MATH 2243: Linear Algebra & Differential Equations

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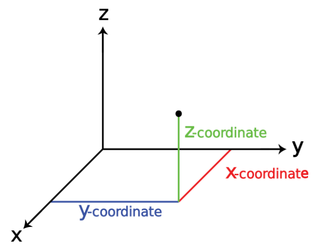
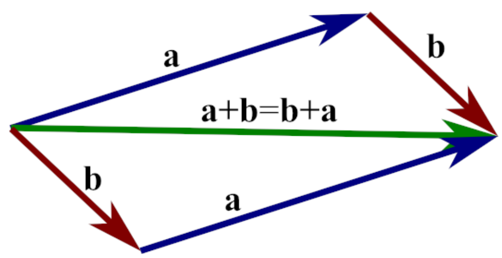
4.1: The Vector Space $\mathbb{R}^n$



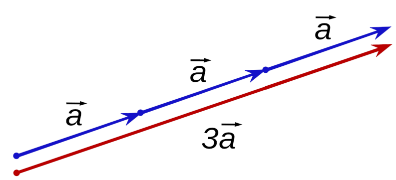
(x,y,z)

If $\vec{a} = (1,2,3)$ and $\vec{b} = (4,5,6)$ are vectors, then:
$\vec{a} + \vec{b} = \left(1+4,\ 2+5,\ 3+6\right) = \left(5,\ 7,\ 9\right)$.



Vector Addition

And: $3\vec{a} = \left(3 \cdot 1,\ 3 \cdot 2,\ 3 \cdot 3\right) = (3,6,9)$.



Scalar Multiplication

Length of vector $\vec{x} := (x_1, x_2, \ldots, x_n)$: Generalization of Pythagorean theorem: $\left|\vec{x}\right|^2 = x_1^2 + x_2^2 + \ldots + x_n^2$ or $\left|\vec{x}\right| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$.

Property: $\left|c\vec{x}\right| = |c|\left|\vec{x}\right|$.

Proof for $n = 2$: $\left|c\vec{x}\right| = |c(x_1,x_2)| = |(cx_1, cx_2)|$

$= \sqrt{c^2x_1^2 + c^2x_2^2}$

$= |c|\sqrt{x_1^2 + x_2^2} = |c|\left|\vec{x}\right|$. ■

Example when $c = -3$: $\left|-3\vec{a}\right| = |(-3,-6,-9)| = \sqrt{3^2 + 6^2 + 9^2} = 3\sqrt{14}$, and

$|-3||(1,2,3)| = 3\sqrt{1^2+2^2+3^2} = 3\sqrt{14}$.

Properties of a Vector Space:

- ♦ $\vec{u}+\vec{v} = \vec{v}+\vec{u}$, [additive commutivity]
- ♦ $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$, for any $\vec{w}$. [additive associativity]
- ♦ $\vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u}$ [additive identity, where $\vec{0} = (0,0,0,\dots)$]
- ♦ $1(\vec{u}) = \vec{u}$ [scalar multiplicative identity]
- ♦ $\vec{u}+(-\vec{u}) = (-\vec{u})+\vec{u} = \vec{0}$ [additive inverse, where $-(1,2,3) = (-1,-2,-3)$]
- ♦ $r(\vec{u}+\vec{v}) = r\vec{u}+r\vec{v}$, for any $r \in \mathbb{R}$ [scalar distributivity over vector addition]
- ♦ $(r+s)\vec{u} = r\vec{u}+s\vec{u}$, for any $r,s \in \mathbb{R}$ [vector distributivity over scalar addition]
- ♦ $r(s\vec{u}) = (rs)\vec{u}$ [scalar multiplicative associativity]

Proving $\mathbb{R}^n$ has the properties of a vector space:

Scalar Distributivity Over Vector Addition: $r(\vec{u}+\vec{v}) = r\vec{u}+r\vec{v}$, for any $r \in \mathbb{R}$

Proof: Given $\vec{u} = \left(u_1, \dots, u_n\right)$ and $\vec{v} = \left(v_1, \dots, v_n\right)$, then:

$$r(\vec{u}+\vec{v}) = r\left(\left(u_1, \dots, u_n\right)+\left(v_1, \dots, v_n\right)\right)$$

$$= r\left(u_1+v_1, \dots, u_n+v_n\right) = \left(r(u_1+v_1), \dots, r(u_n+v_n)\right)$$

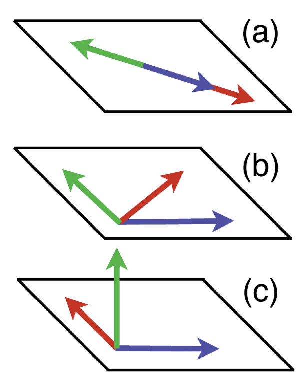
$$= \left(ru_1+rv_1, \dots, ru_n+rv_n\right)$$

$$= \left(ru_1, \dots, ru_n\right)+\left(rv_1, \dots, rv_n\right)$$

$$= r\left(u_1, \dots, u_n\right)+r\left(v_1, \dots, v_n\right) = r\vec{u}+r\vec{v}. \qquad \blacksquare$$

And similarly with the other properties.

Linearly Dependent/Independence



Linearly Dependent Vectors Theorem: Vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ are linearly **dependent** if and only if there exist $a_1, a_2, \dots, a_n \in \mathbb{R}$, such that $a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_n\vec{u}_n = \vec{0}$ and $a_1, a_2, \dots, a_n$ are **not all zero**.

Put another way: Vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ are linearly **dependent** if …
$\vec{u}_k = a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_3\vec{u}_n$, for some $1 \le k \le n$ and $a_1, a_2, \dots a_n \in \mathbb{R}$.

Important Consequence: If you have two linearly independent vectors $\vec{u}_1, \vec{u}_2$ in $\mathbb{R}^2$, then any other $\vec{u}_3$ in $\mathbb{R}^2$ is a linear combination of $\vec{u}_1, \vec{u}_2$. In other words, $\vec{u}_3 = a\vec{u}_1 + b\vec{u}_2$, for some $a, b \in \mathbb{R}$. A similar statement can be made in $\mathbb{R}^3$ for three linearly independent vectors.

Linearly Independent Vectors Theorem: Vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ are linearly **independent**

if and only if $$\left|\vec{u}_1\ \vec{u}_2\ \ldots\ \vec{u}_n\right| := \begin{vmatrix} u_{11} & u_{21} & \ldots & u_{31} \\ u_{12} & u_{22} & \ldots & u_{32} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1n} & u_{2n} & \ldots & u_{3n} \end{vmatrix} \neq 0.$$

Proof for $n = 3$: Definition of independence means $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ are linearly **independent** if and only if

$a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n = \vec{0}$ implies $a_1 = a_2 = \ldots = a_n = 0$.

Written another way: $$\mathbf{U}\vec{a} := [\vec{u}_1\quad \vec{u}_2\quad \ldots\quad \vec{u}_n]\vec{a} = \begin{bmatrix} u_{11} & u_{21} & \ldots & u_{31} \\ u_{12} & u_{22} & \ldots & u_{32} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1n} & u_{2n} & \ldots & u_{3n} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$ implies only the trivial solution.

But by Theorem 7 in section 3.5, the preceding is true if and only if $\mathbf{U}$ is invertible.

That is, if and only if $|\mathbf{U}| \neq 0$. ■

Bases

Basis for Vector Space V: A basis is a set of linearly **independent** vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in V such that

every vector $\vec{v}$ in V can be expressed as a linear combination:

$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n$, for some $a_1, a_2, \ldots a_n \in \mathbb{R}$.

Particularly for $\mathbb{R}^n$, you need:

- Linearly independent vectors,
- (# of vectors) $= n$.

For $\mathbb{R}^3$, a convenient (standard) basis is $\left\{\vec{e}_1, \vec{e}_2, \vec{e}_3\right\} = \left\{(1,0,0),\ (0,1,0),\ (0,0,1)\right\}$,

so that $(a,b,c) = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$, for any $a, b, c \in \mathbb{R}$.

However, $\left\{(1,2,3),\ (1,5,7),\ (3,0,13)\right\}$ is also a basis.

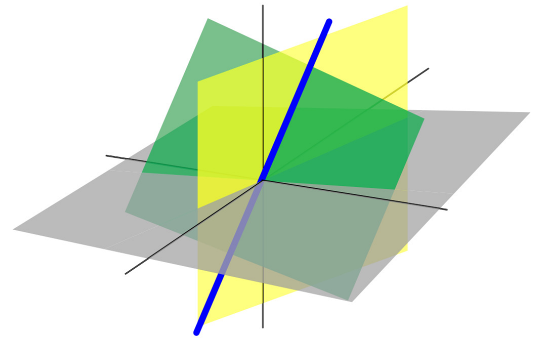
Subspaces

Subspace: A non-empty subset of vectors $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ of vector space V is said to be a subspace if, for all $\vec{v}_i, \vec{v}_j$ in $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ and $c \in \mathbb{R}$, we have:

$\vec{v}_i + \vec{v}_j$ is in V, [**closed** under addition]

and $c\vec{v}_i$ is in V. [**closed** under scalar multiplication]

Fact: Subspaces are vector spaces! Properties of vector space are "inherited."



Examples:

$\{\vec{0}\}$ in $\mathbb{R}^n$,

Any line through the origin in $\mathbb{R}^n$ with $n \geq 1$,

Any plane through the origin in $\mathbb{R}^n$ with $n \geq 2$,

Any m-hyperplane through the origin in $\mathbb{R}^n$ with $n \geq m$.

Video Tutorial (visually rich and intuitive): https://youtu.be/fNk_zzaMoSs

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Exercises

Problem: #24 Determine whether the given vectors $\vec{u} = (1,4,5)$, $\vec{v} = (4,2,5)$, $\vec{w} = (-3,3,-1)$ are linearly independent or dependent. If they are linearly dependent, find scalars $a, b,$ and c not all zero such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$.

So, $\mathbf{A}\vec{z} = 0$, where $\vec{z} = \begin{bmatrix} a & b & c \end{bmatrix}^T$, and $\mathbf{A} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} = \begin{bmatrix} 1 & 4 & -3 \\ 4 & 2 & 3 \\ 5 & 5 & -1 \end{bmatrix}$.

$$\mathbf{A} \Rightarrow \begin{bmatrix} 1 & 4 & -3 \\ 0 & -14 & 15 \\ 0 & -15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & 1 \\ 0 & -15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 29 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system $\mathbf{Ax} = \vec{\mathbf{0}}$ has only the trivial solution $a = b = c = 0$ ($\mathbf{A}$ is invertible), so the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent.

Problem: #32 Show that V, defined as the set of all (x, y, z) such that $z = 2x + 3y$, is closed under addition and under multiplication by scalars, and is therefore a subspace of $\mathbb{R}^3$.

If one were to choose $\vec{u}$ and $\vec{v}$ randomly from V and choose $c \in \mathbb{R}$, we would then need to show that $\vec{u} + \vec{v}$ and $c\vec{v}$ are members of V.

$\vec{u} = \left(u_1,\ u_2,\ 2u_1 + 3u_2\right)$ and $\vec{v} = \left(v_1,\ v_2,\ 2v_1 + 3v_2\right)$ for some $u_1, u_2, v_1, v_2 \in \mathbb{R}$.

Closed Under Addition?

$\vec{u} + \vec{v} = \left(u_1 + v_1,\ u_2 + v_2,\ (2u_1 + 3u_2) + (2v_1 + 3v_2)\right)$

$= \left(u_1 + v_1,\ u_2 + v_2,\ 2(u_1 + v_1) + 3(u_2 + v_2)\right)$ where $u_1 + v_1,\ u_2 + v_2 \in \mathbb{R}$.

$\vec{u} + \vec{v}$ is in the form (x, y, z) such that $z = 2x + 3y,$ so $\vec{u} + \vec{v} \in V.$ ✓

Closed Under Scalar Multiplication?

$c\vec{v} = \left(cv_1,\ cv_2,\ 2cv_1 + 3cv_2\right)$ where $cv_1,\ cv_2,\ 2cv_2 + 3cv_2 \in \mathbb{R}$.

$c\vec{v}$ is also in the form (x, y, z) such that $z = 2x + 3y,$ so $c\vec{v} \in V.$ ✓

Problem: **#33** Show that V, the set of all (x, y, z) such that $y = 1$, is not a subspace of $\mathbb{R}^3$.

Just need an example of vector(s) in V which are not closed under scalar multiplication or under addition.

$(0,1,0)$ is in $V,$ but the sum $(0,1,0) + (0,1,0) = (0,2,0)$ is not in V.

Thus V is not closed under addition, so not a subspace of $\mathbb{R}^3$.