recall: $y_{trial} = Axe^{x} + Bxe^{2x} + Cx^2e^{2x}$

So our particular solution is…

$y_p = -\frac{1}{6}xe^{x} + \frac{19}{144}xe^{2x} - \frac{1}{24}x^2e^{2x}$

And even though the question did not ask for it, our general solution would be…

$y = y_p + y_c = \left(-\frac{1}{6}xe^{x} + \frac{19}{144}xe^{2x} - \frac{1}{24}x^2e^{2x}\right) + (c_1e^{-x} + c_2e^{x} + c_3e^{-2x} + c_4e^{2x}).$ (whew!)

Problem: #34 Solve the initial value problem: $y'' + y = \cos x; \qquad y(0) = 1, \qquad y'(0) = -1.$

$r^2 + 1 = 0, \qquad r = \pm i, \qquad \Rightarrow \qquad e^{ix} = \cos x + i \sin x.$

So, $y_c = c_1 \cos x + c_2 \sin x;$

Finding a trial solution…

$y_i = A \cos x + B \sin x$

$y_{trial} = x(A \cos x + B \sin x)$

Differentiating y_{trial} to plug back into our equation…

$y'_{trial} = (-A \sin x + B \cos x)x + (A \cos x + B \sin x) = (Bx + A) \cos x + (-Ax + B) \sin x$

$$\begin{aligned} y''_{trial} &= B \cos x - (Bx + A) \sin x + (-A) \sin x + (-Ax + B) \cos x \\ &= (-Ax + 2B) \cos x + (-Bx - 2A) \sin x \end{aligned}$$

Plugging them back into $y'' + y = \cos x,$ we get:

$$\begin{aligned} &[(-Ax + 2B) \cos x + (-Bx - 2A) \sin x] + [Ax \cos x + Bx \sin x] \\ &\quad = 2B \cos x - 2A \sin x = \cos x. \end{aligned}$$

Determining our coefficients…

$2B = 1,$ and $-2A = 0. \qquad A = 0$ and $B = \frac{1}{2}.$

$y_p = \frac{1}{2} x \sin x.$

Therefore, our general solution is…

$y_g = y_c + y_p = c_1 \cos x + c_2 \sin x + \frac{1}{2} x \sin x,$ And…

Using the initial conditions to solve for c_1 and c_2…

$1 = c_1 \cos 0 - 1 \sin 0 + \frac{1}{2}(0) \sin 0 = c_1.$

$y' = -c_1 \sin x + c_2 \cos x + \frac{1}{2} \sin x + \frac{1}{2} x \cos x$

$-1 = -c_1 \sin 0 + c_2 \cos 0 + \frac{1}{2} \sin 0 + \frac{1}{2}(0) \cos 0 = c_2.$

Plugging these into y…

$y(x) = \cos x - \sin x + \frac{1}{2} x \sin x.$

Problem: #56 Use the method of **variation of parameters** to find a particular solution to: $y'' - 4y = xe^x.$

Recall: y_p is of the form: $y_p = u_1 y_1 + u_2 y_2$.

So first need to obtain y_1, y_2, from complementary solution.

$r^2 - 4 = 0, \qquad r = \pm 2, \qquad y_c = c_1 e^{2x} + c_2 e^{-2x}$.

So, $y_1 = e^{2x}, \qquad y_2 = e^{-2x}$, and $y_1' = 2e^{2x}, \qquad y_2' = -2e^{-2x}$.

Variation of Parameters involves writing down: $(u_1' y_1 + u_2' y_2 = 0)$ and $(u_1' y_1' + u_2' y_2' = f(x))$. So:

$$u_1' e^{2x} + u_2' e^{-2x} = 0, \qquad (1)$$
$$2u_1' e^{2x} - 2u_2' e^{-2x} = xe^x. \qquad (2)$$

Solving for $u_1', \ u_2'$, first start with the simpler equation (1):

$$u_1' = -\frac{u_2'}{e^{4x}}. \qquad (3)$$

Plugging this into (2), and then solving for u_2':

$$2\left(-\frac{u_2'}{e^{4x}}\right)e^{2x} - 2u_2' e^{-2x} = xe^x$$
$$-2\frac{u_2'}{e^{2x}} - 2\frac{u_2'}{e^{2x}} = -4\frac{u_2'}{e^{2x}} = xe^x$$
$$u_2' = -\tfrac{1}{4}xe^{3x}. \qquad (4)$$

Plugging this into (3) to solve for u_1'...

$$u_1' = -\frac{u_2'}{e^{4x}} = -\frac{\left(-\frac{1}{4}xe^{3x}\right)}{e^{4x}} = \tfrac{1}{4}xe^{-x}. \qquad (5)$$

Now to integrate (4) and (5) to find u_1 and u_2...

$$u_2 = \int u_2' dx = -\tfrac{1}{4}\int xe^{3x}dx = -\tfrac{1}{4}\left[\tfrac{1}{3}xe^{3x} - \tfrac{1}{3}\int e^{3x}dx\right] \qquad \text{(using integration by parts)}$$
$$= -\tfrac{1}{4}\left[\tfrac{1}{3}xe^{3x} - \tfrac{1}{9}e^{3x}\right] = \left(\tfrac{1}{36} - \tfrac{1}{12}x\right)e^{3x}.$$

$$u_1 = \int u_1' dx = \tfrac{1}{4}\int xe^{-x}dx = \tfrac{1}{4}\left[-xe^{-x} + \int e^{-x}dx\right] = \tfrac{1}{4}[-xe^{-x} - e^{-x}] = -\left(\tfrac{1}{4}x + \tfrac{1}{4}\right)e^{-x}.$$

So our particular solution is...

$$y_p = u_1 y_1 + u_2 y_2 = \left[-\left(\tfrac{1}{4}x + \tfrac{1}{4}\right)e^{-x}\right]e^{2x} + \left[\left(\tfrac{1}{36} - \tfrac{1}{12}x\right)e^{3x}\right]e^{-2x}$$
$$= -\left(\tfrac{1}{4}x + \tfrac{1}{4}\right)e^x + \left(\tfrac{1}{36} - \tfrac{1}{12}x\right)e^x = \left(\left(\tfrac{1}{36} - \tfrac{1}{12}x\right) - \left(\tfrac{1}{4}x + \tfrac{1}{4}\right)\right)e^x$$
$$= -\left(\tfrac{1}{3}x + \tfrac{2}{9}\right)e^x.$$

And even though the question did not ask for it, our general solution would be...

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{-2x} - \left(\tfrac{1}{3}x + \tfrac{2}{9}\right)e^x.$$

Alternatively, we can work this problem using the Wronskian method:

Observe the Wronskian is: $W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4.$

So, $y_p = -y_1 \int \frac{y_2 \cdot f(x)}{W} dx + y_2 \int \frac{y_1 \cdot f(x)}{W} dx = -e^{2x} \int \frac{e^{-2x} \cdot xe^{x}}{-4} dx + e^{-2x} \int \frac{e^{2x} \cdot xe^{x}}{-4} dx$

$$= \tfrac{1}{4} e^{2x} \int xe^{-x}\, dx - \tfrac{1}{4} e^{-2x} \int xe^{3x} dx$$

Recall: $\int xe^{-x} dx = -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} = -(x+1)e^{-x}$. (using integration by parts)

and: $\int xe^{3x} dx = \tfrac{1}{3} xe^{3x} - \tfrac{1}{3} \int e^{3x} dx = \tfrac{1}{3} xe^{3x} - \tfrac{1}{9} e^{3x} = \left(\tfrac{1}{3} x - \tfrac{1}{9} \right) e^{3x}$.

So, $y_p = -\tfrac{1}{4} e^{x} (x+1) - \tfrac{1}{4} e^{x} \left(\tfrac{1}{3} x - \tfrac{1}{9} \right) = -\tfrac{1}{4} e^{x} \left(\tfrac{8}{9} + \tfrac{4}{3} x \right) = -\left(\tfrac{1}{3} x + \tfrac{2}{9} \right) e^{x}$, as above.