MATH 1271: Calculus I

Discussion Instructor: Jodin Morey
moreyjc@umn.edu
Website: math.umn.edu/~moreyjc

3.6 - Derivatives of Logarithmic Functions

Review:

$\frac{d}{dx}(\log_e x) = \frac{d}{dx}(\ln x) = \frac{1}{x \cdot \ln e} \cdot \frac{d}{dx}x = \frac{1}{x}$.

But more generally for any function $f(x)$ and log base a:

$$\frac{d}{dx}(\log_a(f)) = \frac{d}{dx}\left(\frac{\ln f}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}\ln f = \frac{1}{\ln a} \cdot \left(\frac{1}{f} \cdot f'\right) = \frac{f'}{f \cdot \ln a}.$$

(using the log rule from algebra, and the chain rule)
For example $\frac{d}{dx}(\log_a x) = \frac{1}{x \cdot \ln a} \cdot \frac{d}{dx}x = \frac{1}{x \cdot \ln a}$.

More Notation/Examples:
$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ or $\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$.

If we have $f(x) = \ln|x|$, then since $f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$, it must be that

$f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0 \end{cases}$, therefore $\frac{d}{dx}\ln|x| = \frac{1}{x}$, when $x \neq 0$.

This is also more generally true in that: $\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}$, when $f(x) \neq 0$.

Steps in Logarithmic Differentiation of an equation: $y = f(x)$
- Take natural logarithm of both sides and use the Laws of Logarithms to simplify.
- Differentiate implicitly with respect to x.
- Solve the resulting equation for y'.

Note the following weird fact: $e = \lim_{x \to 0}(1+x)^{\frac{1}{x}} = \lim_{n \to \infty}(1+\frac{1}{n})^n$, for $n \in \mathbb{N}$.

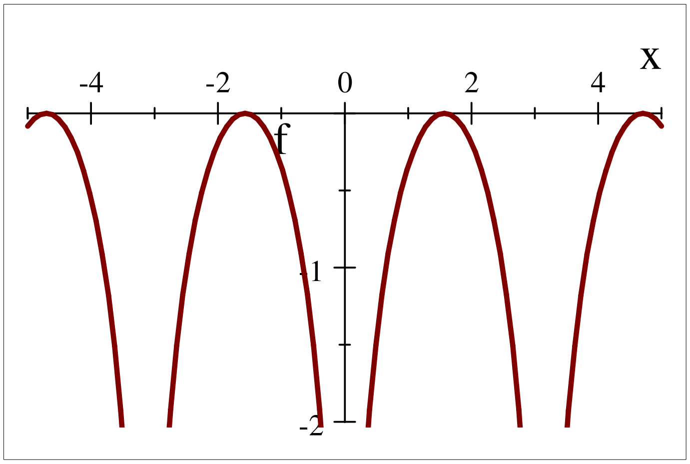
Problem 4. Differentiate the function: $f(x) = \ln[\sin^2 x]$

$f(x) = \ln\left[(\sin x)^2\right]$

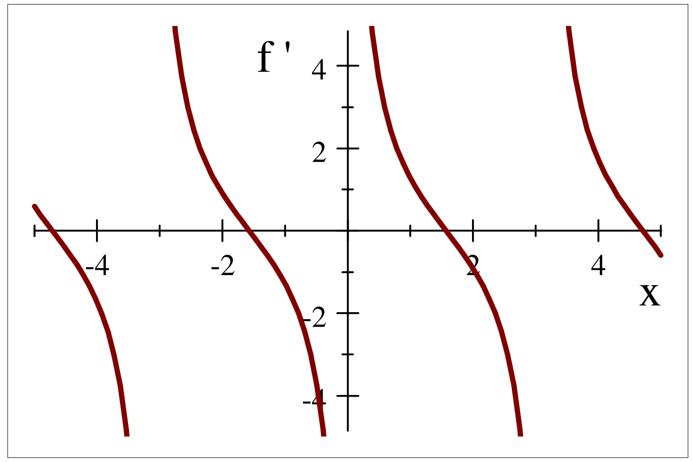
$= 2\ln|\sin x|$.

$f' = 2 \cdot \frac{1}{\sin x} \cdot \cos x$ Recall: $\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}$, when $f(x) \neq 0$.

$= 2\cot x.$



$f(x) = \ln[\sin^2 x]$



$f' = 2\cot x.$

Problem 21. Differentiate the function: $y = 2x\log_{10}\sqrt{x}$

$y = 2x\log_{10}x^{\frac{1}{2}}$

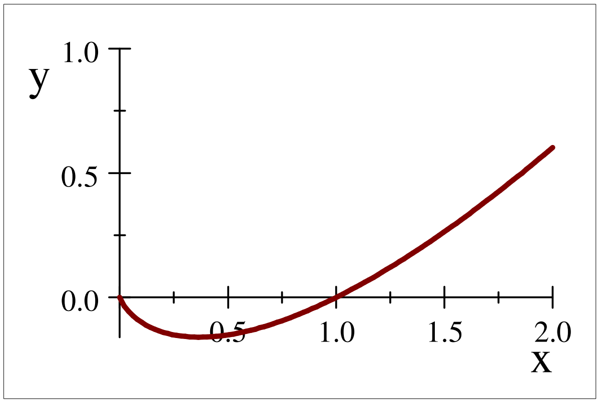
$= 2x \cdot \frac{1}{2}\log_{10}x$

$= x\log_{10}x = x\left(\frac{1}{\ln 10}\ln x\right) = \frac{1}{\ln 10}x\ln x$

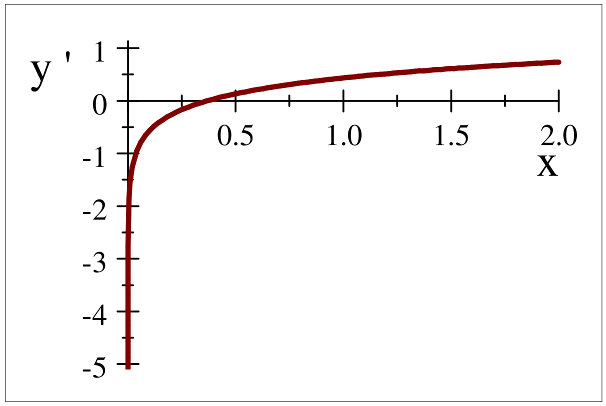
$y' = \frac{1}{\ln 10}(1 \cdot \ln x + x \cdot \frac{1}{x}) = \frac{\ln x}{\ln 10} + \frac{1}{\ln 10}.$

Note: $\frac{1}{\ln 10} = \frac{\ln e}{\ln 10} = \log_{10}e$, so the answer could be written as

$\log_{10}x + \log_{10}e = \log_{10}ex.$



$y = 2x\log_{10}\sqrt{x}$



$y' = \log_{10}ex$

Problem 30. Differentiate $f(x) = \ln(\ln(\ln x))$ and find the domain of f.

$$f' = \frac{1}{\ln(\ln x)} \cdot \frac{d}{dx}\ln(\ln x)$$

$$= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{d}{dx}\ln x$$

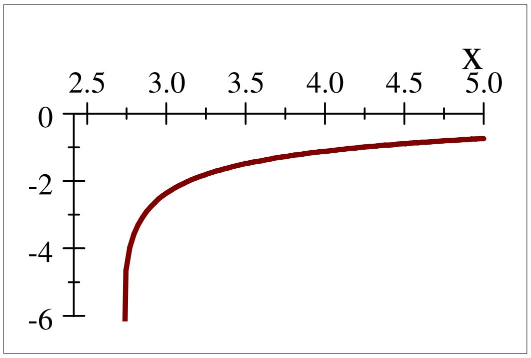
$$= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}.$$

$$Dom(f) = \{x \mid \ln(\ln x) > 0\}$$

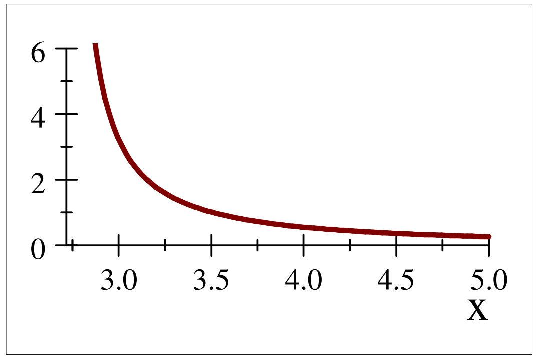
$$= \{x \mid \ln x > 1\}$$

$$= \{x \mid x > e\}$$

$$= (e, \infty).$$



$\ln(\ln(\ln x))$



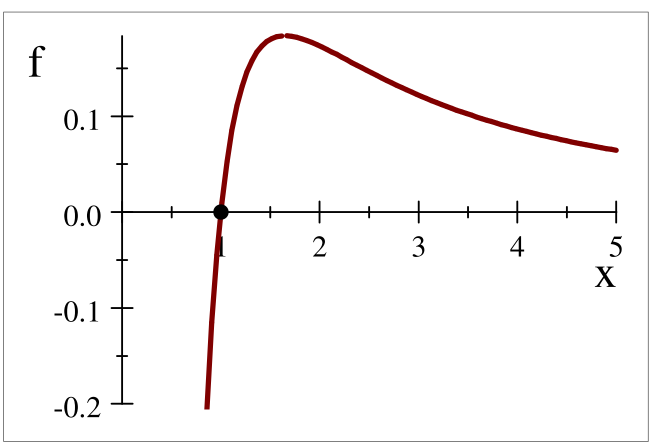
$\frac{1}{\ln(\ln x)} \bullet \frac{1}{\ln x} \bullet \frac{1}{x}$

Problem 31. If $f(x) = \frac{\ln x}{x^2}$, find $f'(1)$.

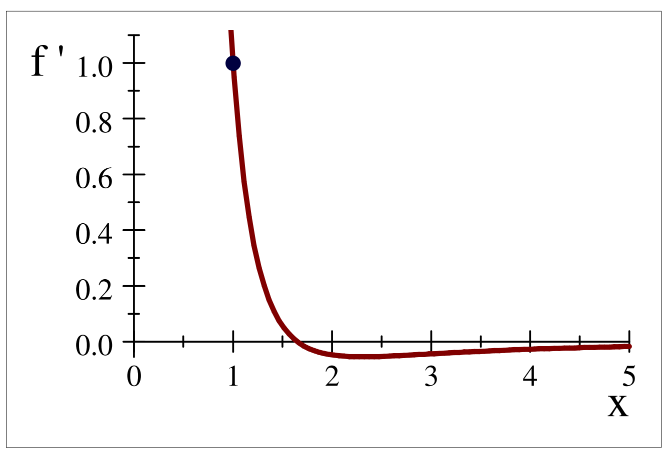
$$f' = \frac{\left(\frac{1}{x}\right)x^2 - (\ln x)(2x)}{(x^2)^2} = \frac{x - 2x\ln x}{x^4}$$

$$= \frac{1 - 2\ln x}{x^3},$$

So $f'(1) = \frac{1 - 2\ln 1}{1^3} = \frac{1 - 2 \cdot 0}{1} = 1.$



$\frac{\ln x}{x^2}$



$\frac{1 - 2\ln x}{x^3}$

Problem 48. Use logarithmic differentiation to find the derivative of: $y = (\sin x)^{\ln x}$.

$$\ln y = \ln\left((\sin x)^{\ln x}\right)$$

$$\Rightarrow \ln y = \ln x \cdot \ln(\sin x) \qquad \text{(now take your derivative!)}$$

$$\Rightarrow \frac{1}{y} y' = \frac{1}{x} \cdot \ln(\sin x) + \ln x \cdot \frac{1}{\sin x} \cdot \cos x \qquad \text{(recall: } \frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}\text{)}$$

$$\Rightarrow y' = y\left(\frac{\ln(\sin x)}{x} + \ln x \cdot \frac{\cos x}{\sin x}\right)$$

This is technically correct. However, since the problem gave us y in terms of x, (if possible) we should provide our y' in terms of x. In other words, we should eliminate the "y" in our equation.

$$\Rightarrow y' = (\sin x)^{\ln x}\left(\frac{\ln(\sin x)}{x} + \ln x \cot x\right).$$