MATH 1271: Calculus I

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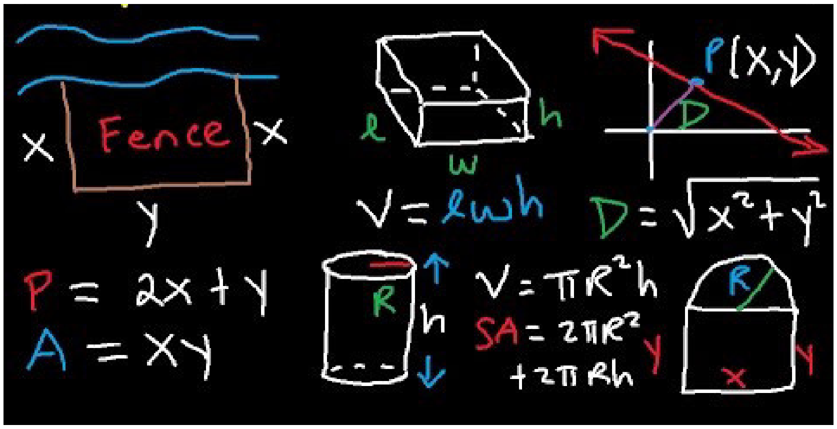
4.7 - Optimization Problems

Review



Steps in Solving Optimization Problems

- **Understand the Problem**: What is the unknown? What are the given quantities? What are the given conditions?

- **Draw a Diagram**: And identify the given and desired quantities on the diagram.

- **Introduce Notation**: Assign a symbol to the quantity that is to be maximized or minimized (for example, f). Select symbols (x, y, h, a, c, etc.) for the other unknown quantities, and label the diagram.

- **Write an expression**: Write $f = \ldots$ in terms of the above symbols (x, a, c).

- **Eliminate Variables**: If f is a function of more than one variable (e.g., $f(x,a,c) = x^2 + 3a + 3c$), use the information given in the problem to find relationships (in the form of equations) among these variables (like the area of a triangle, or the volume of a sphere. For example: $a = 2c,\ c = 5x$). Then, use substitution (or some similar process) to eliminate all but a remaining variable x in the expression for f. Continuing our example:
$$f(x) = x^2 + 3(2c) + 3c = x^2 + 3(2(5x)) + 3(5x) = x^2 + 30x + 15x = x^2 + 45x.$$

- **Find Absolute Max/Min**: Use methods from 4.1 and 4.3 to find the absolute maximum or minimum of f.

Problem 2. Find two positive numbers whose product is 100 and whose sum is minimized.

$xy = 100,$ and $f := x + y$

We want to minimize f, but first we want to simplify the expression with a substitution.

$y = \frac{100}{x}$ (why can we do this?)

$f = x + \frac{100}{x}$

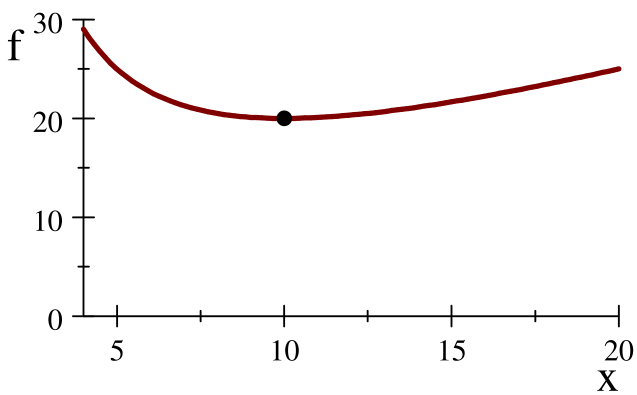
$f' = 1 - \frac{100}{x^2}$

$f' = 0$ when $x^2 = 100$, or $x = 10$ (recall $x > 0$).

Therefore, $y = \frac{100}{x} = 10$.

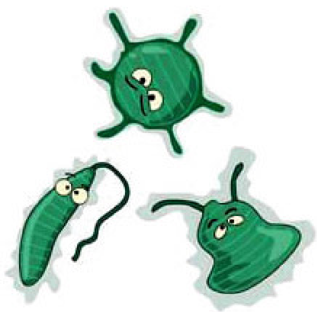
However, is this a minimum, a maximum, or something else? Observe that $f'(1) = 1 - 100 < 0$ and $f'(11) = 1 - \frac{100}{121} \approx 0.17355 > 0$. So by the first derivative test, it is a minimum.

So, it must be that $f = x + y = 10 + 10 = 20$ is the minimum sum of a pair of positive numbers whose product is 100. ■



$x + \frac{100}{x}$

Problem 10. The rate at which photosynthesis takes place for a species of phytoplankton (in milligrams of carbon/m^3/hr) is modeled by the function: $P(I) = \frac{100I}{I^2+I+4}$, where I is the light intensity (measured in thousands of foot-candles). For what light intensity I is P a maximum?



We need to maximize P for $I \geq 0$.

$$P'(I) = \frac{100(I^2+I+4)-100I(2I+1)}{(I^2+I+4)^2} = \frac{100(I^2+I+4-2I^2-I)}{(I^2+I+4)^2} = \frac{-100(I^2-4)}{(I^2+I+4)^2}$$

$P'(I) = 0$ when $-100(I+2)(I-2) = 0$, or at $I = 2$. (Why not $I = -2$?)

(note the use of difference of squares!)

Graphically, if $I = 2$ is a maximum, then we expect the slope of the graph $P'(I)$ to be greater than zero when $I < 2$ and and less than zero when $I > 2$.

So we want to know the sign of $P'(I) = -100(I+2)(I-2)$,

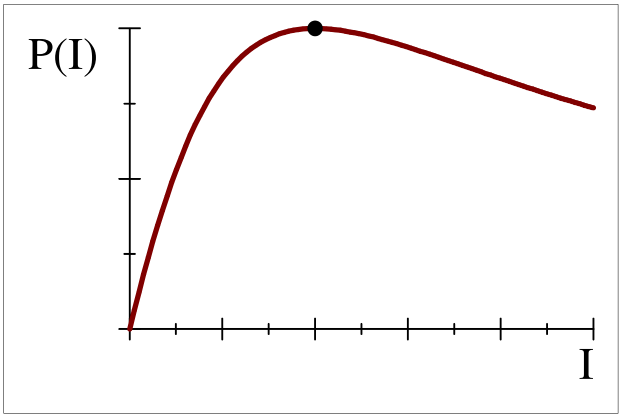
which is the same as the sign of $-(I+2)(I-2)$. (notice that the sign is changing when $I = \pm 2$)

So, we test points $I = 0$, and $I = 3$ (we didn't check $I = -3$ since $I \geq 0$). We discover…

$P'(I) > 0$ when $0 \leq I < 2$.

On the other hand, checking $I = 3$ we have $P'(I) < 0$ for $I > 2$.

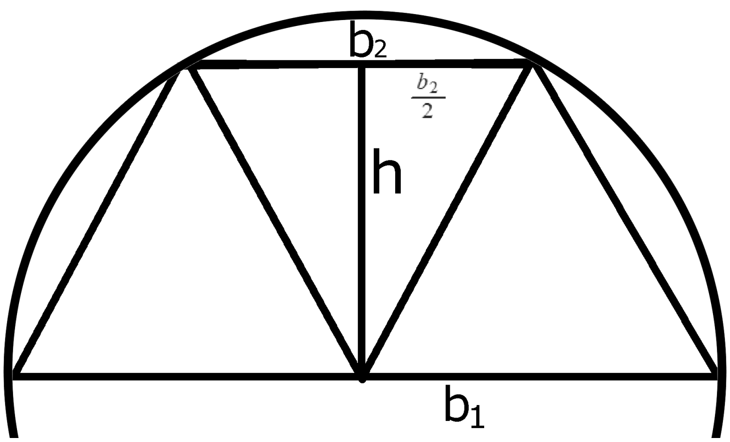
Thus, P has an absolute maximum of $P(2) = \frac{100 \cdot 2}{2^2+2+4} = 20$.



$\frac{100I}{I^2+I+4}$

Problem 26. Find the area of the largest trapezoid that can be inscribed in a circle of radius 1 and whose base is a diameter of the circle?

The area A of any trapezoid is given by $A = \frac{1}{2}h(b_1 + b_2)$.



Observe that, $b_1 = 2 \cdot radius = 2$.

so $A = \frac{1}{2}h(2 + b_2) = h\left(1 + \frac{b_2}{2}\right)$.

Based upon the strategies we talked about in the review, we would ideally eliminate either h or b_2 using some equation which relates the two.

Observe that (due to the triangle in the diagram), we have: $h^2 + \left(\frac{b_2}{2}\right)^2 = r^2 = 1$, or $h = \sqrt{1 - \frac{b_2^2}{4}}$.

So, $A = \sqrt{1 - \frac{b_2^2}{4}}\left(1 + \frac{b_2}{2}\right)$. This is progress, but it looks hard to differentiate.

Observe that it's easier to work with: $A^2 = h^2\left(1 + \frac{b_2}{2}\right)^2$

$$= \left(1 - \frac{b_2^2}{4}\right)\left(1 + \frac{b_2}{2}\right)^2.$$

A common way of dealing with these types of situations is to notice that when A^2 is maximized, so is A (Assuming $A \neq 0$, we have: $(A^2)' = 2AA' = 0$ when $A' = 0$).

Therefore, we can focus on minimizing A^2.

So we have the function: $A^2 := \left(1 - \left(\frac{b_2}{2}\right)^2\right)\left(1 + \frac{b_2}{2}\right)^2.$

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Taking the derivative (with respect to b_2) to find its maximum, we have:

$$\frac{d}{db_2}A^2 = -\frac{b_2}{2}\left(1 + \frac{b_2}{2}\right)^2 + \left(1 - \left(\frac{b_2}{2}\right)^2\right)\left(1 + \frac{b_2}{2}\right)$$
$$= \left(\left(-\frac{b_2}{2} - \frac{b_2^2}{4}\right) + 1 - \frac{b_2^2}{4}\right)\left(1 + \frac{b_2}{2}\right)$$
$$= \left(1 - \frac{b_2}{2} - \frac{b_2^2}{2}\right)\left(1 + \frac{b_2}{2}\right) = -\frac{1}{2}(b_2^2 + b_2 - 2)\frac{1}{2}(2 + b_2)$$
$$= -\frac{1}{4}(b_2 + 2)^2(b_2 - 1)$$

$\frac{d}{db_2}A^2 = 0$ when $b_2 = -2$ or $b_2 = 1$.

Obviously we want a positive length for b_2.

Also observe that $\frac{d}{db_2}A^2 > 0$ if $b_2 < 1$, and $\frac{d}{db_2}A^2 < 0$ if $b_2 > 1$, ...

so we get a maximum at $b_2 = 1$.

Reminder: Our task is to find the maximum area $A = h\left(1 + \frac{b_2}{2}\right)$,

and recall that $h = \sqrt{1 - \frac{b_2^2}{4}}$.

So at the maximum, $h = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$ and the maximum area is...

$$A_{\max} = h\left(1 + \frac{b_2}{2}\right) = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) = \frac{3\sqrt{3}}{4} \approx 1.3.$$

Graph of: $A = \sqrt{1 - \frac{b_2^2}{4}}\left(1 + \frac{b_2}{2}\right)$.