These flashcards were generated to study for a Dynamical Systems oral exam.

Section 1: Examples of DEQs and Basic Concepts

Newton's early method for solving first order ODE w/init value	Power series as ansatz: $y = y_0 + y_1 t + \ldots$ subst in & compare sides. w/init value determine some y_j. Repeat.
Pendulum	**Nonlinear Oscillator**: $\ddot{\varphi} = -\frac{g}{\ell}\sin\varphi$, where ℓ is **length**, g is gravity
Planetary Motion	**Nonlinear Oscillator**: $\ddot{x} = -\nabla V(x)$ Point-mass positions $x = (x^1, x^2, \ldots, x^N)$ for N-bodies, where $x^j \in \mathbb{R}^3$, and potential is $V(x) = -\sum_{1 \le j < k \le N} \frac{m_j m_k}{\lvert x^j - x^k \rvert}$.
Friction for Nonlinear Oscillator	$\ddot{x} = -\nabla V(x) - \gamma\,\dot{x}$, for $\gamma > 0$.
Van der Pol Oscillator	**Nonlinear Nonconservative Oscillator**: $\ddot{x} = \gamma(1 - x^2)\,\dot{x} - x = 0$ γ is scalar parameter indicating strength of nonlinearity and damping. $(x = 0, \dot{x} = 0)$ is unstable. When x large, x^2 term dominates & damping > 0
Predator Prey Model	$x' = (A - By)x$, and $y' = (Cx - D)y$, where y is the predator, and x is the prey
$\exists$ of sol for Autonomous ODE $\dot{x} = f(x)$, $x(0) = x_0 \in R$	$f(x_0) = 0 \Rightarrow x(t) \equiv x_0$ is sol. If $f(x_0) \neq 0$ & $f \in C^0$, then by IFT (since $x \in C^1$) $\tau'(x) = \frac{1}{f(x)} \Rightarrow \tau(x(t)) = \int_{x_0}^{x(t)} \frac{1}{f(y)} dy =: T(x(t); x_0)$. RHS is monotone in $x(t)$ away from $f = 0$ (Need for IFT), so IFT$\Rightarrow x(t; x_0) = T^{-1}(t; x_0)$
! of sol for Autonomous ODE $\dot{x} = f(x)$, $x(0) =: x_0 \in R$	f is Lipshitz. (i) $\forall x_0, \exists \delta > 0$ & $\exists!\, x(t)$ for $\lvert t \rvert < \delta$. (ii) Any 2 sols coincide on common domain of def. (iii) !sols are $x(t) \equiv x_0$ when $f(x_0) = 0$ & o/w implicitly thru $\tau(x(t)) = \int_{x_0}^{x(t)} \frac{1}{f(y)} dy$

Nonuniqueness ODE example $\dot{x} = f(x)$, $x(0) =: x_0 \in R$	$x' = \|x\|^\beta$, which is not differentiable for $\beta < 1$ (unless $x \equiv 0$). Nontrivial sol: $\frac{1}{\|x\|^\beta} dx = t + c$ for $x \neq 0 \Rightarrow \frac{x^{1-\beta}}{1-\beta} = t + c$. Pcwise sol for $x_0 = 0 : x(t) = (1-\beta)^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}},\ t \geq 0,\ \&\ x(t) = 0$ for $t \leq 0$
First Integral	For $x' = f(x)$, FI is $I : \mathbb{R}^n \to \mathbb{R}$ w\ $I(x(t)) \equiv$ const; independent of time for all sols. Equivalently, $0 = \frac{d}{dt} I(x(t)) = \left\langle \nabla I(x(t)), f(x(t)) \right\rangle_{\mathbb{R}^n}$. That is, $\nabla I(x) \perp f(x)$ for all $x \in \mathbb{R}^n$.
Hamiltonian Systems H	$H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and symplectic matrix $J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, with Hamiltonian equation $u' = J\nabla H(u)$ & $H(p;q) = \frac{1}{2m}\|p\|^2 + V(q)$
Hamiltonian Systems Examples	Pendulum, $n = 1$ and $H(\varphi, v) = \frac{1}{2}v^2 - \frac{g}{\ell}\cos\varphi$ Gravitational, $H(x;v) = \sum_j \frac{1}{2} m_j \|v^j\|^2 + V(x)$ $H = Kin. + Pot.$
Proof Hamiltonian is First Integral If f is vector field and H the Hamiltonian	$(\nabla H, f) = (\nabla H, J\nabla H) = (J^T\nabla H, \nabla H) = (-J\nabla H, \nabla H) = (\nabla H, -J\nabla H)$ So, $J\nabla H = -J\nabla H$ or $J\nabla H = 0$, and $(\nabla H, f) = 0$ "Lie derivative of H along flow of f is zero."
Existence of Hamiltonian	Let $f \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ be a vector field with $div(f) = 0$. Then there exists $H \in C^2$ such that: $f = J\nabla H$. Note that $\text{div}(f) = \text{div}\left(\dot{q}, \dot{p}\right) = \text{div}(\partial_p H, -\partial_q H) = \partial_q \partial_p H - \partial_p \partial_q H = 0$.

Section 2: Flows and Vector Fields

Flow	$\Phi : \mathbb{R} \times X \to X,\ (t,u) \to \Phi(t,u) =: \Phi_t(u)$ is a (well defined) flow if it's differentiable satisfies **cocycle property**: $\Phi_0 = id$, $\Phi_t \circ \Phi_s = \Phi_{t+s},\ \forall t,s \in \mathbb{R}$. In particular, Φ_t is invertible with inverse Φ_{-t}
Local flow	Φ is defined in a neighborhood of $\{0\} \times \{x_0\}$ when the cocycle property holds: i.e., when $\Phi_t \circ \Phi_s$ and Φ_{t+s} are defined.
Flow solution to ODE	Let f be vector field for flow Φ_t. Then $x(t) := \Phi_t(x_0)$ solves $\dot{x}(t) = f(x)$. **Proof**: $x'(t_0) = \frac{d}{dt}\vert_{t=t_0}\Phi_t(x_0) = \frac{d}{dt}\vert_{t=t_0}\Phi_{t-t_0}(\Phi_{t_0}(x_0))$ $= \frac{d}{d\tau}\vert_{\tau=0}\Phi_\tau(\Phi_{t_0}(x_0)) = f(\Phi_{t_0}(x_0)) = f(x(t_0))$.
Metric Space	A set X equipped with a metric $d : X \times X \to \mathbb{R}_+$, where $d(x,y) = d(y,x),\quad d(x,z) \le d(x,y) + d(y,z),\quad d(x,y) = 0$ iff $x = y$. The metric defines a topology & convergence, $x_n \to y$ iff $d(x_n,y) \to 0$
Complete Metric Space X	Cauchy sequences converge in the set, that is: $\lim_{N\to\infty} [\sup_{m,n\ge N} d(x_n,x_m)] = 0 \Rightarrow \exists y \in X$ such that $x_n \to y$.
Normed Vector Space	Vector space X with a norm operator $\vert\cdot\vert : X \to \mathbb{R}_+$, where $\vert\lambda x\vert = \vert\lambda\vert\vert x\vert$, $\vert x+y\vert \le \vert x\vert + \vert y\vert,\quad \vert x\vert = 0$ iff $x = 0$. A normed space is a metric space with distance $d(x,y) = \vert x-y\vert$.
Any closed subset of a Banach space	is a complete metric space
Lipshitz continuous	A map $F : X \to Y$ between metric spaces with Lipshitz constant L such that $d_Y(F(x_1),F(x_2)) \le L d_X(x_1,x_2)$, for all $x_1,x_2 \in X$.
Locally Lipshitz Map	If for every $x \in X$ there exists a neighborhood $U(x)$ such that the restriction of F to $U(x)$ is Lipshitz continuous, with a Lipshitz constant $L(x)$.

Contraction	A Lipshitz map $F : X \to X$ on a complete metric space with Lipshitz constant $L < 1$.

Banach's FP Theorem	A contraction possesses a unique fixed point.

Frechet differentiable	Given $U \subset X, V \subset Y$, open subsets. $F : U \to V$. F is **Frechet differentiable** in x if $\exists L(x) \epsilon \mathcal{L}(X,Y)$ (bounded) s.t. $\forall u \neq x \in U$, $F(u) - F(x) - L(x) \cdot (u - x) = o(x - u)$, where $\frac{o(\delta)}{\lvert\delta\rvert} \to 0$ for $\delta \neq 0$. $L(x)$ is the derivative @x & write: $DF(x) := L(x)$

Gateaux differentiable in x	$F : X \to Y$ where for any $x_0 \in X$ the function $f_x : \mathbb{R} \to Y$ with $f_x(t;x_0) = F(x + tx_0)$ is Frechet differentiable in t at $t = 0$, $\forall x$. Note that Frechet $\Rightarrow$ Gateaux. "Directional deriv. in direction of x_0"

Gateaux implies Frechet	Suppose F is Gateaux differentiable and $f_x(t;x_0) = F(x + tx_0) = L_t(x)x_0$ for some bounded operator $L_t(x)$ that depends continuously on x. Then F is Frechet differentiable.

Implicit Function Thm	Suppose $F \in C^k(X \times \Lambda, Y)$, $k > 1$, $F(x_0;\lambda_0) = 0$, & $D_xF(x_0;\lambda_0)$ has bounded inverse. $\exists$Nbhds U of λ_0 & V of x_0, s.t. $\exists!$ sol. to $F(x;\lambda) = 0$, $\forall \lambda \in U$ w/$x \in V$, given thru $x = \varphi(\lambda)$. Moreover, $\varphi \in C^k$, $\varphi(\lambda_0) = x_0$, and $D_\lambda \varphi = -\partial_x F^{-1} \partial_\lambda F$, at $(x,\lambda) = (x_0,\lambda_0)$

Banach's FP Thm with parameters	Suppose $F \in C^k(U_X \times U_\Lambda, U_X)$, on open subsets of Banach spcs X & Λ $k > 1$, F is contraction in 1st argument uniformly wrt 2nd: $\lvert F(x_1,\lambda) - F(x_2,\lambda)\rvert \leq \kappa \lvert x_1 - x_2 \rvert$, for some $\kappa < 1$, and all x_1, x_2, λ. Then $\exists!$ f.p. $x_*(\lambda) \in C^k$.

Picard-Lindelof	Let $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ a locally Lipshitz vector field on U. $\forall x_0 \in U$, $\exists t_b(x_0) > 0$ & sol. $x : [-t_b; t_b] \to U$ of $x' = f(x)$; $x(0) = x_0$. Also, given sol $\tilde{x}(t)$, where $t \in J$ interval, and $\tilde{x}(0) = x_0$, then $\tilde{x}(t) = x(t)$, $\forall t \in J \cap [-t_b; t_b]$

Smooth Dependence on Parameters sols	$f \in C^k(U \times P, \mathbb{R}^n)$, $U \times P \subset \mathbb{R}^n \times \mathbb{R}^p$, for some $1 \leq k \leq \infty, \omega$. Then $\exists!$ sol $x(t; t_0, x_0, \mu)$ and its derivative wrt t are C^k in all vars. "Smooth f gives smooth x & x'."

Peano's Existence	Suppose f continuous @x_0. Then $\exists\delta > 0$ and sol of IVP: $x' = f(x)$, $x(0) = x_0$, on $(-\delta, \delta)$.
Flow Existence? $(t, x_0) \to$ $x(t; x_0) =: \Phi_t(x_0)$	$f \in C^1$ (or Lipshitz) wrt vars &params. Moreover, it has inverse in 2nd argument Φ_{-t}, by uniqueness of sols, which is also differentiable. In particular, the derivative $\partial_{x_0}\Phi_t(x_0)$ invertible & $\Phi_t(\cdot)$ is local diffeo. If global in t, DEQ generates flow in nghbd of $\{0\} \times U$
Sol Blowup Criteria	Let f locally Lipshitz, defined on $U \subset \mathbb{R}^n$. Then maximal time interval of existence $T_{\max}$ is nonempty and open. If $T_{\max} \neq \mathbb{R}$ and $t_* \in \partial T_{\max}$ with $t \to t_*$ in $T_{\max}$ we have that either $x(t) \to \partial U$ or $\lvert x(t) \rvert \to \infty$.
Explicit Euler	Let Φ_t be the flow to an ODE $x' = f(x)$. Since $x(h) = x_0 + \int_0^h f(x(s))ds$ $x(h) \approx \varphi_h(x_0) := x_0 + hf(x_0)$, a 1st order numerical approximation (h^1) to the solution at time h.
Implicit Euler	Let Φ_t be the flow to an ODE $x' = f(x)$. Since $\Phi_h(x_0) = x(h) = x_0 + \int_0^h f(x(s))ds$, then $x(h) \approx x_0 + hf(\varphi_h(x_0))$, a 1st order numerical approximation (h^1) to the solution at time h.

Section 3: Numerical Methods

Order of Numerical Method	We say that a numerical method is of order p if **local error** is of order $p+1$, $\Phi_h(x_0) - \varphi_h(x_0) = O(h^{p+1})$, (as h increases, the error grows as h^{p+1}).

Global Error	Accumulated error over all steps. At step n, for $h = \frac{t}{n}$, global error at a fixed time t is of order p. $E_n = \lvert\varphi_h^n(x_0) - \Phi_{nh}(x_0)\rvert \le nO(h^{p+1}) = O(h^p)$ If f has global Lipshitz constant L, then error estimate is $\left\lvert\varphi_h^{\frac{t}{h}}(x_0) - \Phi_t(x_0)\right\rvert \le Ce^{Lt}h^p$

Backward Error Analysis	Discover that our numer. approx. exactly solves slightly different DEQ $x'=\widetilde{f}(x)$, $x(0) = x_0$; for suitable $\widetilde{f}$. Usually true for smaller errors, $O(e^{-\frac{c}{h}})$ w/analytic vect flds f. OR, exactly solves w/slightly different *initial condition*, $x' = f(x)$, $x(0) = \widetilde{x}_0(h)$.

Runge-Kutta	Assuming autonomous f. Estimate $x(h) = x(0) + \int_0^h f(x(s))ds$, using $x_k = x_0 + h\sum_{i<k} b_j f_i$, where $f_i = f(x_i) = f(x_0 + h(\Sigma_{j<i} a_{ij} f_j)) = \ldots$ where a_{ij}, b_i are taken from a butcher tableau

Backward Differentiation Methd Good for stiff problems	Form kth deg. interpolating poly $y(t)$ w/$y(t_n), y(t_{n-1}), \cdots, y(t_{n-k})$. Differentiate & evaluate it @t_n. Example:interpolate thru $y(t_n)$ & $y(t_{n-1})$ is: $y(t) \approx y(t_n) + (t - t_n)\frac{y_n - y_{n-1}}{t_n - t_{n-1}}$ Approximating $y' = f(y,t)$ gives Backward Euler $y_n = y_{n-1} + (t_n - t_{n-1})f(y_n, t_n)$

Section 4: Qualitative Dynamics

Change of Coordinate	If ψ a smooth diffeo & Φ_t a flow to $x' = f(x)$. Then $\widetilde{\Phi}_t := \psi^{-1} \circ \Phi_t \circ \psi$ is a flow, w/ associated ODE for the variable $y = \psi^{-1}(x)$, as $y' = (D\psi(y))^{-1} f(\psi(y))$
Hartman–Grobman Hyperbolic Linearization Thm	If $\dot{x} = f(x)$ for $f \in C^1$. W/hyperbolic state x_0. i.e. $A = [\frac{\partial f_i}{\partial x_j}]\|_{x_0}$ has no eval w/real part = 0. Then $\exists N$ of x_0 & homeo $h \in C^1$ in N s.t. $h(x_0) = 0$ & s.t. $\forall x \in N, \exists t_0 > 0$ s.t. $\forall t \in (-t_0, t_0)$ we've $h \circ \varphi_t(x) = e^{At} h(x)$, i.e. flow is topo conj to flow of linearizatn
Blowing-Up on $\mathbb{R}^2$ Non-hyperbolic f.p.	Change of coords "blows-up" non-hyperb f.p. into curve on which singularities occur 1. Perform the blowup with polar coordinates. 2. Perform stereographic projection. 3. Use algebra to analyze.
Time Rescaling	Let $\alpha : \mathbb{R}^n \to \mathbb{R}^+$ be smooth, strictly positive. Then $\{x(t), t \in \mathbb{R}\}$ to $x' = f(x)$ are trajectories to $y' = \alpha(y) f(y)$. So, $\exists \gamma : T \to T'$, a diffeo of max existence intervals for $x' = f(x)$ & $y' = \alpha(y) f(y)$ w/InitConds $x(0) = y(0) = x_0$, resp, s.t. $y(\gamma(t)) = x(t)$
Scaling Symmetry For scale invariant equations: $u' = f(u),\ f(\lambda u) = \lambda^p f(u)\ \forall \lambda > 0$	Change to polar, $u = Rv,\ R > 0,\ v \in S^{n-1} \subset \mathbb{R}^n,\ \|v\| = 1$. Since v lives in sphere, we conclude $\langle v', v \rangle = 0$, so we solve for v'. w/substitution we eliminate v' and reduce dimensions of DEQ*s*
Flow-Box	If $f \in C^k$ with $k \geq 1$, $f(x_0) \neq 0$. Then, $\exists$local diffeo $\psi \in C^k$, $\psi : N_{x_0} \to N_0$ s.t. $y := \psi(x)$ satisfies $y' = f(x_0) = const,\ \forall y \in N_0$.
Invariant S	If $\Phi_t(S) \subset S$ for all $t \in \mathbb{R}$, then S is invariant. Invariant sets are unions of trajectories. Level sets of first integrals are invariant.
ω-limit set of x_0	The set of accumulation points as time tends to infinity $\omega(x_0) = \left\{ y \in X \mid \exists t_k \to \infty,\ \Phi_{t_k}(x_0) \to y \right\}$
Examples of ω-limit sets	Equilibrium $\Rightarrow$ $\omega(x_0) = \alpha(x_0) = \{x_0\}$. Homoclinic if $\omega(x_0) = \alpha(x_0) = \{x_*\},\ x_* \neq x_0$. Heteroclinic if $\omega(x_0) = \{x_+\} \neq \alpha(x_0) = \{x_-\}$ Periodic orbit $\Rightarrow$ $\omega(x_0) = \alpha(x_0) = \gamma(x_0) := \{\Phi_t(x_0) \mid t \in \mathbb{R}\}$

Forward orbit of x_0 is bounded. **Then, $\omega(x_0)$ is:**	i) Nonempty, ii) Compact iii) Connected, iv) Invariant, v) $\omega(x_0) = \cap_{T\geq 0} \overline{\cup_{t\geq T} \Phi_t(x_0)}$ and $\lim_{t\to\infty} dist(\Phi_t(x_0), \omega(x_0)) \to 0$, where $dist(y,A) = \inf_{z\in A}\lvert z-y\rvert$. We similarly define $\omega(U)$ for sets $U \subset X$, $\omega(U) = \cap_{T\geq 0} \overline{\cup_{t\geq T} \Phi_t(U)}$.

Counter Example to $\omega(U) = \cup_{x_0\in U}\, \omega(x_0)$	Flow on $[0,1]$ to $x' = x - x^2$, where of course, $\omega([0,1]) = [0,1]$ but $\omega(x) = 1$ for all $x > 0$ and $\omega(0) = 0$ such that $\cup_x \omega(x) = \{0,1\}$

Non-Wandering Set Ω	$x \in \Omega \Leftrightarrow$ for all $U(x)$ there exists $t_k \to \infty$ such that $\Phi_{t_k}(U(x)) \cap U(x) \neq \emptyset$

Chain Recurrent Set	$x \in CR \Leftrightarrow \forall \varepsilon, T > 0$, $\exists\varepsilon$-pseudo-orbit w/$x_n = x$ (endpoint = start point), where ε-pseudo-orbits are piecewise orbits with at most ε-jumps that is, there exists $T_j > T$, x_j, $0 \leq j \leq n-1$, $\lvert\Phi_{T_j}(x_j) - x_{j+1}\rvert < \varepsilon$.

Stability of invariant set $\emptyset \neq M \subseteq X$	Stable if $\forall\varepsilon > 0$, $\exists\delta > 0$ s.t. $\Phi_t(U_\delta(M)) \subset U_\varepsilon(M)$, $\forall t \geq 0$. Set is asymptotically stable if it's stable & if $\forall x_0$ in some nghbd $V(M)$, $\lim_{t\to\infty} dist(\Phi_t(x_0), M) = 0$.

Lyapunov function V	Lyap if V cont & $V(\Phi_t(x))$ non-increasing in t, $\forall x$. If $V \in C^1$, then cond. becomes $\frac{d}{dt}V(\Phi_t(x)) \leq 0$, or equivalently $(\nabla V, f) \leq 0$. Sublevel sets $V_c = \{x \mid V(x) \leq c\}$ are forward invar. If $V(x(t_*)) \leq c$ for t_*, then $\forall t > t_*$, still have $V(x(t)) \leq c$.

LaSalle's Invariance Principle	i) Suppose V Lyapunov, $\omega(x_0) \neq \emptyset$. Then V is *constant* on $\omega(x_0)$, AKA $V(\omega(x_0)) = \{V_0\}$. ii) Suppose V strict Lyapunov. Then $y' = 0$, $\forall y \in \omega(x_0)$, that is, ω-limit set consists of equilibria

Corollary to Morse Lemma	Consider $x' = f(x)$ with $f(0) = 0$, and assume that $V(x)$ is a Lyapunov function near 0, with $V(0) = 0$, $\nabla V(0) = 0$, and $D^2V(0) > 0$. Then 0 is stable.

Lyapunov Function Example	If $x' = Ax$ & neg defint $A = A^T$ (So $\sigma(A) < 0$) we've Lyap $V(x) = -\frac{1}{2}\langle Ax, x\rangle$. $\frac{d}{dt}V(x(t)) = -\frac{1}{2}(\langle A(Ax), x\rangle + \langle Ax, Ax\rangle)$ $= -\lvert Ax\rvert^2 < 0$. Also: $D^2V(x) = -\frac{1}{2}D^2(x^TAx) = -\frac{1}{2}D\left[(Ax + x^TA)\cdot\vec{1}\right] = -\Sigma A_{ij} > 0$ (neg def), $\nabla V(x)\rvert_0 = -\frac{1}{2}(x^T(\nabla Ax) + (\nabla x^T)Ax)\rvert_0 = -(x^TA + Ax)\rvert_0 = 0$. And $f(0) = A(0) = 0$. So stable by Morse.

Section 5: Linear Equations

Matrix Exponential for $\mathbf{x}' = \mathbf{Ax} \in \mathbb{R}^n$, $\mathbf{x(0)} = \mathbf{x}_0$, $\mathbf{A} \in \mathbb{R}^{n \times n}$	Absolutely convergent $e^{At} := \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$ Differentiate termwise and find $\frac{d}{dt}e^{At} = Ae^{At}$. As a consequence, $e^{At}x_0$ is the unique solution to $x' = Ax$
Coordinate Changes for $\mathbf{x}' = \mathbf{Ax} \in \mathbb{R}^n$ $\mathbf{x(0)} = \mathbf{x}_0$, $\mathbf{A} \in \mathbb{R}^{n \times n}$	$S \in GL(X),\ x = Sy,$ which gives: $y' = S^{-1}ASy$ with solutions $y(t) = e^{S^{-1}ASt}y_0$. Note that $e^{S^{-1}ASt} = S^{-1}e^{At}S$, by change of coords rule (or by inspecting cancellations in infinite sum)
Superposition Decomp for $\mathbf{x}' = \mathbf{Ax} \in \mathbb{R}^n$ $\mathbf{x(0)} = \mathbf{x}_0$, $\mathbf{A} \in \mathbb{R}^{n \times n}$	Linear combinations of solutions, $\Sigma_j \alpha_j x_j(t)$ are solutions for any $\alpha_j \in \mathbb{R}$ if $x_j(t)$ are solutions. Therefore, if $X = V \oplus W$, (invariant subspaces under A) it is sufficient to solve DEQ in V and W, separately.
Invariant Subspace Corollary	Suppose $V \subset X$ is a subspace invariant under A, $AV \subset V$. Then V is invariant under the flow $\Phi_t = e^{At}$, or $\Phi_t(x_0) = x_0 e^{At}$, where $x_0 \in V$
Invariant Subspace under Change of Coord in $\mathbb{C}^n$	If you change coordinates $y = S^{-1}x \in \mathbb{C}^n$, then the conjugate matrix $S^{-1}AS$, where $A \in \mathbb{R}^{n \times n}$ possesses (real linear) n-dimensional invariant subspaces $S^{-1}V_{r/i}$. Since $V_{r/i}$ is invariant under A.
If **Spectrum of** $A \in C^{n \times n}$ **is** $\lambda_{1,\ldots,n} = 0$, **then new form**?	There exists a basis w/coordinate change S of $\mathbb{C}^n$ such that in the new basis: $A_{ii} = 0,\ A_{i,i+1} \in \{0,1\}$. For example: $S^{-1}AS = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, for nonsemisimple
Jordan Normal Form	$\forall A \in \mathbb{C}^n$, $\exists$change of coordinates S s.t. $S^{-1}AS$ is block diagonal and each block is of the form $\lambda_j id_{k_j} + N_{k_j}$, where $k_j \geq 1$ is the size of the block and λ_j is an eigenvalue.
JNF Solution Decomp for $x' = Ax$	Since e^{At} leaves subspaces invariant that are A invariant, It has same block diagonal structure as JNF of A. So, it's sufficient to compute exponential of a single block, $\lambda id_k + N_k$. Since identity and N_k commute: $e^{(\lambda id_k + N_k)t} = e^{\lambda t}e^{N_k t} = e^{\lambda t}\sum_{j=0}^{k-1}\frac{t^j}{j!}N_k^j$, since $N^k = 0$.
Jordan Ansatz to $x' = Ax$	$x(t) = \sum_{\lambda \in \sigma(A)} p_\lambda(t)e^{\lambda t}$, where $p_\lambda(t)$ is a vector valued polynomial in t with degree at most the maximal algebraic multiplicity of λ

Discontinuous JNF	Consider $\begin{pmatrix} \varepsilon & 1 \\ 0 & 0 \end{pmatrix}$. At $\varepsilon = 0$, it is JNF, w/transform: $S = id$. For $\varepsilon > 0$, $S = \begin{pmatrix} 1 & 1 \\ -\varepsilon & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \end{pmatrix}\begin{pmatrix} 0 & -\frac{1}{\varepsilon} \\ 1 & \frac{1}{\varepsilon} \end{pmatrix} = SDS^{-1}$ has a pole at $\varepsilon = 0$. Eigenvectors collide.
$\det \mathbf{e}^{At} = ??$	Suffices to triangularize: $A = P^{-1}TP$, w/P invertible & T upper-triangular. $\det e^{Tt} = e^{\lambda_1 t} \ldots e^{\lambda_n t} = e^{(\lambda_1 + \ldots + \lambda_n)t} = e^{tr(T)t}$. Observe $tr(A) = tr(T)$, So: $\det \mathbf{e}^{At} = \vert P^{-1}\mathbf{e}^{Tt}P\vert = \vert P^{-1}\vert \vert \mathbf{e}^{Tt}\vert \vert P\vert = \frac{\vert P\vert}{\vert P\vert}\prod_\lambda e^{\lambda t} = e^{\Sigma \lambda t} = e^{(trA)t}$.
How to Simplify Higher-Order Equations $x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n = 0$	Write as a first order system with characteristic polynomial: $\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$, and eigenvectors $(1, \lambda, \lambda^2, \ldots, \lambda^{n-1})^T$, that is, the geometric multiplicity is always 1.
Spectral Projections	Linear maps w/$P^2 = P \Rightarrow P$ is identity on its range. Range & ker of P span $\mathbb{R}^n$ (or $\mathbb{C}^n$). Interesting are projections that commute w/A, so $PA = AP$. Since such projections leave $P(range(A))$ and $P(\ker(A))$ invariant under A.
Matrix Differential Equations $X' = AX$, where $A \in \mathbb{R}^{n \times n}$ and $X \in \mathbb{R}^{n \times m}$	If $m = n$, & $X(0) = id$, then $X(t) = e^{At}$ is called fund. matrix. sol If $X(t_0)$ invertible, then fund. matrix sol can be calculated as $\Phi(t) = X(t)X^{-1}(t_0)$.
Eigenvalues are completely determined by	Trace and determinant e.g. planar: $\lambda_{1/2} = \frac{tr}{2} \pm \sqrt{\frac{tr^2}{4} - \det}$.
Planar Center Equilibrium	$tr = 0$, $\det > 0$. $\lambda_{1/2} = \pm i\omega$. Solutions are ellipses or (in JNF) circles in the phase plane.
Planar Unstable Focus Equilibrium	$tr > 0$, $\det > 0$, $\det < \frac{tr^2}{4}$. $\lambda_1, \overline{\lambda}_2 = \eta \pm i\omega$, w/$\omega, \eta > 0$. Sols in the complex JNF $z = e^{i\omega t}e^{\eta t}$, hence logarithmic spirals $\vert z\vert \sim e^{arg(z)}$.
Planar Resonant Node Equilibrium	$tr > 0$, $\det > 0$, $\frac{tr^2}{4} = \det$. $\lambda_1 = \lambda_2 > 0$. If A semi-simple, sols are simply $x(t) = x_0 e^{\lambda t}$ radially exponentially outward. A is non-semi simple, in JNF $x_2(t) = e^{\lambda t}x_2^0$, $x_1 = te^{\lambda t}x_2^0 + e^{\lambda t}x_1^0$. Twisted Star

Planar **Unstable Node Equilibrium**	$tr > 0$, $\det > 0$, $\det < \frac{tr^2}{4}$. $\lambda_1 > \lambda_2 > 0$ Solutions in JNF are $x_1(t) = x_1^0 e^{\lambda_1 t}$, $x_2(t) = x_2^0 e^{\lambda_2 t}$. Hence typical solutions are $x_1(t) \sim x_2(t)^{\lambda_1/\lambda_2}$ or parabolae

Planar **Unstable/Zero Equilibrium**	$tr > 0$, $\det = 0$. $\lambda_2 > \lambda_1 = 0$ Solutions in JNF are $x_2(t) = x_2^0 e^{\lambda_2 t}$, $x_1(t) = x_1^0$. Trajectories are parallel to x_2-axis, converging to the x_1-axis in backward time

Planar **Saddle Equilibrium**	tr > 0, $\det < 0$. $\lambda_1 > 0 > \lambda_2$. Solutions in JNF are $x_1(t) = x_1^0 e^{\lambda_1 t}$, $x_2(t) = x_2^0 e^{\lambda_2 t}$. Hence typical solutions are hyperbolae: $x_1(t) \sim x_2(t)^{\lambda_1/\lambda_2}$

Zero Eigenvalues Equilibrium	$tr = 0, \det = 0$. If A semi-simple, $x' = 0$ & flow trivial, all points are EQ. If A non-semisimple, we find in JNF: $x_2(t) = x_2^0$, $x_1(t) = tx_2^0 + x_1^0$, a shear flow parallel to x_1-axis w/equilibria on x_1-axis.

Adjoint Equation: $x' = Ax$, **and** $\psi' = -A^*\psi$. **For any solutions** $x(t), \psi(t)$ ... ?	$\frac{d}{dt}\langle(x(t), \psi(t))\rangle = 0$ **Proof**: $\frac{d}{dt}\langle x(t), \psi(t)\rangle = \left\langle \frac{d}{dt}x(t), \psi(t)\right\rangle + \left\langle x(t), \frac{d}{dt}\psi(t)\right\rangle$ $= \langle Ax(t), \psi(t)\rangle + \langle x(t), -A^*\psi(t)\rangle = \langle x, A^*\psi\rangle + \langle x, -A^*\psi\rangle = 0$

Let $x' = Ax$, **and** $\psi' = -A^*\psi$. **Let** E_λ **be generalized eigenspace to** $\lambda \in \mathbb{R}$ **of** A **Similarly, define** E_λ^*. **Then,** $E_\lambda =$	$E_\lambda = (E_\lambda^{c,*})^\perp$, where E_λ^c the sum of all other generalized eigenspaces. Also: $E_\lambda^c = (E_\lambda^*)^\perp$.

Duhamel formula for **non-autonomous inhomogeneous** $\mathbf{x}' = \mathbf{Ax} + \mathbf{g(t)}$, $\mathbf{x(0)} = \mathbf{x}_0$	Variation of Constant Formula for Non-Autonomous Equations $x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}g(s)ds$. **Proof**: Differentiate!

Solve nonautonomous homog. $x' = A(t)X$, $x(t_0) = x_0$, **for** $A(t)$ **cont**.	Write $\Phi_{t,t_0}x_0$ for the solution, where $\Phi_{t_0,t_0} = id$, and Φ_{t,t_0} is linear. $\Phi_{t,0}$ is fund. sol. since we generate any solution $\Phi_{t,s}$ as : $\Phi_{t,s} = \Phi_{t,0} \cdot (\Phi_{s,0})^{-1}$ Duhamel: $x' = A(t)x + g(t)$, $x(t_0) = x_0$, $\Rightarrow$ $x(t) = \Phi_{t,t_0}x_0 + \int_{t_0}^t \Phi_{t,s}g(s)ds$.

Floquet theory for **non-autonom homog periodic** $\dot{x} = A(t)x$, w/$A(t)$ T-periodic	If $\varphi(t)$ is fundamental matrix sol, then $\forall t$, $\varphi(T+t) = \varphi(t)\varphi^{-1}(0)\varphi(T)$. $\exists B$ s.t. $e^{TB} = \varphi^{-1}(0)\varphi(T)$, $\exists T$-period invertb. $Q(t)$ s.t. $\varphi(t) = Q(t)e^{tB}$, $\forall t$. Also, $\exists R \in \mathbb{R}^{n\times n}$ and $\exists 2T$-periodic $Q(t)$ s.t. $\varphi(t) = Q(t)e^{tR}$, $\forall t$.

Consequence of Floquet thm. $\dot{x} = A(t)x$, **w/**$A(t)$ **T-periodic**	Sol $\varphi(t) = Q(t)e^{tR}$ gives rise to t-dependent change of coordinates ($y = Q^{-1}(t)x$). Original system becomes autonomous linear sys w/real constant coeffs $\dot{y} = Ry$. Stability of the zero solution for $y(t)$ and $x(t)$ is determined by the eigenvalues of R.

Characteristic/ Floquet multipliers $\dot{x} = A(t)x$, **w/**$A(t)$ **T-periodic**	Recall: $e^{TB} = \varphi^{-1}(0)\varphi(T)$ for fundamental matrix sol $\varphi(t)$. Eigenvalues $e^{\mu T} =: \lambda_i$ of e^{TB} (where μ are Floquet Exponents) are the Characteristic/Floquet Multipliers. They are also the eigenvalues of the (linear) Poincaré maps $x(t) \to x(t+T)$.

Floquet Exponents $\dot{x} = A(t)x$, **w/**$A(t)$ **T-periodic**	Recall: $e^{TB} = \varphi^{-1}(0)\varphi(T)$ for fundamental matrix sol $\varphi(t)$. FE are the eigenvalues of B. Floquet exponents are not unique, since $e^{\left(\mu + \frac{2\pi i k}{T}\right)T} = e^{\mu T} =: \lambda_i$, where $k \in \mathbb{Z}$

Section 8: Center Manifolds, Normal Forms, and Bifurcations

Basic Bistability Model and Coupled Model	$u' = u(1-u)\left(u - \frac{1}{2}\right) + a$ $u_1' = d(u_2 - u_1) + u_1(1-u_1)\left(u_1 - \frac{1}{2}\right) + a$ $u_2' = d(u_1 - u_2) + u_2(1-u_2)\left(u_2 - \frac{1}{2}\right) + a$
Weakly Coupled Bistability Model Conclusions. $d << 1$. $u_i' = d(u_i - u_j) + \ldots + a$	9 f.p. in the bistable regime. Linearizing the equation with $u_j' = 0$ at these f.p., we find a diagonal matrix w/nonzero entries. So, by IFT $\exists 9$ unique solutions for $\lvert d \rvert << 1$, around these f.p.
Strongly Coupled Bistability Model Conclusions. $d >> 0$. $u_i' = d(u_i - u_j) + \ldots + a$	All equilibria have $u_1 = u_2$.
Implicit Function Theorem IFT	Suppose $F \in C^k,\ k = 1, \ldots, \infty, \omega$, $F(u_*; \mu_*) = 0$, & $\partial_u F(u_*; \mu_*)$ invertible Then $\exists$locally unique $\varphi(\mu)$ s.t. $F(\varphi(\mu); \mu) = 0$.
Full Rank Theorem (which means the image has full dimensions) $F(u_*) = 0$	Suppose $F \in C^k(\mathbb{R}^m, \mathbb{R}^n)$, $F(u_*) = 0$, and $DF(u_*)$ is onto $(\text{Img}(DF(u_*)\vec{v}) = \mathbb{R}^n$ for $\vec{v} \in \mathbb{R}^m$, in particular $m > n$). Then the set of zeros of F near u_* is a C^k-manifold of dim: $m - n$
Sards Theorem	Suppose $F \in C^k(\mathbb{R}^m, \mathbb{R}^n)$, with $k > 1$ if $m \le n$ & $k > m - n + 1$, otherwise. Let C be critical points, i.e., $DF(u)$ doesn't have max rank for $u \in C$. Then $F(C)$, the set of critical values, has Lebesgue measure zero.
Smooth Bifurcation Curves Corollary **Any smooth one-parameter family of vector fields $F(u;\mu)$ can be …**	approximated by vector fields $F(u;\mu) + \varepsilon_k$, $\varepsilon_k \to 0$, for $\varepsilon_k \in \mathbb{R}^n$ such that the equilibria $F(u;\mu) + \varepsilon_k = 0$ form smooth curves in $\mathbb{R}^n \times \mathbb{R}$ for each ε_k.
Arclength Continuation Algorithm	Find sol. $v_* = (u_*, \mu_*)$; init ds arclength step size. Start loop. Find kernel e of $\partial_v F(v_*)$, w/$\lvert e \rvert = 1$. Step $v_* := v_* + ds \cdot e$. Solve (8.3) $F(v) = 0$, $\langle v - v_*, e \rangle = 0$ for v using Newton w/init guess v_*. $v_* := v$; End loop
Lyapunov-Schmidt Reduction Thm	Suppose $F = F(u;\mu) \in C^k$, $F(0;0) = 0$. Let P & Q be projections onto ker & rng of $\partial_u F(0;0)$. $\exists$locally defined $\varphi \in C^k : RgP \times \mathbb{R}^p \to RgQ$, s.t. sols $(u;\mu)$ to $F(u;\mu) = 0$ in 1-1 corresp. w/sols to $\varphi(\tilde{u};\mu) = 0$.

Lyapunov–Schmidt Constrct $F(x_0,\lambda_0)=0$ $(x_0,\lambda_0) \in X \times \Lambda \to Y$	$Y := Y_1 \oplus Y_2 = range(\partial_x F_0) \oplus range(\partial_x F_0)^{\perp}$. Q:= projection onto Y_1. $X := X_1 \oplus X_2 = \ker(\partial_x F_0) \oplus \ker((\partial_x F_0)^{\perp})$. Decomp $F: QF(x,\lambda) = 0, \ (I-Q)F(x,\lambda) = 0$ Let $x_1 \in X_1$ & $x_2 \in X_2$, then $QF(x_1+x_2,\lambda) = 0$ solved wrt x_2 by IFT.

Bifurcations are failures of one of the three generic properties of vector fields:	Hyperbolic equilibrium points. Hyperbolic periodic orbits Transversal intersections of stable and unstable manifolds of equilibrium points and periodic orbits.

Codimension 1 Bifurcations Exmpls. **Categorized by their vector field failures**:	**EQ**: Saddle-Node, Hopf. **P-Orbits**: Fold Limit Cycle, Flip, Torus. **Global**: Homocl of f.p.s, Homocl tang. of maniflds of p-orbits. Heterocl of f.p.s & p-orbits.

Codimension 2 Bifurcations **Exmpls**	Bogdanov-Takens. Cusp Bifurcation. Fold-Hopf. Hopf-Hopf.

Sols near Saddle-Node Bif L-S Reduc gives $\varphi(Le;\mu) = 0$ **w/**$L \in \mathbb{R}$	Suppose $F(0;0) = 0$ s.t. 1) ker of $\partial_u F(0;0)$ is 1D = span(e). 2) Reduced coeffs in Taylor exp: $\varphi(Le;\mu) = \sum_{j,k=0}^{\infty} \varphi_{jk} L^j \mu^k$ satisfy $\varphi_{01} \cdot \varphi_{20} \neq 0$ Then $\exists$sols near 0 $\Leftrightarrow -sign(\varphi_{01} \cdot \varphi_{20})\mu > 0$. One sol when $\mu = 0$, two o/w.

Saddle-Node Bif	Local bif where 2 f.p.s collide & annihilate 1D - unstable **saddle**, stable **node**. **Normal**: $\frac{dx}{dt} = r \pm x^2$. $\varphi_{20} \cdot \varphi_{01} \neq 0$ (**nondegenerate**)

Sols near Transcrit Bif L-S Reduc gives $\varphi\left(L\vec{e};\mu\right) = 0$ **w/**$L \in \mathbb{R}$	Suppose $\exists$trivial sol $F(0;\mu) = 0, \ \forall \mu$ s.t. ker of $\partial_u F(0;0)$ is 1D = span($\vec{e}$) & $\varphi\left(L\vec{e};\mu\right) = \sum_{j,k=0}^{\infty} \varphi_{jk} L^j \mu^k$ satisfies $\varphi_{11} \cdot \varphi_{20} \neq 0$. Then $\exists! u_{\pm} \neq 0$ to $F(u;\mu) = 0$ for $\mu \neq 0, \ \mu << 1$ w/exp: $u_{\pm}(\mu) = -\frac{\varphi_{11}\mu}{\varphi_{20}}\vec{e} + O(\mu^2)$.

Transcrit Bif	$\exists$ f.p.s $u_{+/-}, \ \forall r$. However, $u_{+/-}$ switch stability as μ varies & they collide. **Normal**: $\frac{du}{dt} = ru - u^2$. $u_{+/-} \in \{0,r\}$. Bif @ $r = 0$. $\varphi_{11} \cdot \varphi_{20} \neq 0$

Vector field f **Equivariant with respect to** Γ, **where** $\Gamma \subset O(n)$	$\forall \gamma \in \Gamma, \ f(\gamma u;\mu) = \gamma f(u;\mu)$ and $L\gamma = \gamma L$, with $L = \partial_u f(0;0)$. So, $Ker(L)$ is invariant under Γ, or Γ acts on $Ker(L)$

Equivariant L-S Lemma for $\Gamma \subset O(n)$, where $L := \partial_u f(0,0)$	If we choose projections P & Q in the L-S Thm as orthogonal projections then L-S Reduction $\varphi(u_0;\mu) = 0$ is equivariant wrt Γ on $Ker(L)$ and $coker(L)$

Sol Near Pitchfork Bifurcation	Suppose $F \in C^3$ s.t. $F(\gamma u;\mu) = \gamma F(u;\mu)$, for a $\gamma \in O(n)$, $F(0;0) = 0$, $Ker(\partial_u F(0;0)) = span(e)$ is 1D, $\gamma e = -e$, $\varphi_{11} \cdot \varphi_{30} \neq 0$, and $\varphi_{10} = \varphi_{20} = \varphi_{01} = 0$. Sols u,μ near 0 include $u \equiv 0$, & $u = \pm\sqrt{\frac{-\mu\varphi_{11}}{\varphi_{30}}}$

Pitchfork Bifurcation w/Normal Form	Local bif w/ 1 f.p. → 3 f.p.s. $\varphi_{11} \cdot \varphi_{30} \neq 0$. $\varphi_{10} = \varphi_{20} = \varphi_{01} = 0$ Supercrit Norm: $\frac{du}{dt} = \mu u - u^3$. Two stable f.p.s @ $u = \pm\sqrt{\mu}$ Subcrit Norm: $\frac{du}{dt} = \mu u + u^3$. Two unstable f.p.s @ $u \pm \sqrt{-\mu}$

Newton Polygon For $f(u;\mu)$. Ex: $\mu^4+\mu u+u^3+\mu^2 u$	Locate terms assoc. w/Leading Order Segment. e.g: u^3, μu. Equate them, divide out common factors, & scale, e.g: $u = u_1\varepsilon$ w/$\lvert u_1\rvert = 1$, so: $\mu = \varepsilon^2$. Substitute into f, simplify, set $\varepsilon = 0$. e.g: $u_1+u_1^3 = 0$ w/3 sols $u_1 = 0, \pm i$. So, $u = \pm i\mu^{1/2} + O(\mu)$. $f \in C^\omega \Rightarrow$ find all sols

Stable Manifld Thm	Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be C^k ($k \geq 1$) w/$f(p) = 0$. $\mathbb{R}^n$ splits into eignspcs of Df_p:$\mathbb{R}^n$=$\mathbb{E}^u \oplus \mathbb{E}^c \oplus \mathbb{E}^s$ for $\lambda >,=,< 1$ (discrete) or $\lambda >,=,< 0$ (cont.). $\exists$nhbd $\mathcal{U}_p \subset U$, s.t. $\mathcal{W}^s$ & $\mathcal{W}^u$ are C^k manif. tang. to $\mathbb{E}^s$ & $\mathbb{E}^u$. $\mathcal{W}^{s/u}$ Pos/Neg Invariant. Given init. cond. $x \in \mathcal{W}^s$, $\lim_{t\to\infty} x(t) = 0$, or $x \in \mathcal{W}^u$, $\lim_{t\to-\infty} x(t) = 0$.

Center Manifold Thm I	Let $f \in C^k$, $k \geq 1$. $\exists nbhd$ $\mathcal{U}$ of 0 in $\mathbb{R}^n \times \mathbb{R}^p$ & $h : \mathcal{U} \cap (\mathbb{E}^c \times \mathbb{R}^p) \to \mathbb{E}^h$, $h(0,0) = 0$, $h \in C^k$, such that $\mathcal{W}^c = graph(h)$, is a center manifold w/ properties described on a different flashcard.

Center Manifold Thm II	Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be C^k w/$k \geq 1$ & $f(0) = 0$. Assume $0 < \mu < 1 < \lambda$. For $r > 0$ define $\bar{f} \in C^k$ w/$\bar{f}\lvert B(0,r) = f\lvert B(0,r)$, $\lvert \bar{f} - Df_0 \rvert_{C^1} < \varepsilon$, & $\bar{f} = Df_0$ off $B(0,2r)$. If ε & r small enough, $\exists$invariant $\mathcal{W}^{cs}(0,\bar{f}) \in C^k$, tang. to $\mathbb{E}^c \oplus \mathbb{E}^s$ at 0, w/ $\mathcal{W}^{cs} = \left\{q : d\left(\bar{f}^j(q),0\right)\lambda^{-j} \to 0 \text{ as } j \to \infty\right\}$, similarly W^{cu}

Center Manifold Properties	h is: **Maximal**: $u(t) \in \mathcal{U}$, $\forall t \Rightarrow u(t) \in \mathcal{W}^c$, $\forall t$. **Tangent** to $\mathbb{E}^c$: $\partial_u h(0,0) = 0$. **Stable**: If $\mathbb{E}^u = \{0\}$: $\exists C, \eta > 0$ s.t. if $u(t) \in \mathcal{U}$, $\forall t \geq 0$, then $\exists z(t) \in \mathcal{W}^c, \forall t \geq 0$ s.t. $\lvert u(t) - z(t)\rvert \leq Ce^{-\eta t}$, $\forall t \geq 0$. CM not unique, unless all sols on CM are bounded in $\mathcal{U}$.

Hopf Bifurcation	Local birth or death of a per-sol from a f.p. as parameter crosses a crit value. Conjugate $\lambda_\pm$ pair of $\partial_u f(0,0)$ $\to \pm i\omega$ as $\mu \to crit$. Need $Re\lambda'(0) \neq 0$. Then $\exists$p-sols. P-sols asy-stable when f.p. unstable, & unstable o/w. Normal: $u' = u(r - \lvert u\rvert^2)$

Normal Form	For systems w/ certain bifs, a NF is a simplified form of the DEQ which is locally topologically equivalent to the original system.

Normal Form Prop 1 $x' = Lx + g(x) \in \mathbb{R}^n$, $g \in C^\infty, \ g = O(\lvert x \rvert^2)$	Let $H_\ell \subseteq Y_\ell, \ \ell \geq 2$, subspace of homog. polys of degree ℓ s.t. $Rg(ad_\ell L) \oplus H_\ell = Y_\ell$. $\exists$ seq of polys transfs $id + \Phi_\ell \cdot y^{(\ell)}$ w/$y^{(\ell)} \in H_\ell$ and $\Phi_\ell \in \mathbb{R}^k$ for some k s.t. we end up with NF: $y' = Ly + g^{nf}(y) + O\left(\lvert y \rvert^{k+1}\right), \quad g^{nf}(y) = \sum_{\ell=2}^{k} g_\ell^{nf} \cdot y^{(\ell)}$, where $g_\ell^{nf} \in \mathbb{R}^k$

Normal Form Prop 2	We can choose $H_\ell = ker(ad_\ell(L^*))$, that is: $ker(ad_\ell(L^*) \oplus R_g(ad_\ell L)) = Y_\ell$.

Normal Form Corollary	A sequence of normal form transformations can achieve $ad_\ell(L^*) = (ad_\ell(L))^*$ up to any order: $g_{new}(e^{L^*\varphi}y) = e^{L^*\varphi} g_{new}(y)$, for all $\varphi \in \mathbb{R}^n$.

Bifurcation w/co-dimension n	Means that n parameters must be varied for all relevant bifurcations to occur. Zero eigenvalue w\algebraic multiplicity n

Bogdanov–Takens Bif $y' = f(y, \beta)$	Bif w/co-dim 2. $f(p) = p$. $\partial_u f(p)$ has double eigenvalue@0. Normal: $y_1' = y_2$. $y_2' = \beta_1 + \beta_2 y_1 + y_1^2 \pm y_1 y_2$. 3 co-dim 1 bifs nearby: SN, Hopf & Homoc.

Separatrix	Boundary separating two modes of behaviour in a differential equation.
Poincaré Bendixson General	Let $\vec{f} \in C^1(E)$ where $E \subseteq \mathbb{R}^2$ open & $\dot{x} = f(x)$ has trajectry Γ w\\$\Gamma^+ \subseteq F$ compct subst of E. Suppose only finite # of f.p.s in F, then $\omega(\Gamma)$ is either f.p., periodic orbit, or consists of finite # of f.p.s $\vec{p}_1, \ldots, \vec{p}_m$ w/countable # of limit orbits whose $\alpha \cup \omega \in \{\vec{p}_1, \ldots, \vec{p}_m\}$.
Devil's Staircase	$f(x) \in C^0, f' = 0$ off Cantor, but rises from 0→1. Take $x_0 \in [0,1]$, express x_0 in base 3. Chop off base 3 expansion after first "1." Change 2s→1s. Now f has only 0's or 1's in expansion, We interpret it as base 2. Call this new number $f(x_0)$. $f(x)$ is the Devils Staircase.
Circle Map	C^1 orientation preserving homeomorphism of the circle, S^1, into itself: $f: S^1 \to S^1$.
Lift of a Circle Map	Let $\Pi(x) : \mathbb{R} \to S^1$, where $\Pi(x) = e^{2\pi i x}$. The map $F : \mathbb{R} \to \mathbb{R}$ is said to be a lift of $f: S^1 \to S^1$ if $\Pi \circ F = f \circ \Pi$. "Lift F accomplishes f, but on $\mathbb{R}$."
Lifts Vary by an Integer Thm	Let $f: S^1 \to S^1$ be orientation preserving homeomorphism of circle. Let F_1 & F_2 be lifts of f. Then $F_1 = F_2 + k$, where k is some integer. **Proof**: Two lifts must satisfy $f \circ \Pi = \Pi \circ F_{1,2} = e^{2\pi i F_1} = e^{2\pi i F_2}$, so $F_1 = F_2 + k$.
Circle Map Lift Iterates Thm	If F is a lift of f, then F^n is a lift of f^n for $n \geq 1$. **Proof**: By definition: $\Pi \circ F = f \circ \Pi$. Therefore $\Pi \circ F^2 = \Pi \circ F \circ F = f \circ \Pi \circ F = f \circ f \circ \Pi = f^2 \circ \Pi$. And similarly for n.
Lift Arguments Expel Integers Thm	Let $f: S^1 \to S^1$ be an orientation preserving homeomorphism of the circle and let F be a lift. Then $F(x+k) = F(x) + k$, for $k \in \mathbb{Z}$
Periodic Functns from Lift Iterates	Let $f: S^1 \to S^1$ be orientation preserving homeomorphism of circle & let F be a lift of f. Then $F^n - id$ is a periodic function with period one for $n \geq 1$.

Lift Rotation Number ρ_0	For orientation preserving homeomorphism $f : S^1 \to S^1$, with F a lift of f: $\rho_0(F) \equiv \lim_{n\to\infty} \frac{\|F^n(x)\|}{n}$
Different Lift $\rho_0 s$ Differ by an Integer	Let $S^1 \to S^1$ be orientation preserving homeomorphism & let F_1 & F_2 be lifts s.t. $\rho_0(F_1)$ & $\rho_0(F_2)$ exist. Then, $\rho_0(F_1) = \rho_0(F_2) + k$, where $k \in \mathbb{Z}$.
Rotation Number	For $f : S^1 \to S^1$ an orientation preserving homeomorphism, with F a lift of f : the rotation number of f, denoted to $\rho(f)$ is the fractional part of $\rho_0(F)$.
Rotation Number Existence	For an orientation preserving homeomorphism $f : S^1 \to S^1$ with F, a lift of f, the rotation number exists and it is independent of x.
Periodic Points from Rotation Numbers	A rotation number is irrational if and only if f has no periodic points
Conjugate Invariance of Rotation Number	Let f & g be orientation preserving homeomorphisms of S^1, then $\rho(f) = \rho(g^{-1}fg)$
Rational Rotation Number: $\frac{p}{q}$	Given init cond, $\exists$three possibilities for orbit. $\frac{p}{q}$ periodic orbit (PO) Homoclinic orbit. Asy approaches PO as $n \to \pm\infty$. Heteroclinic orbit. Asy approaches PO as $n \to -\infty$ & different PO as $n \to +\infty$.
Irrational Rotation Number	Given initial condition, there are three possibilities Orbit that densely fills circle. Orbit that densely fills a Cantor set on circle. Orbit homoclinic to a Cantor set on circle.
Linear Stability Characterization $\dot{x} = Ax$	Origin is linearly stable if $\|e^{At}\|$ is uniformly bounded for $t > 0$; it is asymptotically stable if $\|e^{At}\| \to 0$ for $t \to \infty$.

Linear Asy Stability σ(A)	Asymptotically stable if and only if $Re(\sigma(A)) < 0$. In this case, there exist constants $C, \delta > 0$ such that: $\lvert e^{At} \rvert \leq Ce^{-\delta t}$.
Linear Stability σ(A)	If and only if $Re(\sigma(A)) \leq 0$ and all eigenvalues with $Re(\lambda) = 0$ are semi-simple.
Speed of f.p. Asymptotic Convergence	$\exists C, \varepsilon, \delta > 0$ s.t. $\forall$(init cond) w\\$\lvert x_0 \rvert < \varepsilon$: $\lvert x(t) \rvert \leq Ce^{-\delta t} \lvert x_0 \rvert$. Constant $-\delta$ must be chosen larger than, but arbitrarily close to, $\max Re(\sigma(A))$.
Asymptotic Stability via Poincaré Maps	Assume Floquet multipliers lie inside unit circle except $\lambda = 1$, algebraically simple. Then the Poincaré map is a contraction near $\gamma(0)$ in a suitably defined norm.
Asy stability of periodic orbits	If Floquet exponents λ are s.t. $\{\text{Re } \lambda < 0\}$ except for an algebraically simple $\lambda = 0$, then periodic orbit $\Gamma = \left\{\gamma(t),\ 0 \leq t < T\right\}$ is asymptotically stable. $\exists C, \eta > 0$, a nghbrhd $U(\Gamma)$, & smooth θ: $U \to \mathbb{R}/(T\mathbb{Z})$ s.t. $\forall x_0 \in U, \lvert x(t) - \gamma(t - \theta(x_0)) \rvert \leq Ce^{-\eta t}\ \forall t \geq 0$
Strong -Stable Maniflds	$f(0) = 0$ & $A = Df(0)$ has splitting@$-\eta$, for some $\eta > 0$, $\mathbb{R}^n = E^{ss} \oplus E^{wu}$, w/projection $P^{ss}E^{ss} = E^{ss}$, $P^{ss}E^{wu} = \emptyset$, $AP^{ss} = P^{ss}A$, $Re(\sigma(A))\rvert_{E^{ss}} < -\eta$, $Re(\sigma(A))\rvert_{E^{wu}} > -\eta$. Can characterize E^{ss} as set of x_0 s.t. $\lvert e^{At}x_0 \rvert \leq Ce^{-\eta t}$, for all $t \geq 0$. Strong stable manifold: $W^{ss} = \{x_0 \rvert \Phi_t(x_0) \leq Ce^{-\eta t}\}$.
Center Manifld	$\{$weak-unstable$\} \cap \{$weak-stable$\}$ manflds gives loclly invar. manfld tang. to subspc of λs w\\$-\eta_- < \lambda < \eta_+$ Choosing $\eta_\pm << 1$, subspace contains precisely generalized eigenspace to $\lambda s \in i\mathbb{R}$. Such a manifold tangent to this eigenspace exists, of class C^k, $\forall k < \infty$, if $f \in C^k$.
Structural Stability	One considers perturbations of the vector field, as opposed to perturbations of the initial data.
Conservative System	$\dot{x} = f(x)$, $x \in \mathbb{R}^n$ is considered conservative if there exists a C^1 scalar function $E : \Omega \to \mathbb{R}$ which is not constant on any open set in Ω, but is constant on orbits.

Non-asymptotic stable f.p.s in Conservative Sys	An equilibrium point q of a conservative system cannot be asymptotically stable.

Strong Minimum	$E : \Omega \to \mathbb{R}$ has strong minimum at q if $\exists$nghbrhd N of q s.t. $E(x) > E(q)$ for every $x \in N$ except for $x = q$.

Stability from Strong Minima	Suppose q is f.p. of conservative, autonomous sys & that its integral E has strong minimum there. Then q is stable.

Linear Instb $\not\Rightarrow$ Nonlinear Instb Example	$H = p^4 + q^2$. So $\dot{p} = -\partial H_q$ & $\dot{q} = \partial H_p$, and linearized eqs are $\dot{q} = 4p^3$, $\dot{p} = -2q$. $\lambda = 0$, so: $A := (J_0 - \lambda\mathbb{I}) = J_0$. Calculating gen. evects: $A^2 v_2 := A^2\langle 1\ 0\rangle = 0$ & $Av_2 = \langle 0,-2\rangle =: v_1$ has sols increasng linearly w/t: $\langle q,p\rangle = c_1\langle 0,-2\rangle + c_2(\langle 0,-2\rangle t + \langle 1,0\rangle)$. But $\langle 0,0\rangle$ is strict Min of H

Linear Stb $\not\Rightarrow$ Nonlinear Stb Example	$H = \frac{1}{2}(q_1^2 + p_1^2) - (q_2^2 + p_2^2) - \frac{1}{2}p_2(q_1^2 - p_1^2) - q_1 q_2 p_1$. Linearized sys can be read from EOM. Has periodic sols. For $T > 0$: $p_1 = \sqrt{2}\,\frac{\sin(t-T)}{t-T}$, $q_1 = \sqrt{2}\,\frac{\cos(t-T)}{t-T}$, $q_2 = \frac{\cos(2(t-T))}{t-T}$, $p_2 = \frac{\sin(2(t-T))}{t-T}$ is sol which blows up when $t = T$.

Nonlinear Circle Map	$\theta_{n+1} = \theta_n + \Omega + \frac{K}{2\pi}\sin(2\pi\theta_n) \pmod 1$, where K is the coupling strength which determines the degree of nonlinearity, and Ω is an externally applied driving frequency.

Arnold Tongue AT	Mode-locked region in driven weakly-coupled harmonc oscillatr. $\theta_{n+1} = \theta_n + \Omega + \frac{K}{2\pi}\sin(2\pi\theta_n) \pmod 1$ AT around each $\Omega \in \mathbb{Q}$. External freq. Ω. If $K \neq 0$, motion may be periodic in finite region. $K = 0 \Rightarrow A = \{\mathbb{Q}\}$. $K = 1 \Rightarrow A = \{\text{Cantor}\}$.

Mode locking for Arnold	For $0 < K < 1$, in Mode Locked region, θ_n have a limiting behavior as a rational multiple of n. **Rotation (map winding) number** : $\omega = \lim_{n\to\infty} \frac{\theta_n}{n}$. Regions form V-shape that touch down to rational $\Omega = \frac{p}{q}$ in limit of $K \to 0$.