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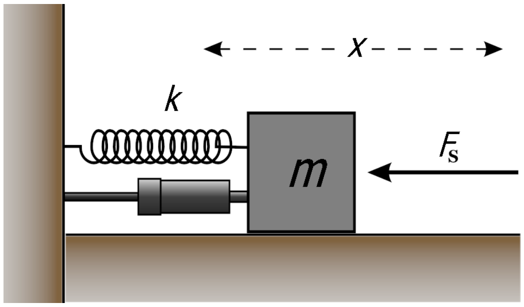
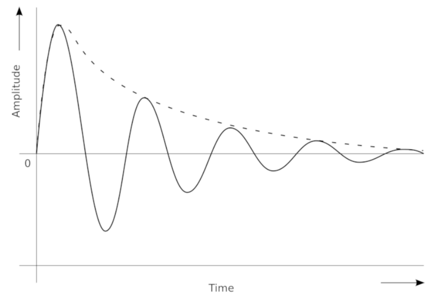
5.4 Mechanical Vibrations



Mass, Spring, Damper Model



Mechanical Vibrations are modeled by the DEQ: $F_T = F_S + F_d + F_e(t)$, where

$F_T = mx''$ represents the **total force** on an object.

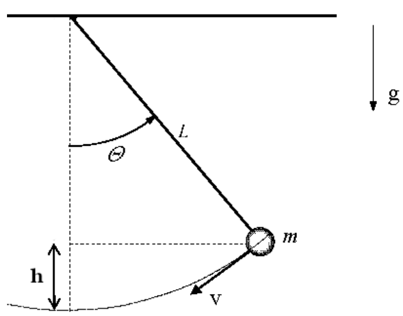
$F_d = -cx'$ represents the **damping force**, $F_S = -kx$ represents the **spring force**, and $F_e(t)$ represents any **external force**.

So our DEQ becomes: $mx'' = -kx - cx' + F_e(t)$.

Rewriting in normal form gives: $x'' + \frac{c}{m}x' + \frac{k}{m}x = \frac{1}{m}F_e(t)$ (it's non-homogenous!).

When $F_e = 0$, we say the DEQ is **free**, otherwise, we refer to it as being **forced**.

Simple Pendulum



Label the counterclockwise angle the pendulum makes with the vertical as function of time: $\theta(t)$.

To determine the DEQs of a physical system, very frequently we start with a conservation law, then derive the DEQs.

Conservation of Mechanical Energy: $T + V = C$, where T, V is kinetic and potential energy, and C is some constant.

Let's calculate kinetic energy: $T = \frac{1}{2}mv^2$.

For this we will need a distance/position function $s(t)$.

Circumference of a circle $2\pi r = 2\pi L$.

Therefore, distance along arc from vertical is $s = L\theta$.

Velocity is $\frac{ds}{dt} = \frac{d(L\theta)}{dt} = L\theta'$ and $T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2$.

Now let's calculate potential energy mgh.

To determine height, we need to know length of the triangle side opposite the mass.

Observe $\cos\theta = \frac{a}{h} = \frac{a}{L}$, where a is the side length of interest.

So, $a = L\cos\theta$ and $h = L - L\cos\theta = L(1 - \cos\theta)$.

Therefore, $T + V = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + L(1 - \cos\theta) = C$.

Taking the derivative with respect to t:

$mL^2\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) + mgL\sin\theta\frac{d\theta}{dt} = 0$,

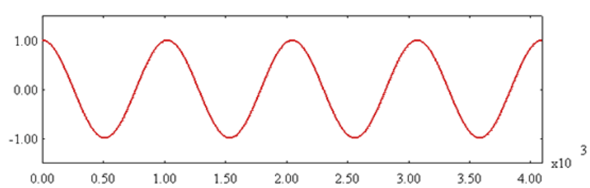
and making the very reasonable assumption that $\frac{d\theta}{dt}, m, L \neq 0$

(pendulum is moving, as a nonzero mass and length), we divide to get:

$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$.

In real situations there is friction $c\theta'$ due to air resistance on m and at the connection where the string is fixed. Also, in many applications, we are most interested in this system when the pendulum is moving only slightly. In such situations, $\theta \approx \sin\theta$. This is an approximation, but making this substitution makes the analysis much simpler. So we get:

$\theta'' + c\theta' + k\theta = 0$, where $k = \frac{g}{L}$.



Free Undamped Motion: $mx'' + kx = 0$. (homogenous)

Normal form:

$\omega_0 := \sqrt{\frac{k}{m}} \quad \Rightarrow \quad x'' + \omega_0^2 x = 0$, where ω_0 is the **circular frequency** in $\frac{rad}{sec}$.

Using our skills from previous section, $r^2 + \omega_0^2 = 0$ when $r = \pm\sqrt{-\omega_0^2} = \pm i\omega_0$.

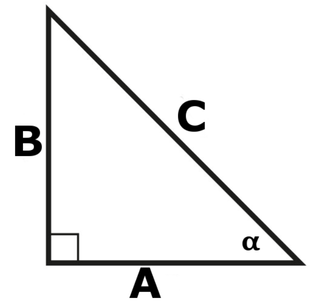
$e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t$.

The Gen. Solution is: $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$.

We wish to alter the solution $x(t)$ to make it simpler.

We want: $x(t) = C\cos(\omega_0 t - \alpha)$, where C turns out to be the amplitude of the vibration!

So, let A and B be the legs of a right triangle, then the hypotenus: $C = \sqrt{A^2 + B^2}$.



With angle α (opposite of B), recall we have: $\cos\alpha = \frac{A}{C}$, $\sin\alpha = \frac{B}{C}$,

$$\text{where } \alpha = \begin{cases} \tan^{-1}\frac{B}{A} & \text{if } A, B > 0 \text{ (1st quadrant)}, \\ \pi + \tan^{-1}\frac{B}{A} & \text{if } A < 0 \text{ (2nd/3rd quadrant)}, \\ 2\pi + \tan^{-1}\frac{B}{A} & \text{if } A > 0, B < 0 \text{ (4th quadrant)}. \end{cases}$$

$$\begin{aligned} \text{Then, } x(t) &= A\cos\omega_0 t + B\sin\omega_0 t = C\left(\tfrac{A}{C}\cos\omega_0 t + \tfrac{B}{C}\sin\omega_0 t\right) \\ &= C(\cos\alpha\cos\omega_0 t + \sin\alpha\sin\omega_0 t). \end{aligned}$$

Recall the Trigonometric Identity: $\cos x\cos y + \sin y\sin x = \cos(x - y) = \cos(y - x)$.

So, $x(t) = C\cos(\omega_0 t - \alpha)$, where C is the **amplitude**,
ω_0 is the **circular frequency** in $\frac{rad}{sec}$, and α is the **phase angle**.

Period of Motion: $T = \frac{2\pi}{\omega_0}$ sec. **Frequency**: $\nu = \frac{1}{T} = \frac{\omega_0}{2\pi}$ in $\frac{cycles}{sec}$.

Free Damped Motion: $x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$
$\Rightarrow$ $x'' + 2px' + \omega_0^2 x = 0$, where $p := \frac{c}{2m} > 0$.

$$r^2 + 2pr + \omega_0^2 = 0 \quad \Rightarrow \quad r = \frac{-2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}.$$

The nature of the roots depend upon the sign of: $p^2 - \omega_0^2 = \frac{c^2}{4m^2} - \frac{k}{m} = \frac{c^2 - 4km}{4m^2}$.

Three situations: $c > \sqrt{4km}$, $c = \sqrt{4km}$, $c < \sqrt{4km}$.

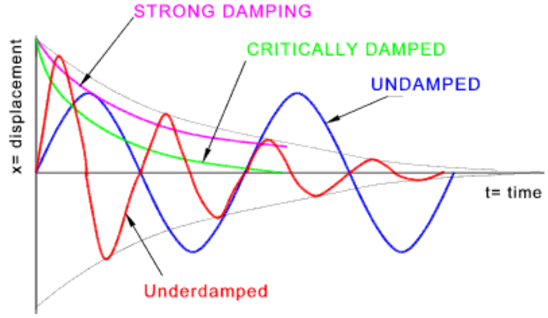
Critical Damping $c_{cr} = \sqrt{4km}$.

♦ **Overdamped Case**: $c > c_{cr}$. $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where $r_1, r_2 < 0$.

♦ **Critically Damped Case**: $c = c_{cr}$. $x(t) = e^{-pt}(c_1 + c_2 t)$.

♦ **Underdamped Case**: $c < c_{cr}$.

$x(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t)$, where $\omega_1 := \sqrt{\omega_0^2 - p^2}$ (damped circ. freq.)
Alternatively: $Ce^{-pt}\cos(\omega_1 t - \alpha)$, (where $C = \sqrt{A^2 + B^2}$, $\cos\alpha = \frac{A}{C}$, $\sin\alpha = \frac{B}{C}$).



Notice that in all three cases, $x(t) \to 0$ as $t \to +\infty$.

Underdamped

Overdamped

Exercises

Problem: ~#17a A mass $m = 81$ is attached to both a spring with spring constant $k = 4$, and a dashpot with damping constant $c = 36$. Find the position function $x(t)$ and determine whether the motion is overdamped, underdamped, or critically damped.

$mx'' + cx' + kx = 0$, $81x'' + 36x' + 4x = 0$

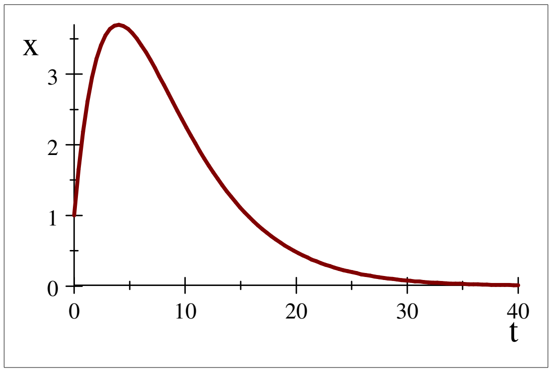
To determine which equation to use: $c_{cr} = \sqrt{4km} = \sqrt{4 \cdot 4 \cdot 81} = 4 \cdot 9 = 36$.

And we see that $c = 36 = c_{cr}$, so …

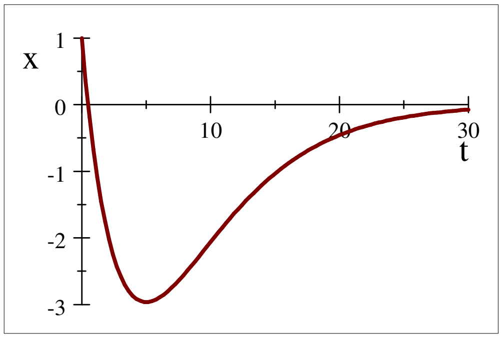
So we are in the critically damped case: $x(t) = e^{-pt}(c_1 + c_2 t)$.

$p = \frac{c}{2m} = \frac{36}{2 \cdot 81} = \frac{2}{9}$.

$x(t) = e^{-\frac{2}{9}t}(c_1 + c_2 t)$.



for $c_1 = 1$ and $c_2 = 2$



for $c_1 = 1$ and $c_2 = -2$

Problem: ~#17b A mass $m = 1$ is attached to both a spring with spring constant $k = 9,$ and a dashpot with damping constant $c = 8.$ Find the position function $x(t)$ and determine whether the motion is overdamped, underdamped, or critically damped.

$mx'' + cx' + kx = 0,\qquad x'' + 8x' + 9x = 0$

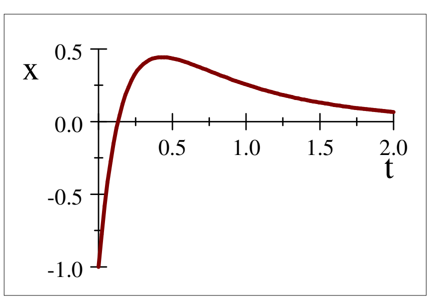
To determine which equation to use: $c_{cr} = \sqrt{4km} = \sqrt{4 \cdot 9 \cdot 1} = 6 < 8 = c.$

And we see that $c = 8 > c_{cr},$ so …

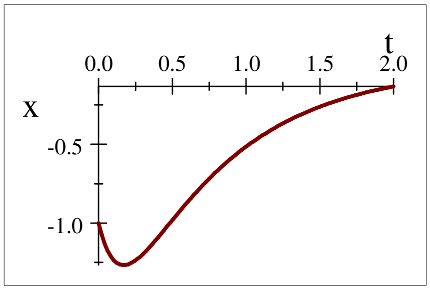
So we are in the over-damped case: $x(t) = c_1 x^{r_1 t} + c_2 x^{r_2 t}.$

$r^2 + 8r + 9 \quad \Rightarrow \quad r = \frac{-8 \pm \sqrt{64 - 4 \cdot 9}}{2} = -4 \pm \sqrt{7}.$

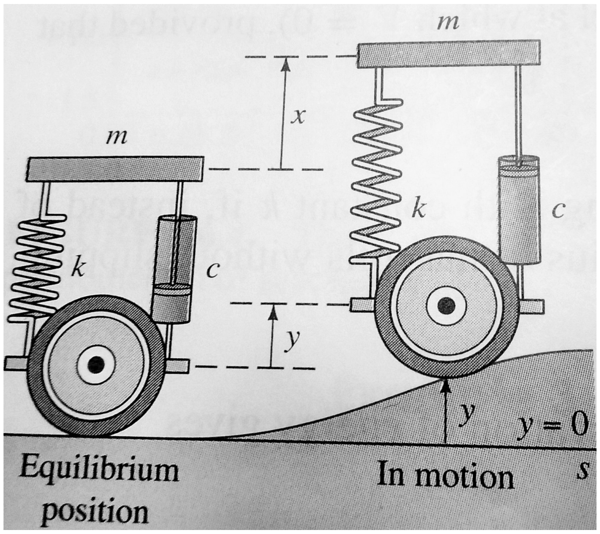
$x(t) = c_1 e^{\left(-4+\sqrt{7}\right)t} + c_2 e^{\left(-4-\sqrt{7}\right)t}$



for $c_1 = 1$ and $c_2 = -2$



for $c_1 = 1$ and $c_2 = -2$



Problem: **#23** This problem deals with a highly simplified model of a car weighing 3,200 pounds (mass $m = 100$ slugs in *fps* units). Assume that the suspension system acts like a single spring, and its shock absorbers (if connected) act like a single dashpot, so that its vertical vibrations (over a smooth flat road) satisfy: $mx'' + cx' + kx = 0$.

a) Find the stiffness coefficient k of the spring if the car undergoes free vibrations (v) of 80 cycles per minute when its shock absorbers are disconnected.

Shock absorbers disconnected? $mx'' + kx = 0$. How to find k?

With $m = 100$ slugs we get: $\omega_0 = \sqrt{\frac{k}{100}}$, $x'' + \omega_0^2 x = 0$.

"cycles per minute" is **frequency** (v), but we need to convert this into ω_0 which is **circular frequency** in units of $\frac{rad}{s}$.

$$\omega_0 = \frac{80\ cycles}{1\ \text{min}} \frac{1\ \text{min}}{60\ \text{sec}} (2\pi) = \frac{8\pi}{3} \frac{rad}{s}.$$

So, $\frac{8\pi}{3} = \sqrt{\frac{k}{100}}$, $\frac{k}{100} = \left(\frac{8\pi}{3}\right)^2$, $k = \frac{6400\pi^2}{9} \approx 7{,}018\ lb/ft$.

b) With the shock absorbers connected, the car is initially set into vibration by driving it over a bump, and the resulting damped vibrations have a frequency of 78 **cycles per minute**.

After how long will the time-varying amplitude be 1% **of its initial value**?

Which equation will we be working with?

Since there are vibrations ...

We are not in the overdamped/critically damped cases. We are dealing with the **underdamped** case.

The gen. solution for this case is : $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$, where $\omega_1 = \sqrt{\omega_0^2 - p^2}$.

"After how long will the **time-varying amplitude** be 1% of its initial value?"

When does $Ce^{-pt} = 0.01Ce^{-p \cdot t_0}$? The initial time value is $t_0 = 0$, becomes: $e^{-pt} = 0.01$.

$$\Rightarrow \quad -pt = \ln(0.01) \quad \Rightarrow \quad t = \frac{\ln(0.01)}{-p}.$$

So, we must first solve for p, where $p = \frac{c}{2m} = \frac{c}{200}$.

Which means we first must solve for c.

$$\omega_1 = \sqrt{\omega_0^2 - p^2}$$

$$= \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \sqrt{\frac{4km - c^2}{4m^2}} = \frac{\sqrt{4km - c^2}}{2m}$$

$$\approx \frac{\sqrt{2{,}807{,}200 - c^2}}{200}.$$

Which means we first must solve for ω_1!!!

However, we are given that the **damped** frequency ν is: $\frac{78 \ cycles}{1 \ \min}$ or $\frac{78 \ cycles}{1 \ \min} \frac{1 \ \min}{60 \ \sec} = \frac{78 \ cycles}{60 \ \sec}$.

So, the damped **circular frequency** ω_1 is: $\frac{78 \ cycles}{60 \ \sec}\left(\frac{2\pi \ rad}{1 \ cycle}\right) \approx 8.1681 \ rad/\sec$.

So, $\frac{\sqrt{2807200 - c^2}}{200} = 8.1681$, $\sqrt{2807200 - c^2} = 1633.6$

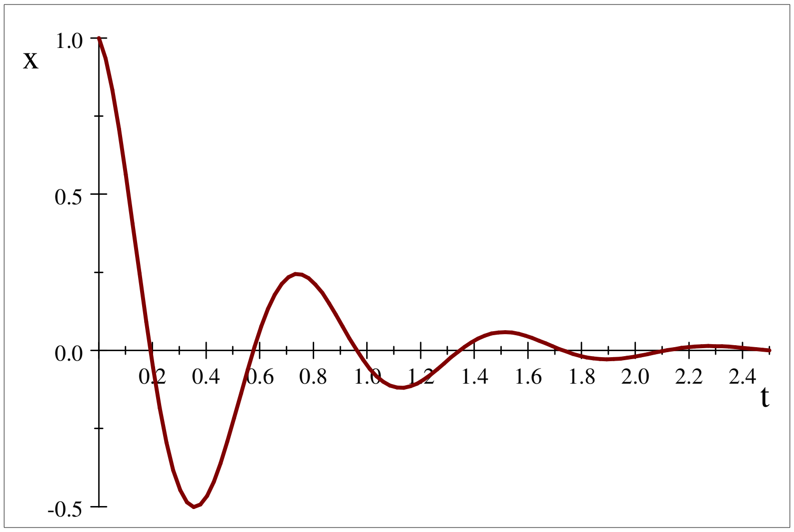
$2807200 - c^2 = 2668648.96$, $c^2 = 2807200 - 2668648.96 = 138551.04$

$c = \sqrt{138551.04} \approx 372.22 \ lb/(ft/\sec)$.

Hence: $p = \frac{c}{2m} = \frac{372.22}{200} \approx 1.8611$.

$t = \frac{\ln(0.01)}{-p} \approx \frac{\ln(0.01)}{-1.8611} \approx 2.47 \ \sec.$ (whew!)

And plugging in our calculations for p and ω_1, we have: $x(t) \approx Ce^{-1.86t}\cos(8.17t - \alpha)$.



$\alpha = 0$ and $C = 1$

For the 3rd **Midterm**/**Final exam**, a less complicated task you should be able to check is whether a damping constant would result in overdamped, underdamped, or critically damped vibrations. You do this by comparing your damping constant c to $\sqrt{4km}$. In the above case, $c = 372.22$ and $\sqrt{4km} = \sqrt{4 \cdot 7018 \cdot 100} \approx 1676$. Therefore $c < \sqrt{4km}$ and the vibrations are underdamped, as we surmised earlier.

Problem: #33 The local maxima and minima of $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$, occur where: $\tan(\omega_1 t - \alpha) = -\frac{p}{\omega_1}$.

Consecutive maxima occur at times $x_1 = x(t_1)$ and $x_2 = x(t_2)$. Assume: $t_2 - t_1 = \frac{2\pi}{\omega_1}$.

Deduce that: $\ln\frac{x_1}{x_2} = \frac{2\pi p}{\omega_1}$ (recall that $p = \frac{c}{2m}$).

If $x_1 = x(t_1)$ and $x_2 = x(t_2)$ are two successive local maxima, then...

$$\cos(\omega_1 t_2 - \alpha) = \cos(\omega_1 t_1 - \alpha)$$

$$\omega_1 t_2 - \alpha = \omega_1 t_1 - \alpha + 2\pi, \text{ and } \omega_1 t_2 = \omega_1 t_1 + 2\pi \text{ so} \ldots$$

$$x_1 = Ce^{-pt_1}\cos(\omega_1 t_1 - \alpha),$$
$$x_2 = Ce^{-pt_2}\cos(\omega_1 t_2 - \alpha) = Ce^{-pt_2}\cos((\omega_1 t_1 + 2\pi) - \alpha) = Ce^{-pt_2}\cos(\omega_1 t_1 - \alpha)$$

Hence, $\frac{x_1}{x_2} = \frac{Ce^{-pt_1}\cos(\omega_1 t_1 - \alpha)}{Ce^{-pt_2}\cos(\omega_1 t_1 - \alpha)} = e^{-p(t_1 - t_2)}$,

and therefore, $\ln(\frac{x_1}{x_2}) = -p(t_1 - t_2) = -\left(\frac{c}{2m}\right)\left(\frac{2\pi}{\omega_1}\right) = \frac{2\pi p}{\omega_1}$.