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MATH 2243: **Linear Algebra** & **Differential Equations**

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4.7: General Vector Spaces

Qualities of a Vector Space:

- ♦ $\vec{u}+\vec{v}=\vec{v}+\vec{u}$, [additive commutivity]
- ♦ $\vec{u}+\left(\vec{v}+\vec{w}\right)=\left(\vec{u}+\vec{v}\right)+\vec{w}$, for any $\vec{w}$. [additive associativity]
- ♦ $\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$ [additive identity $\vec{0}=(0,0,0)$]
- ♦ $1\left(\vec{u}\right)=\vec{u}$ [scalar multiplicative identity]
- ♦ $\vec{u}+\left(-\vec{u}\right)=\left(-\vec{u}\right)+\vec{u}=\vec{0}$ [additive inverse $-\vec{u}=(-1,-2,-3)$]
- ♦ $r\left(\vec{u}+\vec{v}\right)=r\vec{u}+r\vec{v}$, for any $r\in\mathbb{R}$ [distributivity]
- ♦ $(r+s)\vec{u}=r\vec{u}+s\vec{u}$, for any $r,s\in\mathbb{R}$
- ♦ $r\left(s\vec{u}\right)=(rs)\vec{u}$

Matrix Spaces

If you fix $m,n\in\mathbb{N}$, then you will find that the set of all $m\times n$ matrices forms a vector space (satisfies all the qualities above), where $\mathbf{0}^{m\times n}$ is the "additive identity," and for any $\mathbf{A}^{m\times n}$, we have $-\mathbf{A}$ as the additive inverse, etc.

These vector spaces also have bases consisting of the matrices $\mathbf{E}_{ij}$, which are matrices with zeros for every component except the ij^{th} component (which is equal to 1). In this way, any matrix can be formed by creating a linear combination of the $\mathbf{E}_{ij}$. For example, the basis for the matrices of size $\mathbf{A}^{3\times 2}$, is:

$$\{\mathbf{E}_{11},\mathbf{E}_{12},\mathbf{E}_{21},\mathbf{E}_{22},\mathbf{E}_{31},\mathbf{E}_{32}\}=\left\{\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\\0&1\end{bmatrix}\right\}.$$

So, if we choose $m=n$, we see that the set of square matrices, of any particular size (e.g. the 2×2 matrices), is also a vector space.

Function Spaces

Recall the vector space $\mathcal{F}$ of all real valued functions defined on the real line $\mathbb{R}$, where $f(x)=0$ was the additive identity, and for any $g\in\mathcal{F}$, we had $-g$ as the additive inverse, etc.

One subspace of $\mathcal{F}$ is the space $\mathcal{P}$ of all the polynomials. A further subset of $\mathcal{F}$ is this subset $\mathcal{P}_n\subset\mathcal{P}\subset\mathcal{F}$, which is the set of all polynomials of degree at most n. In other words, elements of $\mathcal{P}_2$ include functions like $\left\{0,\ \pi,\ 1+x,\ x^2,\ x-5x^2\right\}$. Observe that polynomials in $\mathcal{P}_2$ have dimension of at most 3 (NOT 2). In general, elements of $\mathcal{P}_2$ can be represented by $p(x)=a_0+a_1x+a_2x^2$. Observe that $\mathcal{P}_2\subset\mathcal{P}_3\subset\ldots$ to infinity! So $\mathcal{P}$, which contains all of these infinite subspaces, is an infinite dimensional subspace!!

Another even larger function space is the space of all continuous functions C^0. We also have the set of functions that have k continuous derivatives C^k. Observe that all polynomials are continuous and have continuous derivatives. Therefore, $\mathcal{P}_0\subset\mathcal{P}_1\subset\ldots\subset\mathcal{P}\subset C^\infty\subset C^k\subset C^0\subset\mathcal{F}$.

Solution Spaces of Differential Equations

We see in other sections of the book that solutions to differential equations often take the form of families of solutions (for example: $x(t) = Ce^{2t}$). These families are subspaces of the function space. We call these **solution spaces**. We will see that we can combine the various solutions in these spaces in linear combinations, and they will still be solutions to our differential equations (closed under addition and scalar multiplication).

Problem 8: Determine whether or not the set of all f such that $f(-x) = -f(x)$ for all x, is a subspace of the space $\mathcal{F}$ of all real valued functions on $\mathbb{R}$.

A function $f : \mathbb{R} \to \mathbb{R}$ such that $f(-x) = -f(x)$ is called an odd function. What is an example?

$\sin x$

Any linear combination $af + bg$ of all odd functions is again odd, because:

$(af + bg)(-x)$

$= af(-x) + bg(-x)$

$= -af(x) - bg(x)$

$= -(af(x) + bg(x))$

$= -(af + bg)(x)$.

Thus the set of all odd functions is a vector space.

Problem 10: You are told that the polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$, has $a_0 = a_1 = 0$. Determine whether or not the set of all polynomials satisfying this condition is a subspace of the space $\mathcal{P}$ of all polynomials.

It is obvious that polynomials of this form exist (x^3, for example). So the set is nonempty.

Choosing two arbitrary polynomials of this form: $f = a_2x^2 + a_3x^3$ and $g = b_2x^2 + b_3x^3$, and putting them in a linear combination we have: $cf + dg = c(a_2x^2 + a_3x^3) + d(b_2x^2 + b_3x^3)$

$= (ca_2 + db_2)x^2 + (ca_3 + db_3)x^3,$

which is an element of the correct form $(a_0 = a_1 = 0)$.

Problem 16: Determine whether the functions $1 + x,\ x + x^2,$ and $1 - x^2$ are linearly independent.

So we would like to know if there are nontrivial coefficients c_i such that:

$c_1(1 + x) + c_2(x + x^2) + c_3(1 - x^2) = 0.$

By inspection I notice:

$(-1)(1 + x) + (1)(x + x^2) + (1)(1 - x^2) = 0,$ so the three given polynomials are linearly dependent.

A way to do this without guessing is to rearrange the equation:

$c_1 + c_3 + (c_1 + c_2)x + (c_2 - c_3)x^2 = 0.$

Recall that in an equation like this, we can equate powers of x.

Therefore, we have a system of three equations in three unknowns, compute!

It turns out that c_2 must equal to c_3, and that c_1 must be equal to $-c_2$.