MATH 2243: **Linear Algebra** & **Differential Equations**

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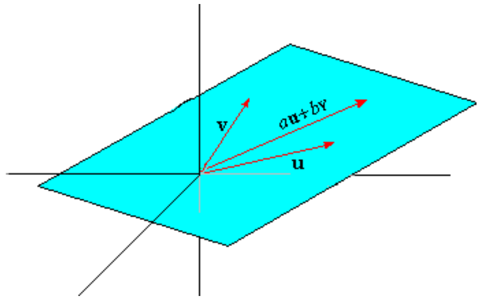
Big idea: Solutions to homogeneous linear systems of equations are subspaces that can be generated (spanned) by a few vectors.

4.3: Linear Combinations and Independence of Vectors



The Span of a Set of Vectors:

Let $V' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a subset of vectors in V. (for example, $V' = \{(-1,2,1),(1,-2,1)\}$ in $\mathbb{R}^3$)

Let W be the set of all linear combinations of V'. (W for our example would be a plane in $\mathbb{R}^3$)

Then, W is a subspace of V.

We write: $W = span(V') = span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\})$.

In particular, recall in the class notes for 4.2 that the homogeneous system:

$$x_1 - 4x_2 - 3x_3 - 7x_4 = 0$$
$$2x_1 - x_2 + x_3 + 7x_4 = 0$$
$$x_1 + 2x_2 + 3x_3 + 11x_4 = 0$$

gives us solution space W consisting of $\vec{x} = a\vec{u} + b\vec{v}$, where $\vec{u} = (-1,-1,1,0)$ and $\vec{v} = (-5,-3,0,1)$.

We can visualize (??) W as a plane in $\mathbb{R}^4$ determined by $\vec{u}, \vec{v}$. In other words, $W = span(\{\vec{u}, \vec{v}\})$.

Is $\vec{w} = (2,-6,3)$ a linear combination of $\vec{v}_1 = (1,-2,-1)$ and $\vec{v}_2 = (3,-5,4)$?

In other words, can we find unknowns c_1, c_2 such that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{w}$?

$$c_1\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ -2 & -5 & -6 \\ -1 & 4 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 7 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 19 \end{array}\right].$$

Alternatively for $\vec{w} = (2,-6,-16)$, $\vec{v}_1 = (1,-2,-1)$ and $\vec{v}_2 = (3,-5,4)$:

$$c_1\begin{bmatrix}1\\-2\\-1\end{bmatrix}+c_2\begin{bmatrix}3\\-5\\4\end{bmatrix}=\begin{bmatrix}2\\-6\\-16\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1&3&2\\-2&-5&-6\\-1&4&-16\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}1&3&2\\0&1&-2\\0&7&-14\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}1&0&8\\0&1&-2\\0&0&0\end{array}\right].$$

$c_1 = 8$ and $c_2 = -2$.

Linear Independence:

The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in a vector space V are said to be linearly independent provided:

$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$ has only the trivial solution: $c_1 = c_2 = \ldots = c_k = 0$.

Corollary: Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly dependent if and only if at least one of them is a linear combination of the others.

Uniqueness of Subspace Linear Combination: Any vector $\vec{w}$ in the subspace W spanned by the independent vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is *uniquely* expressible as a linear combination of these vectors.

Proof: If both $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_k\vec{v}_k$ and $\vec{w} = b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots + b_k\vec{v}_k$, then

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_k\vec{v}_k = b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots + b_k\vec{v}_k$$

$$\Rightarrow \quad (a_1 - b_1)\vec{v}_1 + (a_2 - b_2)\vec{v}_2 + \ldots + (a_k - b_k)\vec{v}_k = \vec{0}.$$

But since $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent, $a_i = b_i$. ■

Standard Unit Vectors for n: $\vec{e}_1 = (1,0,0,\ldots,0)$, $\vec{e}_2 = (0,1,0,0,\ldots,0)$, $\ldots$, $\vec{e}_n = (0,0,\ldots,1)$.

Note: for any $\vec{v} = (a_1, a_2, \ldots, a_n)$, we have $\vec{v} = a_1\vec{e}_1 + a_2\vec{e}_2 + \ldots + a_n\vec{e}_n$, the *unique* linear combination of standard unit vectors for $\vec{v}$.

However, note $\{(5,0,0,0),\ (0,7,0,0),\ (0,0,9,0),\ (0,0,9,1)\}$ and $\{(1,1,1,0),\ (1,0,1,1),\ (1,1,0,1),\ (0,1,1,1)\}$ are (non-standard) bases for $\mathbb{R}^4$.

Linear Independence of $k < n$ Vectors:

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ with $k < n$ are linearly independent if and only if ($\Leftrightarrow$)

there is some $k \times k$ submatrix $\mathbf{B}^{k\times k}$ of $\mathbf{A}^{n\times k} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_k\end{bmatrix}$

with a nonzero determinant ($|\mathbf{B}^{k\times k}| \neq 0$). (justification in the book)

Example : $\{\vec{u}_1, \vec{u}_2\} = \{(1,1,0),\ (2,3,1)\}$

$$\mathbf{A}^{3\times 2} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 1 \end{bmatrix}$$

$$|\mathbf{B}_1| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0, \qquad |\mathbf{B}_2| = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1, \qquad |\mathbf{B}_3| = \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = 2.$$

Alternatively : $\{\vec{u}_1, \vec{u}_2\} = \{(1,1,0),\ (2,2,0)\}$

$$\mathbf{A}^{3\times 2} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$|\mathbf{B}_1| = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0, \qquad |\mathbf{B}_2| = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0, \qquad |\mathbf{B}_3| = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0.$$

Exercises

Problem: **#16** If possible, express $\vec{w} = (7,7,9,11)$ as a linear combination of
$\vec{v}_1 = (2,0,3,1),\ \vec{v}_2 = (4,1,3,2),\ \vec{v}_3 = (1,3,-1,3).$
If not, show that it is impossible.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{w} \qquad c_1(2,0,3,1) + c_2(4,1,3,2) + c_3(1,3,-1,3) = (7,7,9,11)$$

$$\mathbf{A}\vec{c} = \vec{w} \Rightarrow \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 3 \\ 3 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 9 \\ 11 \end{bmatrix} \overset{\text{trust me}}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 3 \\ 0 \end{bmatrix}.$$

Has the unique solution…

$c_1 = 6,\ c_2 = -2,\ c_3 = 3,$ so...

$$\vec{w} = 6\vec{v}_1 - 2\vec{v}_2 + 3\vec{v}_3.$$

Want to be sure you got the right answer? Substitute into this equation the relevant vectors to ensure you get $\vec{w} = (7,7,9,11)$.

Problem: **#22** If the following vectors are linearly independent, show it.

Otherwise, find a nontrivial linear combination of them that is equal to the zero vector.
$\vec{v}_1 = (3,9,0,5),\ \vec{v}_2 = (3,0,9,-7),\ \vec{v}_3 = (4,7,5,0)$

$$\mathbf{A} = \begin{bmatrix} 3 & 3 & 4 \\ 9 & 0 & 7 \\ 0 & 9 & 5 \\ 5 & -7 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & \frac{5}{9} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that the system of 4 equations in 3 unknowns has a one-dimensional solution space.

$c_3 = s, \qquad c_1 = -\frac{7}{9}s, \qquad c_2 = -\frac{5}{9}s$

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{9}s \\ -\frac{5}{9}s \\ s \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ -9 \end{bmatrix}, \text{ when } s = -9.$$

Since s is a parameter, and can therefore be any real number, I have chosen -9 as its value for convenience.

So, we have $c_1 = 7,\ c_2 = 5,$ and $c_3 = -9.$

Therefore $7\vec{v}_1 + 5\vec{v}_2 - 9\vec{v}_3 = \vec{0}$.
(on a test, you will want to double check this by making sure the equality holds by plugging in the vectors)

Problems: **#26** Let's assume the set of vectors $\{\vec{v}_i\}$ are linearly independent. Apply the definition of linear independence to show that the vectors below are also linearly independent.
$\vec{u}_1 = \vec{v}_2 + \vec{v}_3, \qquad \vec{u}_2 = \vec{v}_1 + \vec{v}_3, \qquad \vec{u}_3 = \vec{v}_1 + \vec{v}_2$

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$, has only the trivial solution $c_1 = c_2 = c_3 = 0$.

Want to show that $b_1\vec{u}_1 + b_2\vec{u}_2 + b_3\vec{u}_3 = \vec{0}$, has only the trivial solution $b_1 = b_2 = b_3 = 0$.

$(*) \qquad b_1\vec{u}_1 + b_2\vec{u}_2 + b_3\vec{u}_3 = b_1(\vec{v}_2 + \vec{v}_3) + b_2(\vec{v}_1 + \vec{v}_3) + b_3(\vec{v}_1 + \vec{v}_2)$

$= (b_2 + b_3)\vec{v}_1 + (b_1 + b_3)\vec{v}_2 + (b_1 + b_2)\vec{v}_3.$

Setting this equal to zero, by our previous assumption it must be that $b_2 + b_3 = 0,\quad b_1 + b_3 = 0,$ and $b_1 + b_2 = 0.$

From the first equation we have: $b_3 = -b_2$.

Applying this to the second equation, we have: $b_1 = b_2$.

And then from the third equation,we get: $2b_2 = 0$ or $b_2 = 0$. But then $b_1 = 0$, and $b_3 = 0$.

Therefore, only the trivial solution satisfies the equation $(*)$, and the vectors $\{\vec{u}_i\}$ are therefore linearly independent.

Problem: #28 **Prove**: If a set S of vectors is linearly dependent and a (finite) set T contains S, then T is also linearly dependent. Assume $S = \{\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_k\}$ and $T = \{\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_k, \ldots, \vec{v}_m\}$, with $m > k$.

Because the set S of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is linearly dependent,

there exist scalars $c_1, c_2, \ldots, c_k$ not all zero such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$.

Now let $c_{k+1} = \ldots = c_m = 0$.

So we have: $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} + \ldots + c_m\vec{v}_m = \vec{0}$ with the coefficients $c_1, c_2, \ldots, c_m$ not all zero.

This means that the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ that define T are linearly dependent.