MATH 2243: Linear Algebra & Differential Equations

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5.2: Gen. Solutions of Linear DEQs

Consider: $y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = f(x).$ $\qquad (*)$

Most of the results below are merely extensions of the $n = 2$ case from the previous section, and the related proofs are nearly identical.

Principle of Superposition for Homogeneous DEQs Theorem: Let $y_1, y_2, \ldots, y_n$ be n solutions of the associated *homogeneous* linear DEQ of $(*)$ on the interval I. If $c_1, c_2, \ldots, c_n$ are constants, then the linear combination $y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is also a solution on I.

Existence and Uniqueness for Linear DEQs Theorem: Suppose that the functions $p_1, p_2, \ldots, p_n,$ and f are continuous on the open interval I containing the point a. Then, given n numbers $b_0, b_1, \ldots, b_{n-1}$, the nonhomogeneous DEQ $(*)$ has a unique (that is, one and only one) solution on the *entire* interval I that satisfies the n initial conditions:
$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}.$

Thus we have an nth-order **initial value problem**.

As with the linear 1st order and 2nd order DEQs, the unique solutions to this nth order linear DEQ exist on the whole interval I.

Independence

How do we determine whether n solutions to our DEQ are linearly independent, so that we might form a general solution?

Recall that with two functions, we needed $f_1 = cf_2$ on I for dependence, or $f_1 \neq cf_2$ for independence.

Also recall that with n vectors $\vec{v}_i$, dependence was insured if $c_1\vec{v}_i + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = 0$ with $c_1, c_2, \ldots, c_n,$ not all zero.
And independence was assured if $c_1\vec{v}_i + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = 0$ required that $c_1, c_2, \ldots, c_n,$ all be zero.

But now recall that functions ARE vectors in the real valued function vector space. Therefore, we have the following.

Definition — Linear Dependence of Functions: The n functions $f_1, f_2, \ldots, f_n$ are said to be linearly dependent on the interval I provided that there exists constants $c_1, c_2, \ldots, c_n,$ not all zero, such that $c_1f_1 + c_2f_2 + \ldots + c_nf_n = 0$ on I; that is,
$c_1f_1(x) + c_2f_2(x) + \ldots + c_nf_n(x) = 0$ for all x in I.

Therefore, just as with n-tuple vectors, if functions are dependent, we can solve for one of the functions in terms of a linear combination of the others.

Wronskian of Solutions Theorem: Suppose that $y_1, y_2, \ldots, y_n$ are n solutions of the associated *homogeneous* linear DEQ of $(*)$ on an open interval I, where each p_i is continuous. Let $W = W(y_1, y_2, \ldots, y_n)$.

- ♦ If $y_1, y_2, \ldots, y_n$ are linearly dependent, then $W \equiv 0,$ at each point x in I.
- ♦ If $y_1, y_2, \ldots, y_n$ are linearly independent, then $W \neq 0,$ at each point x in I.

Thus, there are just two possibilities: either $W = 0$ everywhere on I, or $W \neq 0$ everywhere on I.

In the above theorem, let's prove the first bullet point:
that if $y_1, y_2, \ldots, y_n$ are linearly dependent, then $W \equiv 0,$ at each point x in I.

Proof: Since we can assume dependence, we have that $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$ holds at each point x in I for some choice of $c_1, c_2, \ldots, c_n,$ not all zero.

Next, differentiate this equation $n-1$ times in succession, obtaining the equations:

$$\begin{gathered} c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) = 0 \\ c_1 y_1'(x) + c_2 y_2'(x) + \ldots + c_n y_n'(x) = 0 \\ \vdots \\ c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + \ldots + c_n y_n^{(n-1)}(x) = 0 \end{gathered}$$

which still holds at each point x in I.

Observe that the unknowns in the above system are the c_i. Therefore, this can be rewritten as:

$$\mathbf{A}\vec{c} = \vec{0}, \text{ where } \vec{c} := (c_1, \ldots, c_n) \text{ and } \mathbf{A} := \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}.$$

Now recall that a homogeneous $n \times n$ linear system of equations has a nontrivial solution if and only if it's coefficient matrix **A** is not invertible.

We also learned non-invertibility only happens when the determinant of the coefficient matrix |**A**| is zero.

In this case, the determinant is recognizable as the Wronskian $W(x)$ of the y_i.

And since we know that the c_i are not all zero, it follows that $W(x) \equiv 0$, as we wished to prove. ■

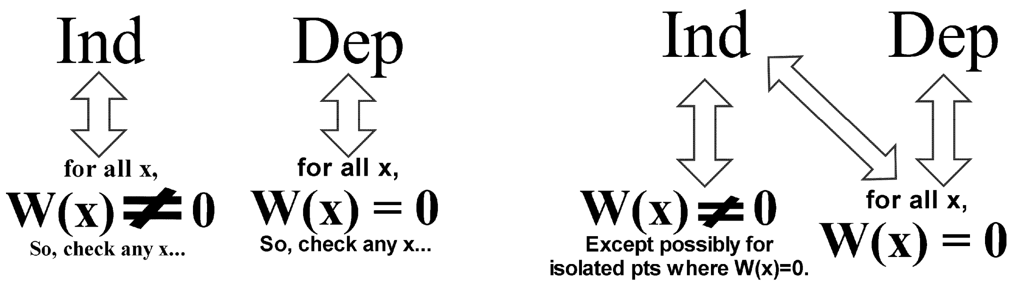
The above is all well-and-good when the functions we are examining are solutions to a homogeneous DEQ. But what if you wish to know the independence of some functions on some open interval I which are not known to be solutions to a DEQ?

Here is a graphic that might clarify (or confuse) things for you…

Given some interval I, we wish to know if some functions {f(x),g(x),h(x), ...} are independent or dependent on I. So form the Wronskian W(x)=W(f,g,h,...) and:

If you are lucky enough to know that your functions are solutions to a linear homogeneous differential equation with continuous coefficient functions.

OTHERWISE...



Consider: $y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = f(x).$ $(*)$

General Solutions of Homogeneous DEQs Theorem: Let's say you know that $y_1, y_2, \ldots, y_n$ are linearly independent solutions of $(*)'$s associated *homogeneous* DEQ on an open interval I, where the p_i are continuous. If Y is any solution whatsoever to the homogeneous DEQ, then there exist numbers $c_1, c_2, \ldots, c_n$ such that $Y(x) = c_1y_1 + c_2y_2 + \ldots + c_ny_n$ for all x in I. (i.e., all other solutions can be characterized as a linear combination of these linearly independent ones)

Solutions to Non-homogeneous DEQs Theorem: Let's say you know that y_p is a **particular solution** for the *non-homogeneous* DEQ $(*)$ on an open interval I, where the p_i and f are continuous. And suppose $y_1, y_2, \ldots, y_n$ are linearly independent solutions of $(*)'$s associated *homogeneous* DEQ. Then if $Y(x)$ is any solution whatsoever to the nonhomogeneous DEQ, then there exist numbers $c_1, c_2, \ldots, c_n$ such that for all x in I we have: $Y(x) = y_p + (c_1y_1 + c_2y_2 + \ldots + c_ny_n)$.

Proof: Let Y and y_p be solutions to $(*)$.

Define $y_c := Y - y_p$. Substituting this into the $(*)'$s associated *homogeneous* DEQ:

$$(Y - y_p)^{(n)} + p_1(x)(Y - y_p)^{(n-1)} + \ldots + p_{n-1}(x)(Y - y_p)' + p_n(x)(Y - y_p)$$

$$= \left(Y^{(n)} + p_1(x)Y^{(n-1)} + \ldots + p_{n-1}(x)Y' + p_1(x)Y\right) - \left(y_p^{(n)} + p_1(x)y_p^{(n-1)} + \ldots + p_{n-1}(x)y_p' + p_n(x)y_p\right)$$

$$= f(x) - f(x) = 0.$$

Therefore, $y_c = Y - y_p$ is a solution to $(*)'$s associated *homogeneous* DEQ.

Recall that the complementary homogeneous solution can be written: $y_c = c_1y_1 + \ldots + c_ny_n$.

But rearranging $y_c = Y - y_p$, we find $Y = y_p + y_c = y_p + (c_1y_1 + \ldots + c_ny_n)$.

Recall our choice of Y as a solution to the nonhomogeneous DEQ was arbitrary.

So we have shown that a *general solution* Y of the nonhomogeneous DEQ is the sum of its complementary function y_c and any particular solution y_p. ■

From this theorem, we see that the general solutions are an "n-fold infinity" of solutions (by choosing $c_1, c_2, \ldots, c_n$). Similarly (and for the same underlying reason), the unique solution given by the existence theorem above implies an "n-fold infinity" of freedom in choosing initial conditions: $y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}$.

Now notice that the trivial solution $y(x) \equiv 0$, is a solution to $y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = 0$.

Furthermore, $y(x) \equiv 0$ is the only solution to the DEQ that satisfies the trivial initial conditions $y(a) = 0, \quad y'(a) = 0, \quad \ldots, \quad y^{(n-1)}(a) = 0$.

Exercises

Problem: **#30** Verify that $y_1 = x$ and $y_2 = x^2$ are linearly independent solutions (on the entire real line) of the equation $x^2y'' - 2xy' + 2y = 0$. Also verify that $W\left(x,\ x^2\right)$ vanishes at $x = 0$. Why do these observations not contradict part (b) of the Wronskian of Solutions Theorem?

Hint: Differentiate y_1 to get y_1' and y_1'', then substitute it into the equation to verify that y_1 is a solution. Do the same thing with y_2. Let's assume we've done that (exercise for home).

To confirm linear independence, it is sufficient to note that you cannot represent x as $x = cx^2$, irrespective of what the constant c is.

Next, create your Wronskian:

$$W\left(x,\ x^2\right) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2, \text{ and verify that the result vanishes at } x = 0.$$

Finally, let's think about the Wronskian of Solutions Theorem: It assumes your equation has the form:

$$y'' + p_1(x)y' + p_2(x)y = 0,$$

where p_1, p_2 **are continuous** functions (on the interval of interest, near the initial condition).

However, if p_1, p_2 are NOT continuous functions there, we should not expect the conclusions of the theorem to hold true.

When the equation $x^2y'' - 2xy' + 2y = 0$ is rewritten in the above form: $y'' + (-\frac{2}{x})y' + \left(\frac{2}{x^2}\right)y = 0$, the coefficient functions $p_1(x) = -\frac{2}{x}$ and $p_2(x) = \frac{2}{x^2}$ are not continuous at $x = 0$. Thus, the assumptions of the theorem are not satisfied.

Problem: **#12** **Use the Wronskian to prove that the functions** $\{x, \cos(\ln x),\ \sin(\ln x)\}$ **are linearly independent on the interval** $x > 0$.

$$W = \begin{vmatrix} x & \cos(\ln x) & \sin(\ln x) \\ 1 & -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \\ 0 & -\frac{\frac{1}{x}\cos(\ln x)(x) - \sin(\ln x)}{x^2} & \frac{-\frac{1}{x}\sin(\ln x)(x) - \cos(\ln x)}{x^2} \end{vmatrix} = \begin{vmatrix} x & \cos(\ln x) & \sin(\ln x) \\ 1 & -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \\ 0 & \frac{-\cos(\ln x) + \sin(\ln x)}{x^2} & \frac{-\sin(\ln x) - \cos(\ln x)}{x^2} \end{vmatrix}$$

$$= x\left(-\frac{\sin(\ln x)}{x}\frac{-\sin(\ln x)-\cos(\ln x)}{x^2} - \frac{\cos(\ln x)}{x}\frac{-\cos(\ln x)+\sin(\ln x)}{x^2}\right) - \left(\cos(\ln x)\frac{-\sin(\ln x)-\cos(\ln x)}{x^2} - \sin(\ln x)\frac{-\cos(\ln x)+\sin(\ln x)}{x^2}\right)$$

$$= \frac{\sin^2(\ln x)+\sin(\ln x)\cos(\ln x)}{x^2} - \frac{-\cos^2(\ln x)+\sin(\ln x)\cos(\ln x)}{x^2} + \frac{\cos(\ln x)\sin(\ln x)+\cos^2(\ln x)}{x^2} + \frac{-\sin(\ln x)\cos(\ln x)+\sin^2(\ln x)}{x^2}.$$

So, $W = x^{-2}[2\cos^2(\ln x) + 2\sin^2(\ln x)]$

$$= 2x^{-2}.$$

And, W is nonzero (and defined) for $x > 0$.

So, the functions $\{x, \cos(\ln x),\ \sin(\ln x)\}$ are linearly independent on the interval $x > 0$.